Supplement to "Historical Reflections on Teaching Trigonometry": Hipparchus

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We do not know the very first problem that used trigonometry in the sense of using the measure of an angle to find the length of a line segment. In the third century BCE, Aristarchus used angle measurements to estimate the distance to the moon and Archimedes estimated the width of the sun (see Van Brummelen, pp. 20–32). But neither of them used a table of values of chords or sines. Both used the fact that the angles they were considering were extremely small, and thus the chord length could be approximated by the arc length. Ptolemy credits the first table of chord lengths to Hipparchus and also credits him with the solution to the problem of the unequal seasons, perhaps the earliest problem that was solved using such a table.

An observation that was noted by Aristotle and that puzzled ancient astronomers is that the seasons are not of equal length. During the course of the year, the sun travels along the *ecliptic*, the circular path through the heavens that proceeds through the constellations that are recorded as the signs of the zodiac. The positions of the sun at the winter and summer solstices were observed to be diametrically opposite points on this circle. Moving out at right angles marked the spring and autumnal equinoxes, the half way points between the solstices. Together, the solstices and equinoxes were chosen to mark the changes of the seasons. Since the earth was assumed to be the center of the universe with the sun making its annual trajectory along this circular path, the seasons should be of equal length. They are not.

Modern calculations give the following approximate values to the lengths of the seasons:

winter 89 days,

spring $92\frac{3}{4}$ days,

summer $93\frac{5}{8}$ days,

fall $89\frac{7}{8}$ days.

Hipparchus of Rhodes (*circa* 190–120 BCE) explained this discrepancy by moving the earth off the center of the universe so that the perpendicular chords marking out the seasons do

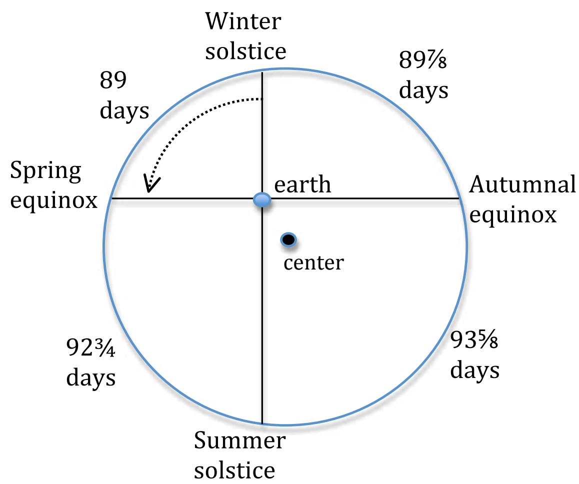

Figure 1: Hipparchus modeled the sun's path by moving the earth away from the center.

not cut arcs of equal length (see Figure 1). That raised the natural question: How far off center *is* the earth?

To answer this, we first convert the length of each season into the length of the corresponding arc, where the full circumference has length 360°. For example, the arc length of winter is $89/365\frac{1}{4}$ of the circumference. In terms of degrees, this is

$$\frac{89}{365.25} \cdot 360 \approx 87 + \frac{43}{60},$$

or approximately $87^\circ\ 43'$. In terms of arc length, the seasons are

winter $87^\circ\ 43'$,

spring $91^\circ\ 25'$,

summer $92^\circ\ 17'$,

fall $88^\circ\ 35'$.

Fall and winter together account for an arc length of $176^\circ\ 18'$, which means that the arc length from the spring equinox to the horizontal diameter of the sun's path is $1^\circ\ 51'$ (see Figure 2). If we can find the chord of $3^\circ\ 42'$, then half that chord is the vertical displacement of the earth from the center of the universe.

Of course, the actual distance depends on the radius of this circle. The radius is the average distance between the earth and the sun, which is 1 *astronomical unit* (au)—known

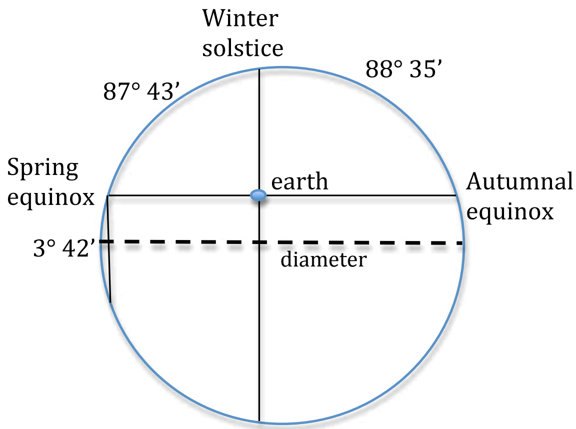

Figure 2: Arc lengths of the sun’s path (not to scale).

