Teaching Absolute

What is the meaning of absolute value? And why do we teach students how to solve absolute value equations? Absolute value is a concept introduced in first-year algebra and then reinforced in later courses. Various authors have suggested instructional methods for teaching absolute value to high school students (Wei 2005; Stallings-Roberts 1991; Friedlander and Hadas 1988), but here we focus on an investigation that will help students make meaning of the absolute value equation in the context of a practical situation. We connect absolute value to the concepts of rate, time, distance, and slope.

TEACHING ALGORITHMS

A geometric definition of *absolute value* is the "distance from zero on a number line" (Murdock, Kamischke, and Kamischke 2007, p. 418). Students quickly understand that $|-3| = |3| = 3$ and move on to solving absolute value equations such as this one:

$$\frac{1}{2} \cdot |x - 15| = 5$$

A typical algebra textbook uses the following algorithm to solve this equation: Treat the absolute value bracket as parentheses and then multiply both sides of the equation by 2:

$$\frac{1}{2} \cdot |x - 15| = 5$$

$$|x - 15| = 10$$

Next, set $x - 15 = 10$ and $x - 15 = -10$ and solve for x, yielding the answer of 25 or 5.

However, there seem to be few situations that can be modeled using absolute value. Although the following class of problems can be solved using the formula rate • time = distance, I have chosen to use absolute value in my solution to demonstrate its usefulness in solving this class of problems.

A MEANINGFUL TASK

The following task was inspired by a problem from *Discovering Algebra: An Investigative Approach* (Murdock, Kamischke, and Kamischke 2002). The problem below demonstrates the meaning of the variables a and b in the following equation: $y = a\,|x \pm b|$.

This activity should be presented in a second-year algebra course. Students should be able to identify an absolute value equation as "a V-shaped graph that points upward or downward" (Bellman et al. 2009, p. 359). In addition, students should be familiar with the process of translating the absolute value parent equation.

Here's the problem:

> A classroom is located on a hallway between the school's front and back stairwells, and students often sprint down the hallway to make it to class on time. How can a student use his or her location to tell just how much time there is until the bell rings?

Assume that the school's hallway is 40 feet long and that the classroom is located 16 feet from the front stairwell and 24 feet from the back stairwell.

