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FURTHER THOUGHTS ON WHY (−1)(−1) = +1

Pseudoreasoning?

I am writing in response to Tina Rapke's article "Thoughts on Why $(-1)(-1) = +1$" (*Mathematics Teacher* 102, no. 5 [December 2008–January 2009]: 374–76) and what I believe to be significant flaws in her argument. Rapke states that she was motivated by a desire to explain to students why $(-1)(-1) = +1$ without falling back on what she refers to as pseudoreasoning. Unfortunately, in her exploration of this issue, she fails to explain why pattern analysis qualifies as pseudoreasoning. And she fails to present a better explanation of this number fact through the use of the distributive property.

Let me address the second issue first. Rapke's final "explanation" of why $(-1) \cdot (-1) = +1$ is nothing more than an arithmetic demonstration of the number fact. It is not an explanation of *why* this number fact is true. Nothing in the needlessly complicated algebraic manipulations helps explain the number fact. In addition, the question of why $(-1)(-1) = +1$ tends to be asked first and most frequently by students early in their study of algebra—in a prealgebra or an introductory course. The algebraic manipulations used in Rapke's demonstration require a sophisticated understanding of the distributive property in reverse—namely, factoring—which most students at this basic level do not possess.

As for pseudoreasoning, I take issue with Rapke on two points. First, I disagree that pattern recognition and analysis in this case qualify as pseudoreasoning. The patterns and reasoning she presents in figure 1 in her article (p. 375) are consistently based on the mathematical definition of multiplication as repeated addition. Thus, the conclusion students reach from these patterns—namely, that the product of two negative integers is a positive integer—is a legitimate conclusion, and the reasoning required to reach the conclusion is sound. It is not pseudoreasoning at all.

Second, Rapke's example of pattern reasoning gone awry is flawed. In her article, Rapke presents the pattern shown in **figure 1 (Rapke/Deutsch)**. From this, Rapke suggests that if students rely on pattern analysis here, they will conclude that $0^0 = 1$. I strongly disagree. Rapke has left out a critical explication in this pattern. The fourth line (the line ending in a question mark) can and should be more fully filled in, as follows: $0^0 = 0^{1-1} = 0^1/0^1 = 0/0$. Even students in a prealgebra or an introductory algebra class know that division by 0 is not allowed. Thus, this line in the pattern does not lead us to the conclusion that $0^0 = 1$; rather, it helps make clear why 0^0 is undefined. It is not, as Rapke suggests, an example of pseudoreasoning.

An aside: The pattern above relies on a strong understanding of negative exponents and exponent properties—again, something that students who are just beginning their study of algebra lack. When I am working with students who are first developing an understanding of zero and negative exponents, I prefer to use the patterns shown in **figure 2 (Rapke/Deutsch)**, which rely on the definition of integer exponents as repeated multiplication. From these patterns, students can see that the definition of exponents remains consistent even for zero and negative integer exponents. Students also conclude that a number raised to the zero power is equal to 1 ($n^0 = 1$), an observation that can lead to a productive discussion as to why 0^0 cannot be computed. Students at this level understand that, consistent with the definition of repeated multiplication, $0^n = 0$ for positive integer expo-

$3^0 = 3^{1-1} = 3^1/3^1 = 1$

$2^0 = 2^{1-1} = 2^1/2^1 = 1$

$1^0 = 1^{1-1} = 1^1/1^1 = 1$

$0^0 = ?$

$(-1)^0 = (-1)^{1-1} = (-1)^1/(-1)^1 = 1$

$(-2)^0 = (-2)^{1-1} = (-2)^1/(-2)^1 = 1$

Fig. 1 (Rapke/Deutsch)

$3^3 = 27$	$2^3 = 8$	$1^3 = 1$
$3^2 = 9\ (27 \div 3)$	$2^2 = 4\ (8 \div 2)$	$1^2 = 1\ (1 \div 1)$
$3^1 = 3\ (9 \div 3)$	$2^1 = 2\ (4 \div 2)$	$1^1 = 1\ (1 \div 1)$
$3^0 = 1\ (3 \div 3)$	$2^0 = 1\ (2 \div 2)$	$1^0 = 1\ (1 \div 1)$
$3^{-1} = 1/3\ (1 \div 3)$	$2^{-1} = 1/2\ (1 \div 2)$	$1^{-1} = 1\ (1 \div 1)$

Fig. 2 (Rapke/Deutsch)

nents. From these arguments, students encounter a dilemma—namely, that for the definition of exponents to be consistent, 0^0 needs to be equal to two values simultaneously. This is a powerful launching point for a discussion of what *indeterminate* means in mathematics and why mathematicians describe 0^0 as an indeterminate form.

