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Algebra on the Number Line

The number line is a powerful instructional tool for teaching the meaning of many mathematical concepts taught in middle school, including numerical operations. The number line can also be used as a tool for thinking about algebraic concepts taught in high school, such as an abstract or algebraic understanding of distance, absolute value, and inequalities.

Inspired by Darley's (2009) article about how to use the number line model as a visual tool to deepen student understanding about algebraic relationships and properties in the middle grades, we developed two activities to engage students in the process of higher algebraic reasoning. The activities have been successfully implemented with algebra learners of all types, including high school teachers and college students.

The activities are intended to meet the following NCTM goals, as outlined in *Principles and Standards for School Mathematics* (2000):

- "Understand the meaning of equivalent forms of equations, inequalities, and relations"
- "Use symbolic algebra to represent and explain mathematical relationships" (Algebra Standard for Grades 9–12, p. 296)

and

- "Create and use representations to organize, record, and communicate mathematical ideas"
- "Select, apply, and translate among mathematical representations to solve problems" (Representation Standard for Grades 9–12, p. 360)

In designing these activities, we have assumed that students have knowledge of integers and variables. Activity 1 reinforces skills with the use of the number line; if your students have had experience with this type of exercise, the activity should take about a day of instructional time. Activity 2 assumes that from activity 1 students have gained understanding of the meaning of variable on the number line—in particular, that $-a$ is a positive number when a is less than 0—and thus can explain why when $a < b$, then $-a > -b$. Activity 2 can take two days of instruction, depending on the time spent on discussion.

"Activities for Students" appears six times each year in *Mathematics Teacher*, often providing in reproducible formats activity sheets that teachers can adapt for use in their own classroom. Manuscripts for the department should be submitted via http://mt.msubmit.net. For more information on the department and guidelines for submitting a manuscript, please visit http://www.nctm.org/publications/content.aspx?id=10440#activities.

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TEACHER NOTES FOR ACTIVITY 1

For “Variables on the Number Line” (**activity sheet 1**; see pp. 383–84), students gain experience with the number line, learn the effect of scalar multiplication and sign change, and discover the effect of sign change on inequalities. In the first section (“Numbers on the Number Line”), students build their own number lines and work simple problems with given integers. Students should discover that the arbitrary placement of the zero (location) and 1 (scale) on the number line determines the position of all other numbers. The purpose of this portion of the exercise is to get students thinking about the number line and visualizing the distance between numbers.

In the second section (“Variables on the Number Line”), students work with numbers of unknown value but of known sign. Using the number line, students visualize the effect of scalar multiplication and sign change. For many students, the realization that a number labeled $-a$ can actually be positive if $a < 0$ is a significant obstacle to overcome. For positive numbers, we often say that “a is smaller than b” when $a < b$. After students complete problem 4, teachers may want to discuss with students why this language is problematic when a and b are negative numbers.

In the third section (“Inequalities on the Number Line”), students start with the inequality $a < b$, investigate the different situations that can occur, and discover the effect of sign change. The point of these exercises is to broaden student understanding of the meaning of $a < b$. Our experience indicates that most students think only of the case where both a and b are positive and use expressions such as “a is smaller than b,” which is easily confused with “a is closer to 0 than b.”

TEACHER NOTES FOR ACTIVITY 2

For “Absolute Value and the Concept of Distance” (**activity sheet 2**; see pp. 385–86), students gain more experience with using the concept of distance on the number line and understanding its relationship to absolute value. We find that absolute value is a tricky concept. Nearly all students know that $|-3| = 3$, but they have a hard time writing an expression for the function $f(x) = |x|$ and solving inequalities involving absolute values.