today to be about 150 million km. Using the values of the sine function, we can calculate the chord length (crd) to be

$$\text{crd } (3^\circ\ 42') = 2\sin(1^\circ\ 51') = 2\sin 1.85^\circ = 0.065 \text{ au.}$$

The vertical displacement is half that, about 0.032 au.

Winter and spring account for $179^\circ\ 08'$. The arc length from the summer solstice to the vertical diameter of the sun’s path is only 26 minutes. The horizontal displacement is approximately half the chord of $52'$:

$$\text{crd } (52') = 2\sin(26') = 2\sin 0.433^\circ = 0.015 \text{ au,}$$

for a horizontal displacement of 0.0075 au.

By the Pythagorean theorem, the distance from the earth to the center of the universe is about $\sqrt{0.0075^2 + 0.015^2} = 0.017$ au.

Exactly the same mathematics can be used if we assume that the earth circles the sun in a circular orbit at constant angular velocity. The sun would be about 2.55 million km from the center of the earth’s orbit. But that is not what actually happens. The earth’s orbit is elliptical, and the earth speeds up as it gets closer to the sun and slows down as it recedes. In fact, the sun, which is located at one of the foci of the earth’s orbit, is just a little less, about 2.5 million km, from the center of the earth’s orbit.

Supplement to "Historical Reflections on Teaching Trigonometry": Euclid

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When Ptolemy constructed his table of chords, he was able to start with the chords of arc lengths 180°, 90° and 60° as well as two chords that can be found in Euclid's *Elements*:

$$\text{crd } 36^\circ = \frac{\sqrt{5}-1}{2}\,R, \quad \text{and} \quad \text{crd } 72^\circ = \sqrt{\frac{5-\sqrt{5}}{2}}\,R,$$

where R is the radius of the circle. This is not quite the way that Euclid stated these results. These results are contained in Book XIII, Propositions 9 and 10:

Book XIII, Proposition 9. *If the side of the hexagon and that of the decagon inscribed in the same circle are added together, then the whole straight line has been cut in extreme and mean ratio, and its greater segment is the side of the hexagon.*

Book XIII, Proposition 10. *If an equilateral pentagon is inscribed in a circle, then the square on the side of the pentagon equals the sum of the squares on the sides of the hexagon and the decagon inscribed in the same circle.*

We first will see how to interpret these statements as chord lengths of the respective angles. We then will prove the propositions.

The side of the inscribed hexagon is the chord of 60°, which is R, the radius of the circle. The chord of the inscribed decagon is the chord of 36°. To say that a line segment has been cut in "mean and extreme proportion" means that the ratio of the longer to the shorter length is the golden ratio: $(1+\sqrt{5})/2$. Proposition 9 states that

$$\frac{R}{\text{crd } 36^\circ} = \frac{1+\sqrt{5}}{2}.$$

Equivalently,

$$\text{crd } 36^\circ = \frac{R}{(1+\sqrt{5})/2} = \frac{2R}{\sqrt{5}+1}\cdot\frac{\sqrt{5}-1}{\sqrt{5}-1} = \frac{2R(\sqrt{5}-1)}{5-1} = \frac{\sqrt{5}-1}{2}\,R.$$

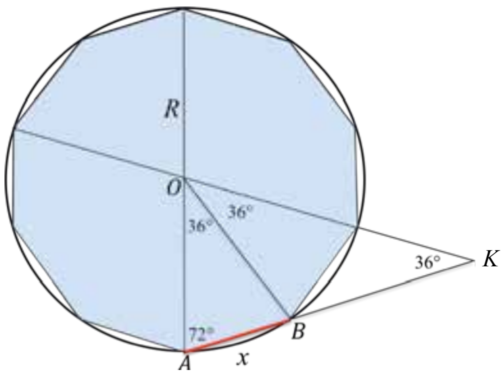

Figure 1: The ratio of the radius to the chord of 36°, R/x, is equal to $(1+\sqrt{5})/2$.