F64/GETTY IMAGES

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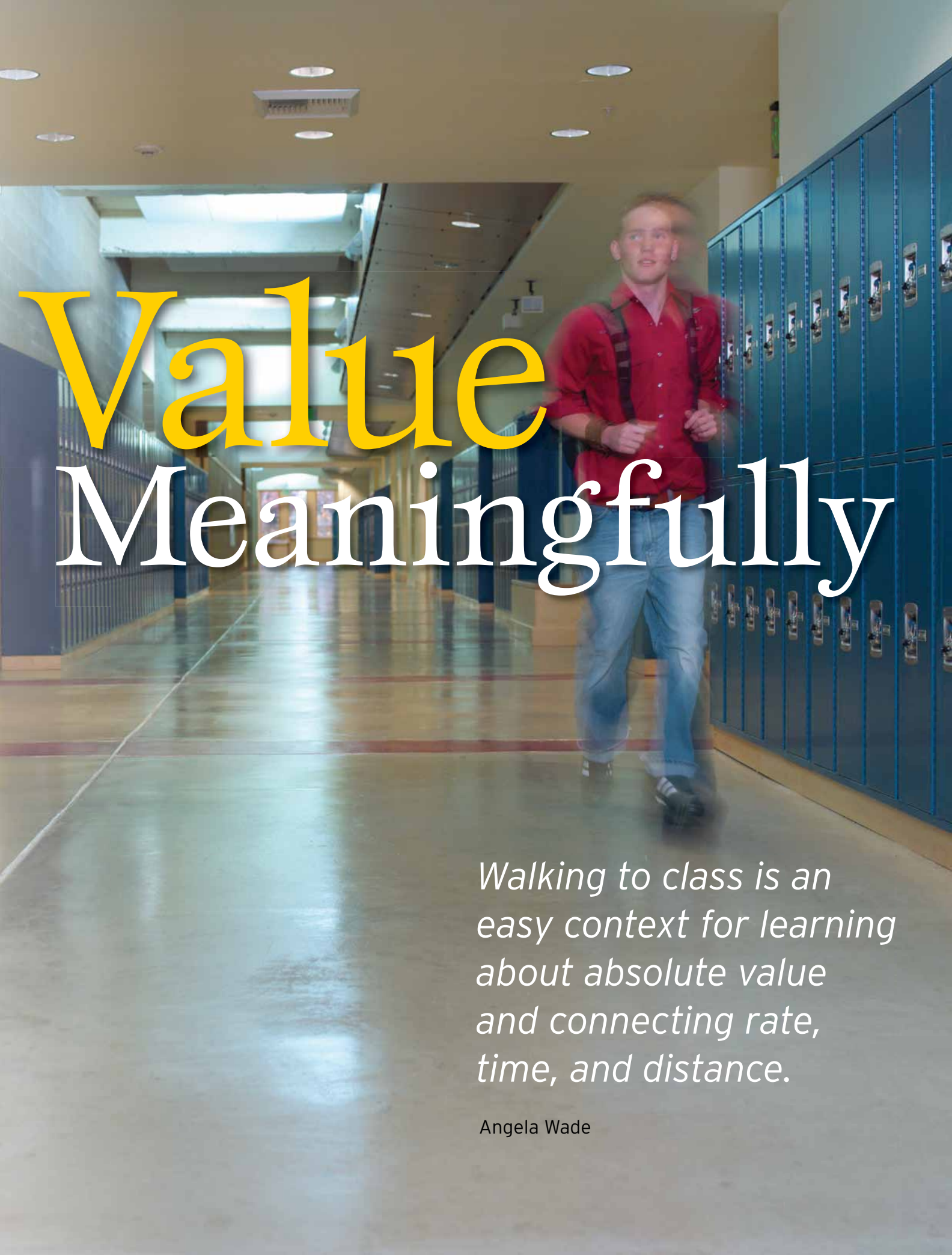


Value Meaningfully

Walking to class is an easy context for learning about absolute value and connecting rate, time, and distance.

Angela Wade

Dividing the class into groups of three students seems to work well for this activity. First, students time a group member walking at a constant pace from one end of the hallway to another, recording to the nearest second the time that he or she reaches the center of the classroom door. Then students can fill out a chart similar to that shown in **figure 1**. The times given here are hypothetical and were created to be used as an example; in actuality, students' times will vary. Distances and times should be recorded to the nearest foot and second, respectively.

Distance	Measurement
Total length of hallway	**40 feet**
Distance from front stairwell to classroom	16 feet
Distance from back stairwell to classroom	24 feet
Total time from front stairwell to back stairwell	80 seconds
Time from front stairwell to classroom	32 seconds
Time from classroom to back stairwell	48 seconds

Fig. 1 The chart lists all the required distances and times.