In the first section (“Absolute Value and the Distance between Two Numbers”), in problem 1, we expect students to discover that $|a| = -a$ when $a < 0$. In problems 2 and 3, the students are guided to discover that the distance between two numbers a and b is found either by subtracting the lesser number from the greater one or by taking the absolute value of the difference, regardless of order. In problems 4 and 5, students are expected to use the previous exercises to realize where the number $|b - a|$ can lie, depending on the values of a and b. One important misconception that must be overcome is the difference between the location of the number $d = |b - a|$ and the length of the line segment from a to b. If both a and b are large numbers but close to each other, d will be close to the origin and far away from a and b. For arbitrary values, it is impossible to say where d will lie in relation to a and b.

After these exercises to strengthen their intuition about distance on the number line, students solve equations and inequalities involving absolute value in the second section (“Equations with Absolute Value”). The intent of these exercises is to encourage students to think of distance, particularly on a number line, to help them solve equations. When students translate $|x - 3| = 2$ as “the distance between x and 3 is 2,” they should simply be able to “see” the solutions on a number line. When students use this approach, problem 8, which is very difficult to introduce when using the usual techniques of high school algebra, becomes accessible to students at this level. Essentially, students need to find only numbers that are closer to 4 than to -3.

CONCLUSION

The number line is a useful tool for thinking about some of the fundamental but tricky concepts in algebra. Once students complete the exercises included in these activities, they will be armed with visual images applicable to many of the symbolic manipulations done in algebra. These visual representations will become even more useful as students progress to calculus and more advanced mathematics.

REFERENCES

Darley, Joy W. “Traveling from Arithmetic to Algebra.” *Mathematics Teaching in the Middle School* 8, no. 8 (April 2009): 458–64.

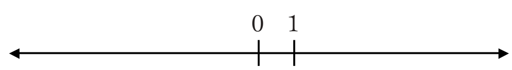
National Council of Teachers of Mathematics (NCTM). *Principles and Standards for School Mathematics*. Reston, VA: NCTM, 2000.

SOLUTIONS

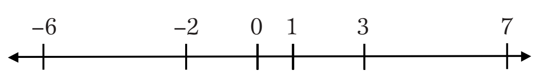
Activity 1: Variables on the Number Line

Numbers on the Number Line

1. Throughout the exercises, it is assumed that all number lines are oriented in the traditional way, with positive numbers to the right of zero.



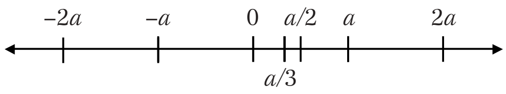
2.



3. The purpose of this problem to check for proper scale on the number line. The distance between 3 and 7 must be the same as the distance between –2 and –6. The common distance is 4 units. If students did not use proper scaling, the distances may not be the same.

Variables on the Number Line

1\.



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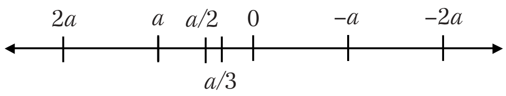

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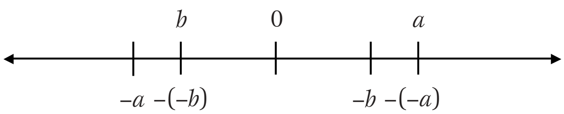
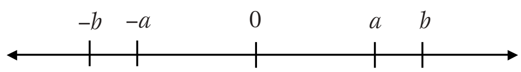
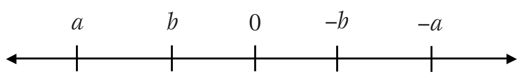


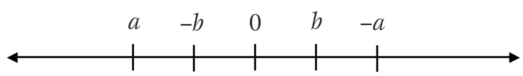
4. If $a < 0$, then $2a < a$, but a is closer to 0 than $2a$ is.