The side of the inscribed pentagon is the chord of 72°. Proposition 10 implies that

$$\begin{aligned}(\text{crd } 72^\circ)^2 &= R^2 + \left(\frac{\sqrt{5}-1}{2}\,R\right)^2 \\ &= \left(1 + \frac{(\sqrt{5}-1)^2}{2^2}\right) R^2 \\ &= \left(1 + \frac{6-2\sqrt{5}}{4}\right) R^2 \\ &= \frac{10-2\sqrt{5}}{4}\,R^2 \\ &= \frac{5-\sqrt{5}}{2}\,R^2.\end{aligned}$$

The value of crd 72° is found by taking the square root of each side.

The proofs of these propositions, while following those of Euclid, have been cast into modern terminology. A direct translation of Euclid's proofs can be found at http://aleph0.clarku.edu/~djoyce/java/elements/toc.html

Proof of Proposition 9. See Figure 1. The length of $\overline{AB}$, one side of the inscribed decagon, is denoted by x. Since triangle OAB is isosceles, $\angle OAB = 72^\circ$. Extend the line segment $\overline{AB}$ and the radial line to the next vertex of the decagon so that they meet at point K. The angle at K is $\angle AKO = 36^\circ$.

Since triangle OBK is isosceles, $BK = R$. By the similarity of triangles OAK and

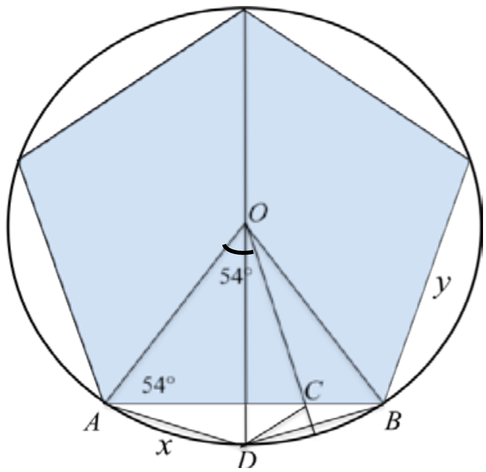

Figure 2: The square of the chord of 72° is equal to the sum of the squares of the radius and of the chord of 36°: $y^2 = R^2 + x^2$.

ABO, we obtain the relationship

$$\frac{R}{x} = \frac{R+x}{R} = 1 + \frac{x}{R}.$$

Let $z = R/x$. The equation is $z = 1 + z^{-1}$, which Euclid immediately recognized as the equation of mean and extreme proportion. In modern algebraic notation, we convert this to the quadratic equation $z^2 - z - 1 = 0$ which can be solved for z:

$$\frac{R}{x} = z = \frac{1+\sqrt{5}}{2}.$$

□

We now know that the chord of 36° is $R\left(\sqrt{5}-1\right)/2$.

Proof of Proposition 10. See Figure 2. The length of $\overline{AB}$, one side of the inscribed pentagon, is denoted by y. The length of $\overline{AD}$, one side of the inscribed decagon, is denoted by x. Draw the perpendicular bisector of $\overline{DB}$, and denote by C the point at which it intersects $\overline{AB}$. It follows that $\angle OAB = \angle AOC = 54°$, and therefore triangles OAB and COA are similar isosceles triangles. Therefore,

$$\frac{R}{y} = \frac{AC}{R} \quad \Longrightarrow \quad R^2 = y \cdot AC. \tag{1}$$

Triangles DAB and CDB are also similar isosceles triangles, and therefore

$$\frac{x}{BC} = \frac{y}{x} \quad \Longrightarrow \quad x^2 = y \cdot BC. \tag{2}$$

Combining equations (1) and (2) yields the desired result,

$$R^2 + x^2 = y\,(AC + BC) = y^2. \tag{3}$$

$\square$

This establishes that the chord of 72° is $\sqrt{\dfrac{5-\sqrt{5}}{2}}\,R$.

Supplement to "Historical Reflections on Teaching Trigonometry": Ptolemy

David M. Bressoud

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Claudius Ptolemy of Alexandria (*circa* 85–165 CE) established the basis for Western, South Asian, and Middle Eastern astronomy that would last until the 16th century. Ptolemy's book, known originally as "The Mathematical Treatise," would come to be known as "The Great Treatise" or, in Arabic, *Kitāb al-majisī*, which was translated into Latin as the *Almagest*. One of the basic challenges that Ptolemy had to face was constructing a table of chords corresponding to various arc lengths.

