The Shape of an Ellipse

Gregory D. Foley

REPRESENTING AN ELLIPSE USING POLAR COORDINATES

In 1609 Johannes Kepler published his first law of planetary motion: Planets follow elliptical paths around the sun, with the sun serving as a focus of the ellipse. Before Kepler, orbits were assumed to be circular. In today's world, Kepler's first law is used in combination with polar coordinates and computer technology. When studying the motion of planets, dwarf planets, asteroids, and comets, physicists and astronomers use polar equations because they provide the distance r and direction θ from the sun, which serves as the *pole*, or origin, of the polar coordinate system (see **fig. 1**).

The polar equations of parabolas, ellipses, and hyperbolas are based on the following unifying definition of these three interrelated curves: A *conic section* is the set of all points in a plane whose distances to a fixed point (the focus) and a fixed line (the directrix) in the plane have a specific ratio (the eccentricity). Students may find it interesting to know that *directrix* is the Latin feminine form of *director* (Schwartzman 1994). The directrix does determine the direction of the conic's focal axis, which is perpendicular to the directrix. This focus-directrix definition is an extension of the standard definition of a *parabola*—the set of points in a plane equidistant from a fixed point and a fixed line in the plane.

Notice that the unifying definition for conic sections includes a definition of *eccentricity* as a ratio of distances. It turns out that this definition of eccentricity is equivalent to the $e = c/a$ definition, typically used for ellipses and hyperbolas. As always, the value of the eccentricity is related to the type of conic: If $e > 1$, the conic section is a hyperbola; if $e = 1$, the conic is a parabola; and if $e < 1$, the conic is an ellipse.

As shown in **figure 1**, the focus-directrix definition of conics works well in combination with polar coordinates. Recall that in polar coordinates the origin becomes the pole and the positive x-axis

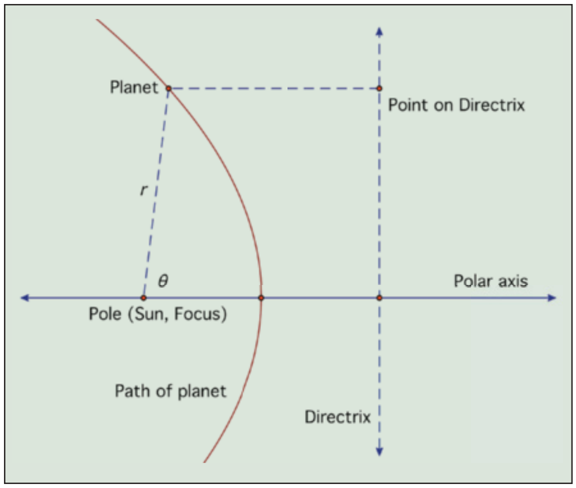

Fig. 1 Part of a planet's elliptical orbit can be represented using polar coordinates.

becomes the polar axis. To obtain a polar equation for a conic section, we can position the pole at the conic's focus and the polar axis along the focal axis, with the directrix to the right of the pole and perpendicular to the polar axis. Details are available in standard textbooks (e.g., Demana et al. 2011).

THE TWO-BODY PROBLEM

Celestial mechanics is the branch of astrophysics that investigates the motion of moons, planets, stars, and other heavenly bodies. In the two-body problem of celestial mechanics, the pole can be thought of as the center of the more massive body, such as Earth, with the smaller body, such as the moon, following a polar conic around the larger one. The polar coordinates given by the equation tell us the distance r between the two bodies and the direction θ from the larger body to the smaller body in relation to the axis of the conic path of motion. For example, the polar equation for an ellipse can be written as

$$r = \frac{a\left(1-e^2\right)}{1+e\cos\theta}.$$

Table 1 Semimajor Axes and Eccentricities of the Planets and Dwarf Planet Pluto		
Planet	**Semimajor Axis (Gm)**	**Eccentricity**
Mercury	57.9	.2056
Venus	108.2	.0068
Earth	149.6	.0167
Mars	227.9	.0934
Jupiter	778.4	.0484
Saturn	1427	.0542
Uranus	2871	.0472
Neptune	4498	.0085
Pluto	5906	.2488

Source: http://ssd.jpl.nasa.gov/; for table, follow links to http://ssd.jpl.nasa.gov/txt/p_elem_t1.txt.