Inequalities on the Number Line

1\. (*a*)

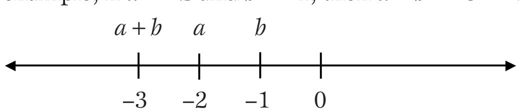




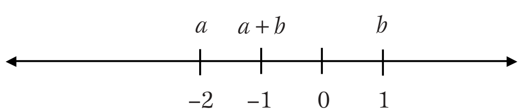


(*b*) For each of the three cases, when a is less than b, $-a$ is *greater* than $-b$ because $-a$ lies to the right of $-b$ on the number line. That is, if $a < b$, then $-a > -b$.

2. If b is negative and $a < b$, then $a + b < a$. For example, if $a = -2$ and $b = -1$, then $a + b = -3 < -2$.



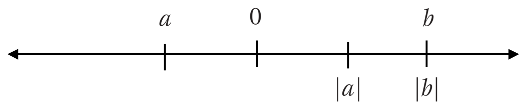
3. If a is negative and b is positive, then $a < b$, and $a < a + b < b$. For example, if $a = -2$ and $b = 1$, then $a + b = -1$ and $-2 < -1 < 1$.



Activity 2: Absolute Value and the Concept of Distance

Absolute Value and the Distance between Two Numbers

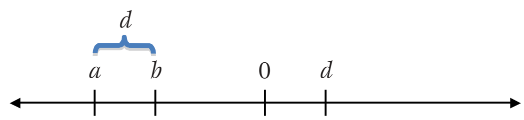
1\.

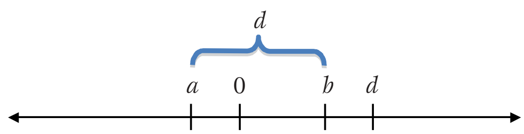


We also call $|a|$ the opposite of a or $-a$ when a is negative.

2. (*a*) 1, 5, 5, 1
   (*b*) 5, 13, 5, 13
   (*c*) 24, 0

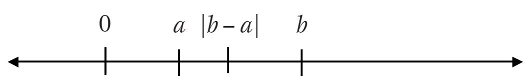
3. (*a*)





(*b*) The distance between two numbers can be found either by subtracting the lesser number from the greater or by taking the absolute value of their difference, regardless of order. That is, if a and b are two numbers, then the distance between them is given by $d = |b - a| = |a - b|$.

4. Answers will vary, but a and b must be positive.

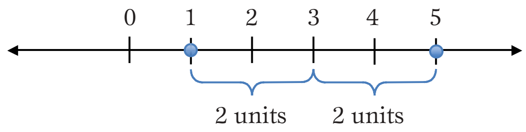


5. Answers will vary. Following are two possibilities: $a = 10$, $b = 19.9999$, $d = 9.9999$; and $a = -1{,}000{,}000$, $b = 0$, $d = 1{,}000{,}000$.

Equations with Absolute Value

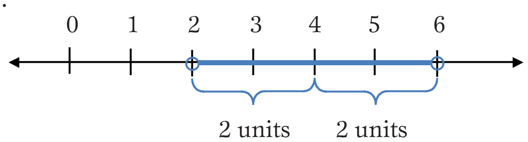
1. The distance between the number x and 3 is equal to 2.

2.



The number x is equal to 1 or 5 because the distance between 1 and 3 is 2 and the distance between 5 and 3 is also 2.

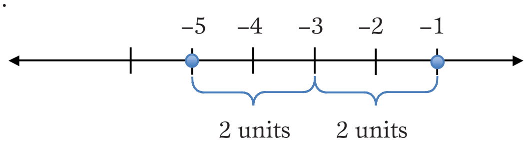
3.



The solution to the inequality $|x - 4| < 2$ is the set of all values of x such that the distance between x and 4 is *less* than 2—that is, all the values less than 6 but greater than 2 or $2 < x < 6$.

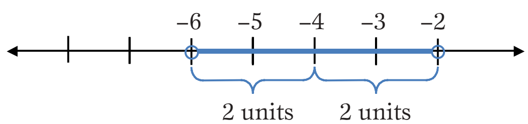
4. The distance between the number x and –3 is equal to 2.

5.