First he had to choose the radius of his circle, recognizing that chord lengths would need to be scaled when applied to specific circles. He chose a radius of $R = 60$. Since, following the practice of the ancient Mesopotamians, each degree was subdivided into 60 minutes and each minute into 60 seconds and so on, the choice of 60 simply makes it easy to scale, much as choosing a radius of 100 would be convenient for our decimal system. We will explain his results in terms of an arbitrary radius, R.

As explained in the print article and the supplement on Euclid, Ptolemy started with a knowledge of the chords (crd) of several arc lengths, especially crd $60° = R$ and crd $72° = \sqrt{(5-\sqrt{5})/2}\, R$. The next step in building his table was to show how to find the chord for an arc length that is the sum or difference of arc lengths for which the chords are known. This would enable him to find the chord length for 12°, and then successively cut that in half to get down to the chord of $3/4^{\text{ths}}$ of a degree, not quite what he needed, but getting close. The key to the sum and difference of arc lengths formula is what we now know as Ptolemy's Theorem.

Ptolemy's Theorem. *Given any quadrilateral inscribed in a circle, the product of the diagonals equals the sum of the products of the opposite sides.*

Proof. The proof relies on Euclid's result, Book III, Proposition 21, that if we take any chord $\overline{AB}$ of a circle and any third point C on the circle, then the angle $\angle ACB$ depends only on the chord $\overline{AB}$ and not on the choice of C. In fact, $\angle ACB$ is exactly half the length of the arc from A to B, a result that we shall need later. It follows that in Figure 1,

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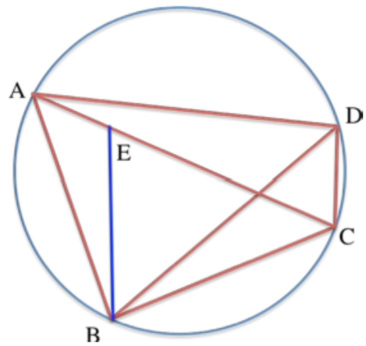

Figure 1: For any quadrilateral inscribed in a circle, the product of the diagonals is equal to the sum of the products of the opposite sides: $AC \cdot BD = AB \cdot CD + AD \cdot BC$.

$\angle BAC = \angle BDC$. We draw a line segment from B, meeting $\overline{AC}$ at E, so that $\angle ABE = \angle DBC$. It follows that triangles ABE and DBC are similar, and, therefore

$$\frac{AE}{AB} = \frac{CD}{BD} \quad \Longrightarrow \quad BD \cdot AE = AB \cdot CD.$$

Again invoking Book III, Proposition 21, we see that $\angle ADB = \angle ACB$. From the construction of $\overline{BE}$, we also have that $\angle ABD = \angle CBE$. Now it follows that triangles ADB and ECB are similar, and, therefore

$$\frac{BD}{AD} = \frac{BC}{EC} \quad \Longrightarrow \quad BD \cdot EC = AD \cdot BC.$$

Combining these results, we obtain

$$BD \cdot AC = BD \cdot (AE + EC) = BD \cdot AE + BD \cdot EC = AB \cdot CD + AD \cdot BC. \qquad (1)$$

□

To get the sum and difference of angles formulas, we consider the special case of this theorem in which one of the diagonals of the quadrilateral is a diameter of the circle of radius R (see Figure 2 in which α is the arc length from A to D and β is the arc length from A to B. The diameter is $AC = 2R$). Note that for any arc length α,

$$(\text{crd } \alpha)^2 + \big(\text{crd } (180^\circ - \alpha)\big)^2 = (2R)^2.$$

If we know the lengths of chords $\overline{AB}$ and $\overline{AD}$, then we know the lengths of all chords except $\overline{BD}$, and Ptolemy's Theorem can be used to find this chord length (crd):

$$\text{crd } (\alpha) \cdot \text{crd } (180^\circ - \beta) + \text{crd } (\beta) \cdot \text{crd } (180^\circ - \alpha) = 2R\, \text{crd } (\alpha + \beta). \qquad (2)$$