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Peterson Defended

After finishing Rapke's article on avoiding pseudoreasoning for explaining why $(-1)(-1) = +1$, I felt that perhaps the intent of Peterson's "Fourteen Different Strategies for Multiplication of Integers or Why $(-1)(-1) = +1$" (1972) was misunderstood.

I agree with Rapke's stance that pseudoreasoning should not replace a mathematically correct and logical explanation, and I feel Peterson's article advocates the same idea. The linear pattern that Rapke references from Peterson's article (pattern B, fig. 1 in Rapke's article) is

$-3 \times 2 = -6$
$-3 \times 1 = -3$
$-3 \times 0 = 0$
$-3 \times -1 = 3$

I believe that this pattern is less misleading and more helpful than the illustration Rapke prefers. In addition, I believe that there is a small error in the powers of zero pattern: 0^0 is an indeterminate form, not an undefined form.

Peterson's reason for providing fourteen strategies is that no one strategy can work for all people. He argues that a good teacher will have available a wide range of techniques ready to teach the idea behind this product. Also, some of the fourteen strategies establish a context for why the product might be positive. Although the distributive property argument given by Rapke is mathematically sound, it does not provide a context for why the product is what it is.

Finally, Peterson does present both a deductive and a distributive argument very similar to those Rapke presents. Peterson indicates that these arguments are the most mathematically sophisticated and require assumptions or a certain level of mathematical understanding to make sense. For me, this is the strength of Peterson's original article—not that pseudoreasoning replaces mathematical argument but that the teacher who is teaching students of differing ability levels has at the ready a multitude of mathematically sound reasons or contexts to justify that the product of two negatives is positive.

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Rapke replies: The publication of my article "Thoughts on Why $(-1)(-1) = +1$" generated many additional interesting thoughts. I write this response to touch on and share the most frequently questioned matters that may be of interest to *MT* readers.

Some readers had questions about pattern analysis and what I meant by pseudoreasoning with regard to Peterson's (1972) table, discussed in the article. I feel the need to clarify my introduction of pseudoreasoning. I meant to draw attention to the many ways in which it is possible to mask, or distract students from, an important piece of mathematical reasoning to point to a perhaps more palatable, simple, or obvious piece of reasoning. I will try to demonstrate more clearly what I mean by referring again to an example from Peterson. He presents the following pattern:

$3 \times 2 = 6$
$3 \times 1 = 6 - 3 = 3$
$3 \times 0 = 3 - 3 = 0$
$3 \times (-1) = ?$

Peterson claims that "it is necessary to rely only on students' assumptions of linearity" (p. 399). If we consider pattern A (fig. 1 of my article), we notice the pattern that each solution is obtained by subtracting 3 from the previous one. Using this logic, we may reasonably conclude that $3 \times (-1) = 0 - 3 = -3$.

One might notice that there is something missing—the distributive property! The distributive property *is* the reasoning behind the equal signs, or pattern. Here is a version that takes into account the distributive property:

Pattern A	
$3 \times 2 = 6$	subtract 3 from each side of the first to get:
$3 \times 2 - 3 = 6 - 3$	
$3 \times (2 - 1) = 6 - 3$	by the distributive property, or cleaned up:
$3 \times 1 = 6 - 3 = 3$	the second entry in Peterson's pattern

For the third entry, we subtract 3 from both sides of the second (previous) entry:

$3 \times 1 - 3 = 3 - 3$	
$3 \times (1 - 1) = 3 - 3$	by the distributive property, or cleaned up looks like:
$3 \times 0 = 3 - 3 = 0$	the third entry in Peterson's pattern

The distributive property was masked by pattern logic.

In the article, I then went on to play devil's advocate and provided an example of how relying on pattern logic might go wrong. I presented the pattern shown in **figure 1 (Rapke/Deutsch)**. I suggested that it would be reasonable for students to conclude, on the basis of pattern recognition, that $0^0 = 1$. Deutsch suggests that if the question mark were replaced by 0/0, then it would not be possible for a student to conclude that 0/0 = 1, because even students in prealgebra and introductory algebra know that division by zero is not allowed. This is an interesting statement and perhaps takes us to the crux of the matter. I refer to Ball (1991), who discusses the misconceptions of division by zero that are held by preservice teachers. Research shows, then, that many preservice teachers, who themselves were once prealgebra and introductory algebra students, still do not "know that division by zero is not allowed." Thus, it should not surprise us that many students of algebra, although they may have been told or shown this, do not "know" it in any rich and meaningful sense of the word. Results of a study involving 153 students from Israel "indicate that about a third performed the division of a non-zero number by zero and reached zero, the dividend or 'the number infinity'" (Tsamir and Tirosh 2002, p. 341).