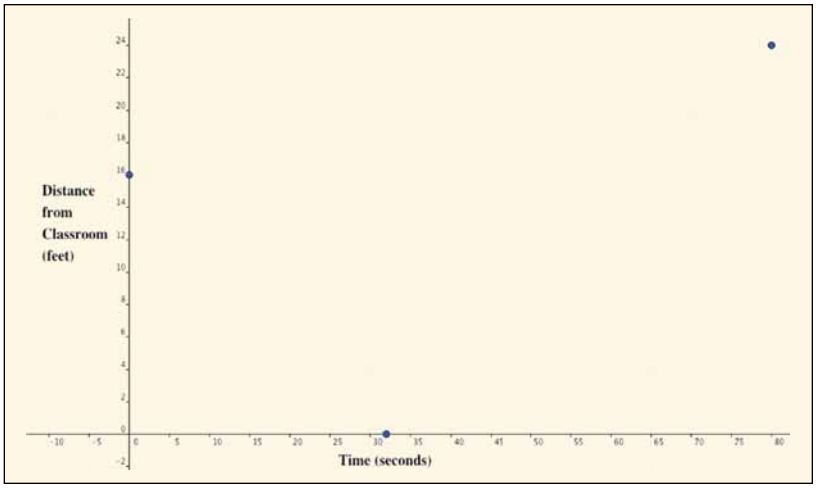


Fig. 2 When the three known points are plotted, the graph does not appear linear.

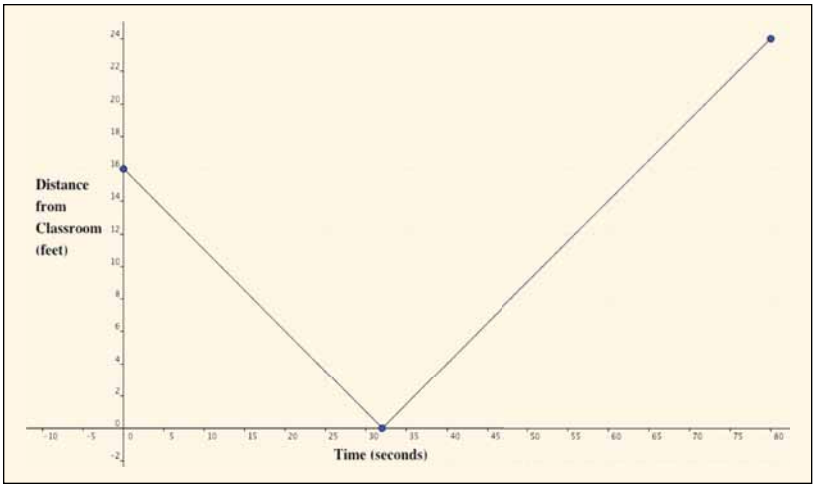


Fig. 3 The function $d = (1/2)|t - 32|$ for $0 \le t \le 80$ represents the student's walking path.

After collecting data, students will plot distance from the classroom, d, as a function of time, t. Students already know three points: At the front stairwell, a student is 16 feet from the classroom (after 0 seconds); at the classroom, a student is 0 feet from the classroom (after 32 seconds); and at the back stairwell, a student is 24 feet from the classroom (after 80 seconds). We thus have three ordered pairs: (0, 16), (32, 0), and (80, 24). Students can plot these points to begin considering a function (see **fig. 2**).

To find a function for the walker's pattern, students must consider several ideas. Knowing that the walker maintains a constant rate, students can deduce that a straight line would model the walking path from the front stairwell to the classroom and from the classroom to the back stairwell. Students can then find the slope of these two lines and relate it to walking rate. Students will discover that the absolute value of the slopes of each line is the same as the walking rate, which in the example above is 1/2 ft./sec.

Finally, combining knowledge of slope, absolute value, and horizontal translation of the absolute value equation, students can deduce the equation $d = (1/2)|t - 32|$ where $0 \le t \le 80$. The graph of this function is shown in **fig. 3**. Note that the maximum time is the total time it takes to walk from one end of the hallway to the other.

From this activity, students can discover the general formula $d = r \cdot |t - a|$ where d is the distance from the classroom, r is the rate of the walker, t is the time, and a is the time it takes for the walker to reach the classroom from the front stairwell. In addition, this activity helps make a discussion of domain and range more meaningful; it is impossible to be more than 24 feet away from the classroom without walking through a wall. Also, each group's graph will be unique because each is dependent on time and walking rate. This activity provides a forum for discussing the translation of graphs to the left and right and how the coefficient before the absolute value bracket affects the equation.