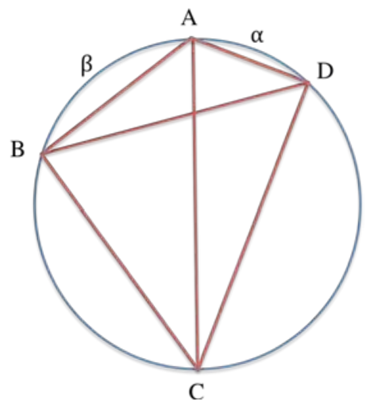

Figure 2: In terms of chords (crd), the sum of angles formula for the sine translates as crd $(\alpha) \cdot$ crd $(180^\circ - \beta) +$ crd $(\beta) \cdot$ crd $(180^\circ - \alpha) = 2R\,$crd $(\alpha + \beta)$, a direct consequence of Ptolemy's Theorem.

Similarly, if we know the chord of α and the chord of $\alpha + \beta$, Ptolemy's Theorem enables us to find the chord of β.

Ptolemy could have derived the half angle formula by setting $\alpha = \beta$ in equation (2) and solving for crd α in terms of crd 2α. He chose instead to derive the formula

$$(\text{crd } \alpha)^2 = 2R^2 - R\,\text{crd } (180^\circ - 2\alpha) \tag{3}$$

as follows (see Figure 3). We draw a diameter $\overline{AB}$ and place points C and D so that the arc length from B to C and the arc length from C to D both equal α. We drop a perpendicular from C to $\overline{AB}$, meeting $\overline{AB}$ at F, and locate the point E so that $AD = AE$. Since $\angle DAC = \angle CAE$, triangles ADC and AEC are congruent. Since $CE = CD = BC$, triangles CFE and CFB are congruent, implying that

$$BF = \frac{1}{2}BE = \frac{1}{2}(AB - AE) = \frac{1}{2}(AB - AD). \tag{4}$$

Triangles ACB and CFB are similar, so

$$\frac{BC}{BF} = \frac{AB}{BC} \quad \Longrightarrow \quad BC^2 = AB \cdot BF = \frac{AB}{2}(AB - AD). \tag{5}$$

Using the fact that $AB = 2R$, $BC =$ crd α, and $AD =$ crd $(180^\circ - 2\alpha)$, we get equation (3).

The results obtained so far enabled Ptolemy to find the exact value of the chord of any arc of length $3 \cdot 2^k$ degrees where k can be any integer. In the 12th century, this fact would lead Al-Samawal to argue that the circle should be divided into 480° rather than 360°,

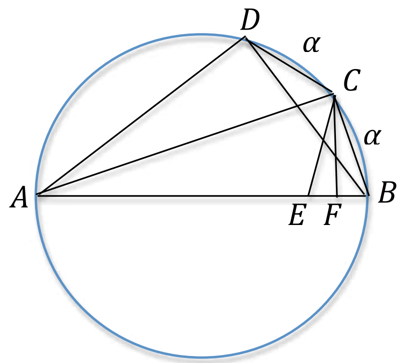

Figure 3: In terms of chords, the half angle formula can be expressed as $(\text{crd } \alpha)^2 = 2R^2 - R\,\text{crd } (180° - 2\alpha)$.

because it *is* possible to find an exact value for the chord of $1/480^{\text{th}}$ of the circumference of a circle. He seems to have convinced no one to change the definition of a degree.

Although Ptolemy could not find an exact value for the chord of $1°$, he was able to create a table of chord lengths for all arc lengths from $0°$ to $90°$ in increments of half a degree and to within an accuracy of one part in $216{,}000 = 60^3$. The following proposition enabled him to calculate crd $1°$ and crd $30'$ to the desired accuracy.

Proposition. *If* $0 < \alpha < \beta < 180°$ *are arc lengths, then*

$$\frac{\text{crd } \beta}{\text{crd } \alpha} < \frac{\beta}{\alpha}. \tag{6}$$

In particular, this implies that

$$\frac{2}{3}\,\text{crd } 1°\,30' < \text{crd } 1° < \frac{4}{3}\,\text{crd } 45',$$

bounds that produce the desired accuracy. In fact, these bounds differ by less than one part in 2,600,000.

Proof. See Figures 4 and 5. Let α denote the arc length from A to B and β the arc length from B to C. Draw the line segment $\overline{BD}$ that bisects $\angle ABC$, and mark E as the point of intersection of $\overline{AC}$ and $\overline{BD}$.