Some readers also questioned whether 0^0 is undefined. From my experience, students seem to have difficulty with 0^0 precisely because of all the complexities involved. Indeed, there is something wrong with the teaching and learning of this concept if it is merely accepted as a "fact" of mathematics that 0^0 is undefined. Consider a discussion provided by Weisstein:

> 0^0 itself is undefined. The lack of a well-defined meaning for this quantity follows from the mutually contradictory facts that a^0 is always 1, so 0^0 should equal 1, but 0^a is always 0 (for $a > 0$), so 0^0 should equal 0. It could be argued that $0^0 = 1$ is a natural definition since
>
> $$\lim_{n\to 0} n^n = \lim_{n\to 0^+} n^n = \lim_{n\to 0^-} n^n = 1. \qquad (2)$$
>
> However, the limit does not exist for general complex values of n. Therefore, 0^0 is usually defined as indeterminate. (Weisstein n.d.)

We are now on very fruitful grounds that, for teachers, have the potential to lead to rich student discussions involving undefined quantities, division by zero, and exponentiation.

Some readers suggested that I misunderstood Peterson. Although Peterson did reason similarly on the basis of the distributive property, I felt that further discussion was called for. Peterson warned that his similar deductive and distributive arguments are mathematically sophisticated but did not provide a discussion surrounding the idea that the number systems we use have axioms at their heart. Because these axioms can be transferred to new domains, they provide the structure on which reasoning can be built. It is therefore important for students to develop an intuitive feel for the places in mathematics where such axioms might be significant.

I was gratified that my article generated so much interest, created discussion, and pressed readers (and myself) to think further about some of these ideas. I offer this response in the spirit of such ongoing conversation.

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AN APPLICATION OF THE GREATEST INTEGER FUNCTION

Suppose you are given a large rectangular box with dimensions length = l, width = w, height = h. You also have small boxes with length = a, width = b, and height = c. What is the largest number of small boxes that can be placed entirely inside the large box if they must all have the same orientation?

Trying to imagine the ways to pack the boxes can be confusing without a systematic way of keeping track of the packing. The greatest integer function, written $f(x) = \text{int}(x)$ or $f(x) = [x]$, will return the largest integer less than or equal to x. This function provides a quick way to answer the packing question.

Put the volume of the large box as a product of length, width, and height in the numerator of a fraction and put the volume of a small box as the product of its length, width, and height in the denominator of the same fraction. Now permute the denominators in the six possible arrangements:

$$\frac{lwh}{abc}, \frac{lwh}{acb}, \frac{lwh}{bac}, \frac{lwh}{bca}, \frac{lwh}{cab}, \text{ and } \frac{lwh}{cba}.$$

The number of boxes possible in the first arrangement can be found by computing

$$\left[\frac{l}{a}\right] \bullet \left[\frac{w}{b}\right] \bullet \left[\frac{h}{c}\right].$$

The number of boxes possible in the second arrangement can be found by computing

$$\left[\frac{l}{a}\right] \bullet \left[\frac{w}{c}\right] \bullet \left[\frac{h}{b}\right].$$

Continue this pattern by taking the greatest integer function of the ratio of each of the top variables to the variable directly beneath it. There will be six possible products, and the largest product indicates how to orient the small boxes in the large box. For example, suppose that the large box has dimensions 5 in. × 6 in. × 12 in. and that each small box has dimensions 2 in. × 3 in. × 4 in. The possible arrangements are the following:

$$\left[\frac{5}{2}\right]\cdot\left[\frac{6}{3}\right]\cdot\left[\frac{12}{4}\right]=2\cdot 2\cdot 3=12$$

$$\left[\frac{5}{3}\right]\cdot\left[\frac{6}{2}\right]\cdot\left[\frac{12}{4}\right]=1\cdot 3\cdot 3=9$$

$$\left[\frac{5}{2}\right]\cdot\left[\frac{6}{4}\right]\cdot\left[\frac{12}{3}\right]=2\cdot 1\cdot 4=8$$

$$\left[\frac{5}{3}\right]\cdot\left[\frac{6}{4}\right]\cdot\left[\frac{12}{2}\right]=1\cdot 1\cdot 6=6$$

$$\left[\frac{5}{4}\right]\cdot\left[\frac{6}{2}\right]\cdot\left[\frac{12}{3}\right]=1\cdot 3\cdot 4=12$$

$$\left[\frac{5}{4}\right]\cdot\left[\frac{6}{3}\right]\cdot\left[\frac{12}{2}\right]=1\cdot 2\cdot 6=12$$

Thus, we see that the maximum number of boxes that can be packed inside the large box is 12 and that the packing can be done in three ways.