After completing the activity, students can answer the question, How can a student know by his or her location just how much time is left before he or she is late to class? Using the function above, we can substitute any distance for d and solve the function for t. A worksheet containing additional questions that link the concepts of rate, time, distance, slope, and absolute value can be found on the web at **www.nctm.org/mt027**.

A QUESTION OF DIFFERENT RATES

We can apply the general formula $d = r \cdot |t - a|$ to more complex walking problems:

Student 1 walks from the front stairwell to the classroom at a rate of 1/2 ft./sec. Student 2 walks from the back stairwell. At what rate does student 2 need to walk to meet student 1 at the classroom door? Keep in mind that the hallway is 40 feet long and that the distance from the front stairwell to the classroom is 16 feet.

To find the rate at which student 2 must walk, we must solve the following system of equations:

$$\text{Student 1: } 16 = \frac{1}{2} \cdot |t - 32| \qquad (1)$$

$$\text{Student 2: } 24 = r \cdot |t - 32| \qquad (2)$$

We first solve for $|t - 32|$ in each equation:

$$|t - 32| = 32 \qquad (3)$$

$$|t - 32| = \frac{24}{r} \qquad (4)$$

Setting equations (3) and (4) equal to each other, we get

$$32 = \frac{24}{r} \rightarrow r = \frac{3}{4} f/s.$$

Therefore, student 2 must walk at a rate of 3/4 ft./sec. to reach the classroom at the same time as student 1. **Figure 4** shows the walking paths of student 1 and student 2.

Notice that at 0 seconds student 1 and student 2 are 16 feet and 24 feet, respectively, from the classroom, but both reach the classroom after 32 seconds. This situation can be generalized to show that the amount of time it takes for each student to walk to the classroom is irrelevant as long as both take the same amount of time.

A QUESTION OF DIFFERENT RATES AND DIFFERENT TIMES

However, rarely do two students arrive at exactly the same time; consequently, we want to explore what happens if one student arrives before the other. Let's consider the following problem:

A student walks from the front stairwell to the classroom at a rate of 1/2 ft./sec. Another student walks from the back stairwell. At what rate is student 2 walking if she reaches the classroom 6 seconds after student 1? If she reaches the classroom 6 seconds before student 1? The distances remain the same as in the original problem.

To solve the problem in which student 2 arrives 6 seconds after student 1, we have two equations:

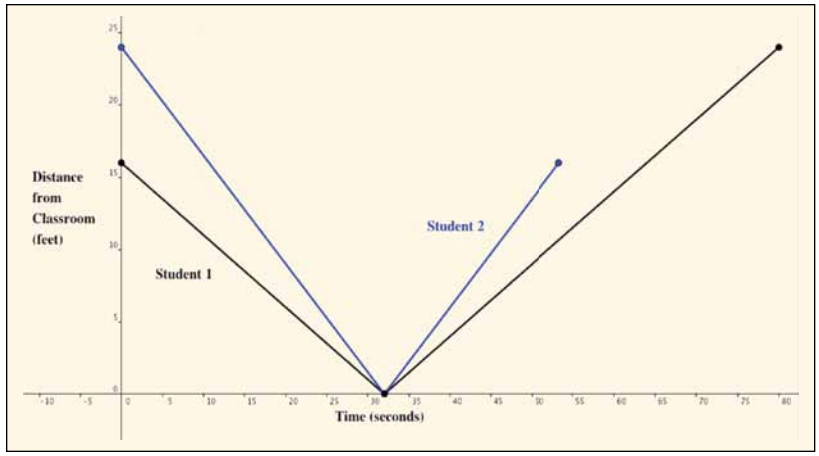


Fig. 4 The graph shows the walking functions for two students: $d = (1/2)|t - 32|$ for $0 \le t \le$ for student 1 and $d = (3/4)|t - 32|$ for $0 \le t \le 160/3$ for student 2. Both students arrive at the classroom at the same time.

$$d_1 = \frac{|t - 32|}{2} \qquad (7)$$

$$d_2 = r \cdot |t - (32 + 6)| = r \cdot |t - 38| \qquad (8)$$

We want to find the time it takes to get to the classroom, or t when $d_1 = 16$ ft.:

$$16 = \frac{|t - 32|}{2} \qquad (7)$$

$$32 = |t - 32|$$

$$32 = t - 32 \text{ or } -32 = t - 32$$

$$t = 64 \text{ or } t = 0$$

Student 1 takes 64 seconds to get to the classroom.