We will need the fact that, since $\angle ABE = \angle EBC$,

$$\frac{AB}{AE} = \frac{BC}{CE}. \tag{7}$$

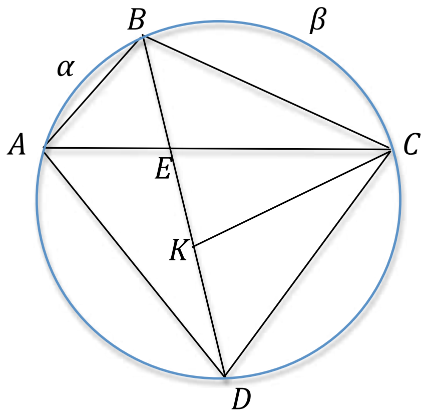

Figure 4: If ABC is any triangle and $\overline{BE}$ bisects the interior angle at B, then the ratio of AB to AE is equal to the ratio of BC to CE: $\dfrac{AB}{AE} = \dfrac{BC}{CE}$.

We can see why this is true if we add a point K on the segment $\overline{BD}$ so that $CE = CK$. Since triangle ECK is isosceles, $\angle EKC = \angle KEC = \angle AEB$, and therefore triangles ABE and CBK are similar. From this it follows that

$$\frac{AB}{AE} = \frac{BC}{CK} = \frac{BC}{CE}.$$

We draw a perpendicular from D to $\overline{AC}$, meeting $\overline{AC}$ at F (see Figure 5). Since the arc from A to D is equal to the arc from D to C, F is at the midpoint of $\overline{AC}$. We draw the arc of the circle centered at D with radius DE and mark its intersection with $\overline{AD}$ as G and its intersection with the extension of $\overline{DF}$ as H.

Since triangles EFD and AED have the same heights, the ratio of their areas is equal to the ratio of their bases, EF/AE. The ratio of the areas of these triangles is less than the ratio of the areas of the sectors EHD to GED, and the ratio of the sectors is equal to the ratio of the angles at D. We see that

$$\frac{EF}{AE} = \frac{\text{area of } \Delta EFD}{\text{area of } \Delta AED} < \frac{\text{area of sector } EHD}{\text{area of sector } GED} = \frac{\angle EDF}{\angle ADE}. \tag{8}$$

We observe that

$$EC = EF + FC = EF + AF = 2EF + AE,$$

and, similarly, $\angle EDC = 2\angle EDF + \angle ADE$. Now we put it all together,

$$\begin{aligned} \frac{\text{crd } \beta}{\text{crd } \alpha} &= \frac{BC}{AB} = \frac{EC}{AE} = 2\frac{EF}{AE} + \frac{AE}{AE} \qquad (9) \\ &< 2\frac{\angle EDF}{\angle ADE} + \frac{\angle ADE}{\angle ADE} = \frac{\angle EDC}{\angle ADE} = \frac{\beta/2}{\alpha/2} = \frac{\beta}{\alpha}. \end{aligned}$$

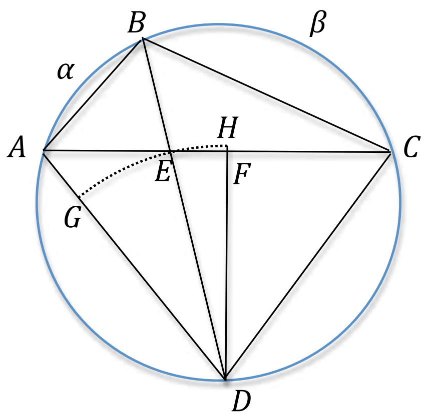

Figure 5: For ratios larger than 1, the ratio of the chord lengths is strictly less than the ratio of the corresponding arc lengths: $\dfrac{\text{crd }\beta}{\text{crd }\alpha} < \dfrac{\beta}{\alpha}$.

$\square$

Now that Ptolemy knew the chord of $\frac{1}{2}^\circ = 30'$ and had exact values at each multiple of $1^\circ\,30'$, he could find very accurate values for chords at any multiple of half a degree. Of course, as we saw in the problem of locating the position of the earth, we need finer values than this. In his table, Ptolemy reported the value of one sixtieth of the difference between each pair of successive chord values. With this information, anyone using his table could employ linear interpolation to find the intermediate chord values for arc lengths in increments of half a minute.