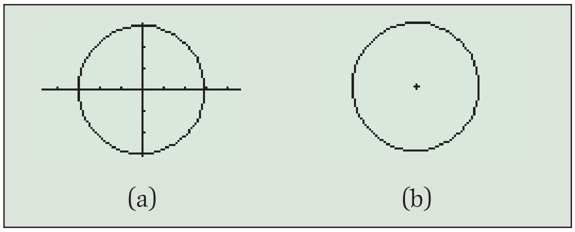

Fig. 2 Graphs of Earth's orbit around the sun indicate that the orbit is almost circular.

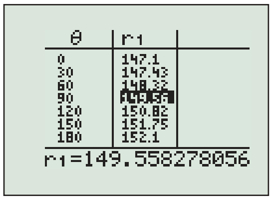

Fig. 3 Sun-to-Earth distances are shown at various orbital positions.

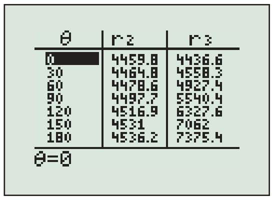

Fig. 4 Sun-to-object distances are shown for Neptune (r_2) and Pluto (r_3) at various orbital positions.

Astronomers use this equation to model the elliptical orbits of planets, asteroids, and comets around the sun. For example, using the data listed in **table 1**, the orbit of Earth can be represented as

$$r = \frac{149.6\left(1-0.0167^2\right)}{1+0.0167\cos\theta},$$

where r is measured in gigameters (Gm, millions of kilometers). **Figure 2** shows two versions of the graph of this equation in the window [–235, 235] by [–155, 155] using $0 \le \theta \le 2\pi$—with and without axes. These graphs are scale models of Earth's elliptical path with the sun as one focus, located at the pole. They look like circles! This perception is not the result of viewing-window distortion; Earth's orbit really is almost circular.

To see just how circular the orbit is, recall that the aspect ratio of an ellipse is $b/a = \sqrt{1 - e^2}$. Thus, the aspect ratio for Earth's orbit is

$$\frac{b}{a} = \sqrt{1-0.0167^2} \approx 0.99986.$$

This implies that the semiminor axis is only 0.014% shorter than the semimajor axis. The sun, however, is about 2.5 million kilometers off-center ($c = ea = 0.0167 \cdot 149.6 \approx 2.498$). This distance c is more than 100 times that of the distance $a - b$. In this case, thinking of the eccentricity as a percentage is illuminating. The eccentricity is 1.67%; thus, the sun's location within Earth's orbit is about 1.67% off-center, while Earth's orbit is only 0.014% out of round.

Another way to get a sense of the shape of Earth's elliptical orbit is to look at a table of values for the polar function $r(\theta)$ for values of θ in 30° increments (see **fig. 3)**. This figure shows that Earth's distance from the sun varies from a minimum (perihelion) distance of $a - c \approx 147.1$ Gm to a maximum (aphelion) distance of $a + c \approx 152.1$ Gm.

Many possible student activities related to celestial mechanics and polar equations of ellipses can help develop reasoning and sense making (see, e.g., NCTM 2009). Ask students to develop polar models for the orbits of Neptune and Pluto and then create tables like the one in **figure 4** to show that at times during its 247.9-year orbit Pluto is closer to the sun than is Neptune (as was the case between February 1979 and February 1999). Have students explore the orbits of other planets, dwarf planets, asteroids, or comets about the sun or have them investigate the orbits of moons about their planets. Data sources for such research include **table 1**, Sagan and Druyan (1997), and various websites.

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