This method will work even when one or two dimensions of the small box are greater than one dimension of the large box. Suppose that one ratio is int(5/6). The result is 0, so no boxes can be stacked in that orientation.

After choosing the best results, determine whether additional boxes having a different orientation can be packed into the large box. It is easy to calculate the size of any available space and then determine whether additional boxes can be fit into the space.

Students can investigate the possibilities that can occur when the small-box dimensions are factors of the large-box dimensions or when all possible pairings of the dimensions are relatively prime.

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TANGENTS AND TRIANGLE AREA, CONTINUED

Using Coordinate Geometry

In "Tangents and Triangle Area" ("Reader Reflections," *Mathematics Teacher* 101, no. 9 [May 2008]: 628–30), Cindy Sung presents a nice discussion of how she solved a calculus homework exercise. She was challenged to find the area of the triangle formed by two points on the parabola $y = ax^2$ and the point at which the lines tangent to the parabola at those two points meet. She used Microsoft Excel, with $a = 1$, to find the general formula and then used integration to prove it. The area can be found in a more straightforward manner by using the coordinates of the vertices of the triangle.

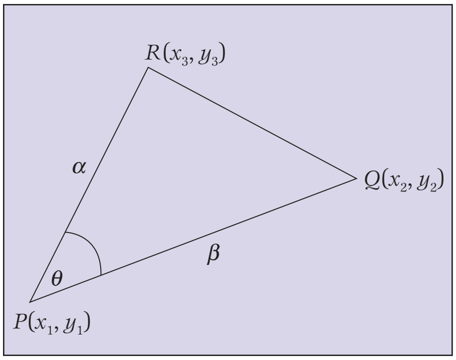

Fig. 1 (Goehl)

In **figure 1 (Goehl)**, let α be the length of side PR and β be the length of side PQ. Then twice the area of the triangle is given by $2A = \alpha\beta\sin\theta$. If θ_a and θ_b are the angles that α and β make with a line parallel to the x-axis, $2A = \alpha\beta|\sin(\theta_a - \theta_b)| = \alpha\beta|\sin\theta_a\cos\theta_b - \sin\theta_b\cos\theta_a|$. In terms of the coordinates, this may be written as $2A = |(y_3 - y_1)\cdot(x_2 - x_1) - (y_2 - y_1)(x_3 - x_1)|$ or, in symmetrical form, as $2A = |y_1(x_3 - x_2) + y_2(x_1 - x_3) + y_3(x_2 - x_1)|$.

In the present case, take points P and Q to be on the parabola and point R to be the point of intersection of the tangent lines. As shown by Sung, the coordinates of the point of intersection are given by $x_3 = (x_1 + x_2)/2$ and $y_3 = ax_1x_2$. Now $x_3 - x_2 = x_1 - x_3 = (x_1 - x_2)/2$, and so $4A = |(x_1 - x_2)(y_1 + y_2 - 2y_3)| = |a(x_1 - x_2)(x_1^2 + x_2^2 - 2x_1x_2)| = |a(x_1 - x_2)^3|$.

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Using Matrices

In "Tangents and Triangle Area," Cindy Sung provides a neat and very detailed approach to finding the triangular area formed by two points of a specific quadratic function and the point of intersection of the two tangents at the given points. Another method of establishing her area formula would be to use the determinant formula to find the area of a triangle given its three points. In this particular case, the formula is as follows:

$$A(x_1,x_2)=\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1\\ x_2 & y_2 & 1\\ \dfrac{x_1+x_2}{2} & x_1x_2 & 1\end{vmatrix}$$

Taking the absolute value of the right side may be necessary if the determinant is negative. This situation can be avoided if the points are entered in counterclockwise order. Also, the third row is determined by finding the point of intersection of the two tangent lines. Expanding this determinant yields Sung's area formula.

If we consider the triangle formed by two points on the graph of $f(x) = ax^2 + bx + c$ with x-coordinates x_1 and x_2 and the intersection of the two lines tangent to f at those two points, the area formula becomes

$$A(x_1,x_2)=\left|\frac{a}{4}(x_1-x_2)^3\right|.$$

This result follows from the use of the determinant for finding the area of a triangular region when given three points. Finally, consider the quadratic relation, $x = Ay^2 + By + C$. As before, the area formed can be found by using the determinant formula for the area of a triangle:

$$A(y_1, y_2)=\left|\frac{A}{4}(y_1 - y_2)^3\right|$$

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SPACE GEOMETRY AND VECTORS INSTEAD OF ALGEBRA

I read with interest Robert Ryden's reflection "*N*-Mids Follow-Up" (*Mathematics Teacher* 101, no. 5 [December 2007–January 2008]: 324–25), and here I give two other proofs of the three-dimensional analog to the Pythagorean theorem. This theorem originates with Johann Faulhaber (1580–1635).