Next, we can substitute the values of t into equation (8) when d_2 is 24 feet (the distance from the back stairwell to the classroom). First, let $t = 64$ and solve for r:

$$24 = r \cdot |t - 38| \qquad (8)$$

$$24 = r \cdot |64 - 38|$$

$$24 = 26r$$

$$r = 24/26 = (12/13) \text{ ft./sec.}$$

Next, let $t = 0$ and solve for r:

$$24 = r \cdot |t - 38| \qquad (8)$$

$$24 = r \cdot |0 - 38|$$

$$24 = r \cdot |-38|$$

$$24 = 38r$$

$$r = 24/38 = 12/19 \text{ ft./sec.}$$

These two rates produce two functions:

$$d_2 = \frac{12}{13} \cdot |t - 38| \qquad (9)$$

$$d_2 = \frac{12}{19} \cdot |t - 38| \qquad (10)$$

We know that at $t = 0$, student 2 is 24 feet away from the classroom. Substituting 0 into equations (9) and (10), we get the following:

$$d_2 = \frac{12}{13} \cdot |0 - 38| = \frac{456}{13} \approx 35.0769 \qquad (9)$$

$$d_2 = \frac{12}{19} \cdot |0 - 38| = 24 \qquad (10)$$

Therefore, student 2 will arrive 6 seconds after student 1 when walking at a rate of 12/19 ft./sec.

Similarly, we can use the equations below to find the rate at which student 2 must walk to reach the classroom 6 seconds before student 1:

$$d_1 = (1/2)|t - 32| \qquad (7)$$
$$d_2 = r \cdot |t - (32 - 6)| = r \cdot |t - 26| \qquad (11)$$

Solving these equations produces the same rates. The walking functions for both students are graphed in **figure 5**.

GENERALIZING

We can also generalize by introducing more abstraction:

> Student 1 walks from the front stairwell to the classroom at a rate of r_1 ft./sec. and arrives at the classroom after a seconds. Another student walks from the back stairwell. At what rate (r_2) is student 2 walking if she reaches the classroom at $a + b$ seconds (where b can be positive or negative)? Note that student 1 starts d_1 feet away from the classroom and that student 2 starts d_2 feet away from the classroom.

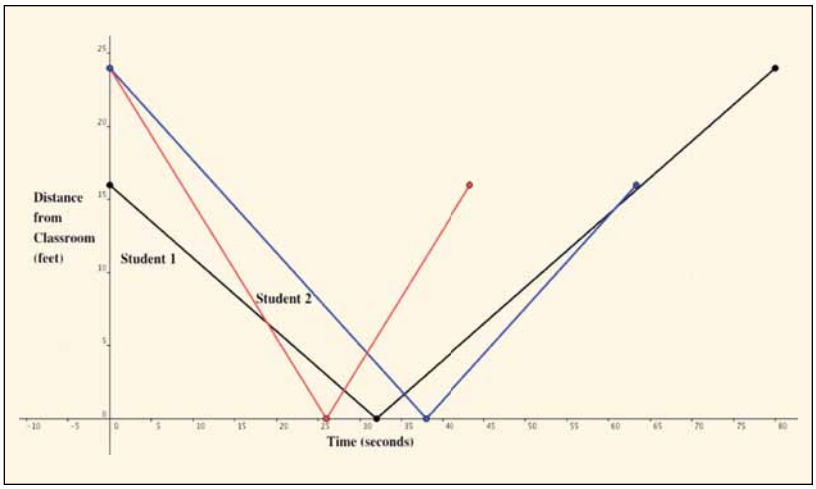


Fig. 5 The functions $d_1 = (1/2)|t - 32|$ on the interval [0, 80]), $d_2 = (12/19)|t - 38|$ on [0, 190/3], and $d_2 = (12/13)|t - 26|$ on [0, 138/3] are graphed in black, blue, and red, respectively.