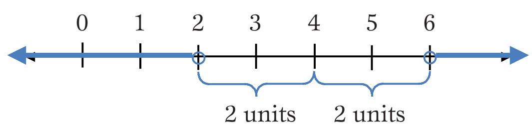
The number x is equal to –5 or –1 because the distance between –5 and –3 is 2 and the distance between –1 and –3 is also 2.

6.



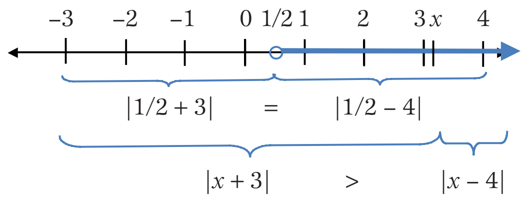
The solution to the inequality $|x + 4| < 2$ is the set of all values of x such that the distance between x and –4 is *less* than 2—that is, all the values less than –2 but greater than –6 or $-6 < x < -2$.

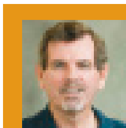
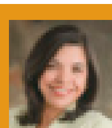
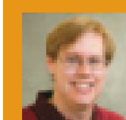
7.



The solution to the inequality $|x - 4| > 2$ is the set of all values of x such that the distance between x and 4 is *greater* than 2—that is, all the values less than 2 or greater than 6 or $x > 6$ or $x < 2$.

8. The solution to the inequality $|x + 3| > |x - 4|$ is the set of all values of x such that the distance between x and –3 is *greater* than the distance between x and 4—that is, all numbers that are closer to 4 than they are to –3. Note that for $x = 1/2$, the distances are the same and equal to 3.5. As we move to the right of 1/2, then we are getting closer to 4 and away from –3. When we reach 4, the distance between x and 4 is 0, and the distance between x and –3 is 7, so the inequality still holds. What about beyond 4? For any value greater than 4, its distance from 4 will still be less than the distance from –3. Hence, the solution is the set of all values greater than 1/2.



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Activity 1: Variables on the Number Line

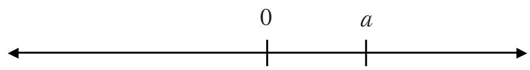
Numbers on the Number Line

1. Draw a number line. Mark a point near the middle of the number line and label it 0. Next, locate, mark, and label the point representing 1.
2. Locate, mark, and label the numbers 3, 7, –2, and –6.
3. Use a ruler to compare the distance between 3 and 7 with the distance between –2 and –6. Are these distances the same? If not, why not?

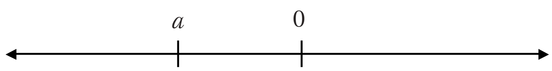
Variables on the Number Line

Given each representation below, locate the points that represent the indicated numbers.

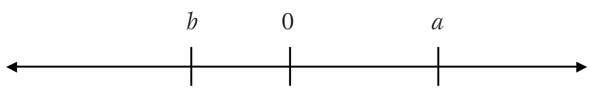
1. Plot and label each of the following points: $2a$, $a/2$, $a/3$, $-a$, and $-2a$.



2. Plot and label each of the following points: $2a$, $a/2$, $a/3$, $-a$, and $-2a$.



3. Plot and label each of the following points: $-a$, $-(-a)$, $-b$, and $-(-b)$.



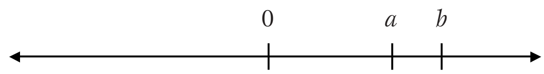
4. If a is a positive number, then $2a > a$ and a is closer to 0 than $2a$. How does this relationship change if a is negative?

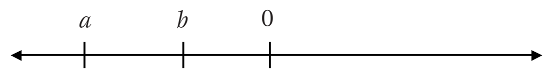
Activity 1: Variables on the Number Line (continued)

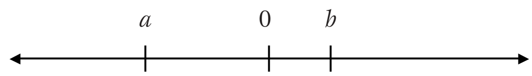
Inequalities on the Number Line

1. Suppose that each of a and b are numbers so that $a < b$.

   (a) Using the three possible locations for a and b, plot the locations of the points representing $-a$ and $-b$ in each case.