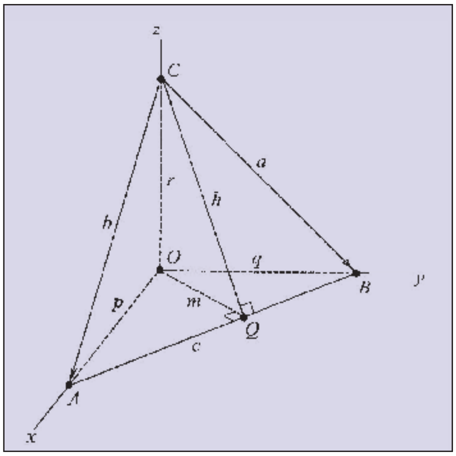

Fig. 1 (Kuřina)

In my diagram (**fig. 1 [Kuřina]**), h is the length of altitude CQ drawn to AB, and m is the length of altitude OQ drawn to AB. The length of AB is c.

Since $mc = pq$, $c^2 = p^2 + q^2$, and $h^2 = r^2 + m^2$, then

$$m^2c^2 = p^2q^2 \to m^2 = \frac{p^2q^2}{p^2+q^2}.$$

If we use this value for m^2,

$$h^2 = r^2 + \frac{p^2q^2}{p^2+q^2} = \frac{p^2q^2+q^2r^2+r^2p^2}{p^2q^2},$$

and the area of triangle ABC can be written as follows:

$$\begin{aligned}\left[A_{\Delta ABC}\right]^2 &= \frac{1}{4}c^2h^2 \\ &= \frac{1}{4}(p^2+q^2) \bullet \frac{p^2q^2+q^2r^2+r^2p^2}{p^2+q^2} \\ &= \frac{1}{4}(p^2q^2+q^2r^2+r^2p^2) \\ &= \left[A_{\Delta AOB}\right]^2 + \left[A_{\Delta BOC}\right]^2 + \left[A_{\Delta COA}\right]^2.\end{aligned}$$

For the points $A(p, 0, 0)$, $B(0, q, 0)$, $C(0, 0, r)$ (see **fig. 1 [Kuřina]**), the vectors

$$\overrightarrow{CA} = (p, 0, -r), \overrightarrow{CB} = (0, q, -r),$$

and their vector product is

$$\overrightarrow{CA} \times \overrightarrow{CB} = (rq, pr, pq)$$

with

$$\overrightarrow{CA} \times \overrightarrow{CB} = \sqrt{p^2q^2+q^2r^2+r^2p^2}.$$

Using the formula

$$A = \frac{1}{2}|\vec{u} \times \vec{v}|$$

for the area of a triangle, we obtain

$$\begin{aligned}\frac{1}{4}(p^2q^2+q^2r^2+r^2p^2) &= \left[A_{\Delta ABC}\right]^2 \\ &= \left[A_{\Delta AOB}\right]^2 \\ &= \left[A_{\Delta BOC}\right]^2 + \left[A_{\Delta COA}\right]^2.\end{aligned}$$

For this last proof of our three-dimensional analog to the Pythagorean theorem, I thank Svatopluk Zachariáš, Plzeň. The elaboration of this note was supported by grant GAČR 406/08/0710.

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PROBLEM 18, MARCH 2008

Another Way to Score 22 Points in a Football Game

The "Calendar" problem for March 18, 2008 (*Mathematics Teacher* 101, no. 7), asked readers to determine how many different ways a football team could score 22 points. One solution was omitted: two 8-point scores (two touchdowns with 2-point conversions) and one 6-point score (a touchdown without an extra point).

In general when doing problems like this, students will find it much easier to start with the larger number and work down, using as many numbers as possible until the score is too big. For example:

8 7 6 3 2
2 0 1 0 0 [This is the solution that was omitted.]
{Two 8s and a 7 is too much, so go to two 8s and a 6. Then trade the 6 for two 3s, and then trade two 3s for three 2s.}

2 0 0 2 0
2 0 0 0 3
1 2 0 0 0 [The combination of one 8 and two 7s works.]
1 1 0 1 2 [The combination 8 + 7 + 6 does not work because one 3 is needed to get an even number, so reduce the 6 to a 3, and then two 2s works.]

Carry on with the same argument down to 0 0 0 0 1 1.

There are 30 possible ways a football team could score 22 points. These could all be listed, but the reader ought to do that.

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Another Few Ways to Score 22 Points

Could the answer be 1338?

Occasionally, I have students who like to play mental games with a problem I give them. If their solution meets the requirements of the question as worded, then I feel obligated to give them credit. So I try to word my problems carefully, not wanting to tangle with the "lawyers" in the class.

So I found myself trying to determine the level of difficulty intended for this problem and assumed, in the end, that *MT* would provide the answer given. But I thought this *might* be a trick question; typically, wording such as "How many ways …" could imply that order is important. If we assume that the order in which these points were scored is significant (and that I punched the buttons in the right order), another answer might be 1338 ways.

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[Can readers use Unger's method on Askey's result to find a better answer?—*Ed.*]

PROBLEM 22, MARCH 2008

The "Calendar" problem for March 22, 2008 (*Mathematics Teacher* 101, no. 7), asked readers to find the difference in the cubes of two numbers whose difference and product are both 1. The given solution shows that the numbers involved are related to the golden mean.

A more efficient derivation that the difference in the cubes is 4 is obtained by writing the two numbers as x and y, so that $x - y = 1$ and $xy = 1$. Factoring the difference between the two cubes and rearranging gives the following:

$$\begin{aligned} x^3 - y^3 &= (x - y)(x^2 + xy + y^2) \\ &= (x - y)\left((x - y)^2 + 3xy\right) \\ &= (1)\left((1)^2 + 3(1)\right) \\ &= 4 \end{aligned}$$

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[Similar letters were received from Anita Schuloff, New York City, and Lawrence J. Cohen, Port Jefferson, New York.—*Ed.*]

RAPAPORT'S AREA RATIOS

Jonathan Rapaport's reflection (*Mathematics Teacher* 101, no. 9 [May 2008]: 630) presents formulas for the ratio of the area, A, of a regular n-gon to the area, I, of an inscribed circle and to the area, C, of its circumscribed circle. He also notes a formula for the ratio I/C.

Herein, we correct his formula for A/C. Instead of resorting to a geometric diagram, we substitute his two correct formulas into the relationship

$$\frac{A}{C} = \left(\frac{A}{c}\right)\left(\frac{c}{C}\right).$$

Then

$$\begin{aligned} \frac{A}{C} &= \frac{n\tan\left(\frac{\pi}{n}\right)}{\pi} \cdot \cos^2\left(\frac{\pi}{n}\right) \\ &= \frac{n\sin\left(\frac{\pi}{n}\right) \cdot \cos\left(\frac{\pi}{n}\right)}{\pi}. \end{aligned}$$

Because of the double angle formula for the sine, this can be simplified to

$$\frac{A}{C} = \frac{n\sin\left(\frac{2\pi}{n}\right)}{2\pi}.$$

(Rapaport's formula does not contain the two 2s.)

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A VISUAL PROOF

In Roger Nelson's excellent book *Proof without Words* (Mathematical Association of America 1993), several sections are devoted to proofs of sums of integers. These demonstrations inspired me to develop a visual proof of the formula for the sum of the first n terms of an arithmetic series. My precalculus students enjoyed both the graphical and the analytical components of this proof.

Consider the arithmetic series $a_1 + a_2 + \ldots + a_n$. Each term, a_i, is represented by a rectangle with dimensions $1 \times a_i$. Therefore, the area of each rectangle is a_i. (See **fig. 1 [Hurwitz]**.)

Let d represent the common difference between successive terms. Thus, $a_2 = a_1 + d$ and $a_3 = a_2 + d$. (See **fig. 2 [Hurwitz]**.) Join the upper-right corners of each rectangle with line segments, noting that because these line segments have a common slope ($-1/d$), their union is the entire segment.

Notice that a trapezoid is formed: $a_1 + a_2 + \ldots + a_n$ = area of trapezoid – area of the n triangles with base d and height 1. (See **fig. 3 [Hurwitz]**.) Since

$$\text{area of trapezoid} = \frac{n}{2}\left(a_1 + a_n + d\right)$$

and

$$\text{area of the } n \text{ triangles} = n \cdot \frac{1 \cdot d}{2} = \frac{nd}{2},$$

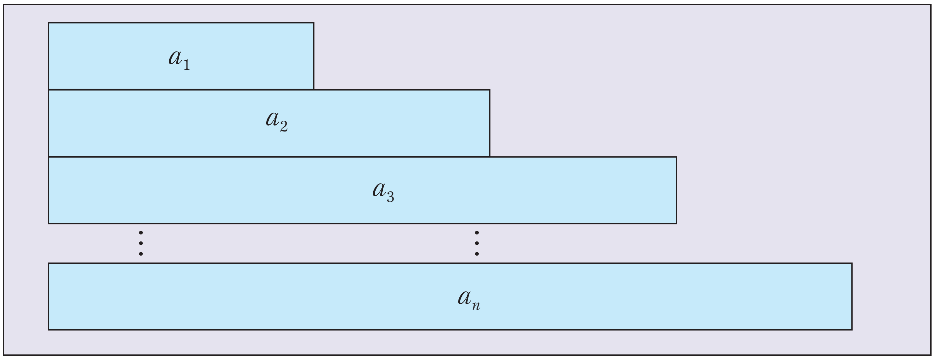

Fig. 1 (Hurwitz)