   (b) If $a < b$, write a general rule that relates $-a$ and $-b$. Explain how the number lines above illustrate your rule.

2. Find an example of a pair of numbers a and b for which each of the following is true: $a < b$ and $a + b < a$. Illustrate this situation on the number line.

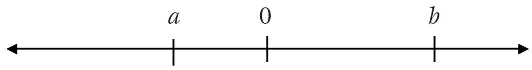
3. Find an example of a pair of numbers a and b for which each of the following is true: $a < b$ and $a < a + b < b$. Illustrate this situation on the number line.

From the December 2010/January 2011 issue of **MATHEMATICS** teacher

Activity 2: Absolute Value and the Concept of Distance

Absolute Value and the Distance between Two Numbers

1. Suppose x is a number. The absolute value of x, denoted by $|x|$, is the distance that x is from the number 0. Locate and label the number $|b|$ on the number line below.



   Now locate and label the number $|a|$ on the number line. What is another name for the number $|a|$?

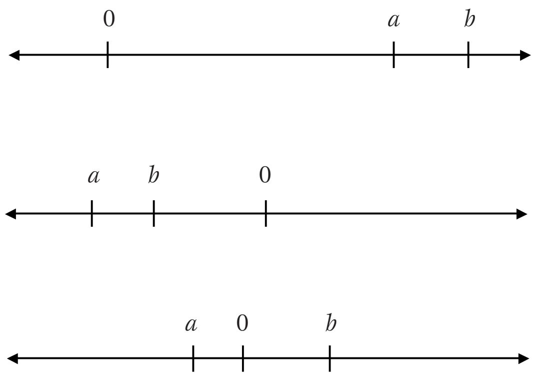
2. Use the number line to find the distance between the following pairs of numbers:

   (*a*) 2 and 3
   2 and –3
   –2 and 3
   –2 and –3

   (*b*) 4 and 9
   –4 and 9
   –4 and –9
   4 and –9

   (*c*) –12 and 12
   12 and 12

3. The number lines below show three possible cases: (1) two positive numbers; (2) two negative numbers; or (3) one positive and one negative number.



   (*a*) For each case, determine the distance d between each pair of nonzero numbers. Locate, mark, and label the value of d on the number line.

   (*b*) Describe a process for finding the distance between two numbers.

4. Draw a number line and locate a, b, and 0 on the number line so that the distance from a to b when placed on the number line falls *between* a and b.

5. In the examples above, we notice that the distance between a and b can be less than a, between a and b, or greater than b. Give a numerical example in which the distance is slightly less than a. Give a numerical example in which the distance is much greater than b.

Activity 2: Absolute Value and the Concept of Distance (continued)

Equations with Absolute Value

1. Suppose that x is a number. Write a sentence that uses the word *distance* and that expresses the meaning of the equation $|x - 3| = 2$.

2. Use the concept of distance and the number line below to show what number(s) x make the equation $|x - 3| = 2$ true.

3. Use the concept of distance and the number line below to show what number(s) x make the inequality $|x - 4| < 2$ true.

4. Suppose that x is a number. Write a sentence that uses the word *distance* and that expresses the meaning of the equation $|x + 3| = 2$. Hint: $|x + 3| = |x - (-3)|$.

5. Use the concept of distance and the number line below to show what number(s) x make the equation $|x + 3| = 2$ true.

6. Use the concept of distance and the number line below to show what number(s) x make the inequality $|x + 4| < 2$ true.

7. Use the concept of distance and the number line below to show what number(s) x make the inequality $|x - 4| > 2$ true.

8. Use the concept of distance and the number line below to show what number(s) x make the inequality $|x + 3| > |x - 4|$ true.