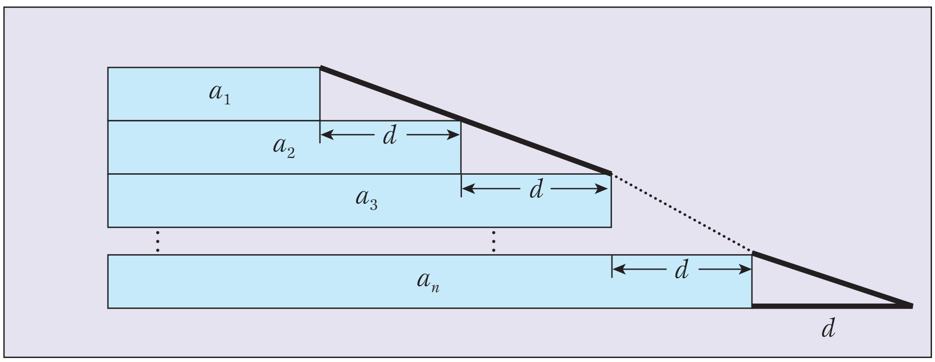

Fig. 2 (Hurwitz)

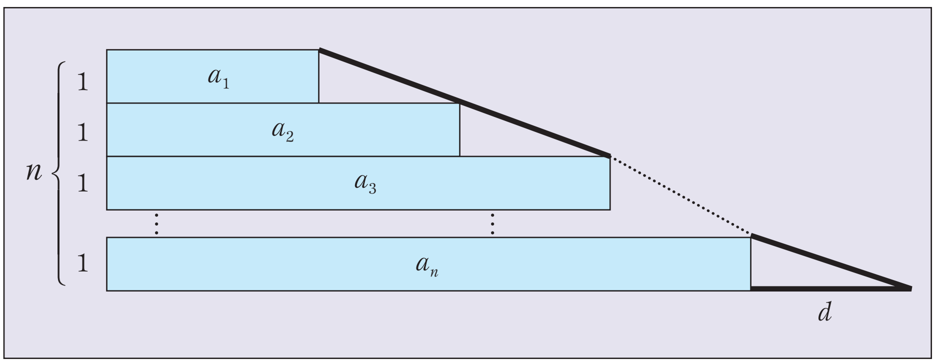

Fig. 3 (Hurwitz)

therefore,

$$a_1 + a_2 + \cdots + a_n = \frac{n}{2}\left(a_1 + a_n + d\right) - \frac{nd}{2}.$$

Factoring out $n/2$ yields

$$\begin{aligned} a_1 + a_2 + \cdots + a_n &= \frac{n}{2}\left(a_1 + a_n + d - d\right) \\ &= \frac{n}{2}\left(a_1 + a_n\right). \end{aligned}$$

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MORE ON DIVISIBILITY TESTS

In testing a number, N, for divisibility by 9 by casting out nines, the sum, T, of the digits of N will be seen to have the same remainder as N, which gives more information about the division of N by 9 than mere divisibility. The basis for this fact is the proposition that for any divisor D, N and T will have equal remainders if and only if $N - T$ is divisible by D, shown by subtracting $T = DQ_1 + R_1$ from $N = DQ_2 + R_2$. Elementary school students are taught to continue the long-division algorithm until the remainders get to be zero or a number smaller than the divisor. Knowing about negative numbers, we can go another step to obtain a negative remainder with absolute value less than the divisor. Sometimes it is more convenient to choose the remainder with the smaller absolute value.

Some books on mathematical recreations have a similar test for the divisor 11: If the digits of N are d_0, d_1, d_2, … (i.e, if $N = d_0 + 10d_1 + 10^2d_2 + \ldots + 10^nd_n$), then calculate $T = d_0 - d_1 + d_2 - d_3 + \ldots + (-1)^nd_n$, casting out or adding 11s at each step to obtain totals whose absolute values are less than 11. T will have the same remainder as N. This result leads to a method somewhat different from the one that Redmond has demonstrated to determine divisibility (in fact, the remainders) for any divisor. For the divisor $D = 7$ and with $T = a_0d_0 + a_1d_1 + a_2d_2 + \ldots$, we can obtain $N - T = (1 - a_0) \cdot d_0 + (10 - a_1)d_1 + (10^2 - a_2)d_2 + \ldots$. If we choose a_k so that $10^k - a_k$ is divisible by 7 for $k = 0, 1, 2, \ldots, n$, then $N - T$ will be divisible by 7. If we use our basic proposition, N and T would have equal remainders.

Calculating a_k is fairly simple; it is the remainder on dividing 10^k by 7. In fact, calculating can be streamlined. Calculate any a_k so that $(10^k - a_k)$ is divisible by 7 (e.g., $a_0 = 1$). If we multiply by 10, $(10^{k+1} - 10a_k)$ is also divisible by 7, showing that a_{k+1} is the remainder of $10a_k$ on division by 7 ($a_1 = 3$). Doing so gives the following sequence of remainders for 7: 1, 3, 2, 6, 4, 5, 1, 3, …. The calculation is the same as that for finding the decimal equivalent of 1/7. The sequence has repetitions because its only possible numbers are 1, 2, …, 6 (0 is absent since 7 is not a divisor of any power of 10). When a remainder is repeated, the calculation again gives the remainders that followed its previous appearance, so the series repeats a finite sequence of remainders corresponding to the repeating decimal for 1/7.

For example, find the remainder of $N = 18{,}254$ on division by 7. We calculate the sum (mod 7) (i.e., by casting out 7s): $1 \times 4 + 3 \times 5 + 2 \times 2 + 6 \times 8 + 4 \times 1 = 5$. That 5 is indeed the remainder can be verified on a calculator by carrying out the division to get 2607.7142. Subtract the integer part and multiply the fraction by 7 to get 4.9994. The remainder should be a number between 0 and 6, but the calculator does not have room to show all the digits of the infinite decimal for 1/7, so the remainder must be 5.

Calculating the decimal equivalent of 1/9 gives as remainders a sequence of 1s. Calculating it for 1/11 gives repetitions of 1, 10, 1, 10, … or using remainders of smallest absolute value: 1, –1, 1, –1, 1, …, thus supporting the tests of division by 9 and 11. For 13 we get the sequence 1, 10, 9, 12, 3, 4, 1, 10, … or 1, –3, –4, –1, 3, 4, 1, –3, …. Either series can be used to calculate T. For division of, say, $N = 13{,}872{,}143$ by 13, write the digits of N backward and their coefficients forward below them. Add the products (mod 13); the sum is 12. This remainder can be verified on a calculator.

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MORE ON LEAST SQUARES

Mathematics teachers know the importance of carefully distinguishing among *must*, *can*, and *cannot* (i.e., *always*, *sometimes*, and *never*, respectively). In their article "Applied Algebra: The Modeling Technique of Least Squares" (*Mathematics Teacher*, vol. 102, no. 1 [August 2008]: 46–51), Jeremy Zelkowski and Robert Mayes state: "Summing errors when some residuals are positive and some are negative can result in a total error that is not representative of how well a line fits the data, because the errors can cancel one another out"

(p. 49). It could have been made more clear that the “when” condition holds in *every* situation except the contrived scenario where the least-squares line hits every point. But whether or not that condition happens, the n errors *must* (not just *can*) sum to zero:

$$\begin{aligned}\sum(\text{observed values} - \text{fitted values}) \\ &= \sum(y_i - (a + bx_i)) \\ &= \sum y_i - \sum a - b\sum x_i \\ &= n\overline{y} - na - bn\overline{x}\end{aligned}$$

To see that this last expression must equal zero, note that the line going through $(\overline{x}, \overline{y})$ implies that $\overline{y} = a + b\overline{x}$.

Also, despite what exercise 5 suggests, the nonuniqueness pitfall of minimizing the sum of absolute errors is *not* limited to data with no trend, as I wrote in *Teaching Statistics* (Summer 1999, pp. 54–55), *Mathematics Teacher* (November 2005, pp. 228–29), and the algebra curriculum that I co-wrote with Mayes, *ACT in Algebra: Applications, Concepts, and Technology in Learning Algebra* (McGraw-Hill 1998, p. 281).

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TURNING THE CUBIC CURVE INTO AN ADDING MACHINE

Let f be a cubic polynomial with quadratic term zero; that is, $f(x) = A + Bx + Cx^3$, with $C \neq 0$. Let u and v be positive real numbers. Let $L = mx + b$ denote the line through the points $(-u, f(-u))$ and $(-v, f(-v))$. Now L intersects the graph of f at the point $(w, f(w))$.

Remarkably, $u + v = w$. To prove this, note that $g(x) = f(x) - mx - b$ is a cubic polynomial with quadratic term zero and two roots $x = -u$ and $x = -v$. Since the sum of the zeros of g is 0, it follows that $x = u + v$ is also a zero of g.

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