To solve this problem, we once again have two equations:

$$d_1 = r_1 \cdot |t - a| \qquad (12)$$
$$d_2 = r_2 \cdot |t - (a + b)| \qquad (13)$$

First, we solve for t in equation (12):

$$d_1 = r_1 \cdot |t - a|$$
$$\frac{d_1}{r_1} = |t - a| \rightarrow$$
$$\frac{d_1}{r_1} = t - a \text{ or } \frac{d_1}{r_1} = a - t$$

We now have these two equations:

$$t = a + \frac{d_1}{r_1} \qquad (14)$$

$$t = a - \frac{d_1}{r_1} \qquad (15)$$

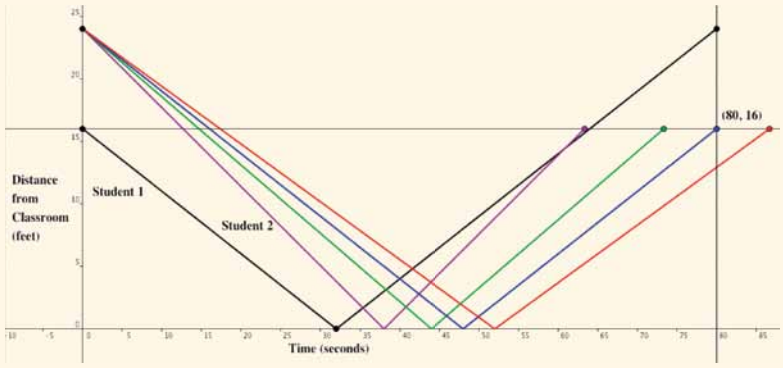
Substituting (14) into (13), we get the following:

$$d_2 = r_2 \cdot |t - (a + b)|$$
$$= r_2 \cdot \left| a + \frac{d_1}{r_1} - (a + b) \right|$$
$$= r_2 \cdot \left| \frac{d_1}{r_1} - b \right| \rightarrow$$
$$r_2 = \frac{d_2}{\left| \frac{d_1}{r_1} - b \right|} \qquad (16)$$

Next, substituting (15) into (13), we get the following:

$$d_2 = r_2 \cdot |t - (a + b)|$$
$$= r_2 \cdot \left| a - \frac{d_1}{r_1} - (a + b) \right|$$
$$= r_2 \cdot \left| -\frac{d_1}{r_1} - b \right| \rightarrow$$
$$r_2 = \frac{d_2}{\left| -\frac{d_1}{r_1} - b \right|} \qquad (17)$$

In the case where student 2 reaches the classroom at $a - b$ seconds, $-b$ can be substituted for b in equations (16) and (17), and the same rates would result. The correct rate would then need to be assessed according to the distance at $t = 0$. Hence, it can be seen that student 2's walking rate can be generalized, whether or not she arrives b seconds before or after student 1.



Color	Time to Classroom (sec.)	Walking Rate of Student 2 (ft./sec.)	Is Student 2 Walking Faster than Student 1?
Black	32	1/2 (student 1)	No
Purple	38	12/19 ≈ 0.632	Yes
Green	44	6/11 ≈ 0.545	Yes
Blue	48	1/2 = 0.5	No, the rates are the same.
Red	52	6/13 ≈ 0.462	No

Fig. 6 The black function, $d_1 = (1/2)|t - 32|$ on the interval [0, 80], and the blue function, $d_2 = (1/2)|t - 48|$ on [0, 96], show the paths of student 1 and student 2, respectively, when they walk at the same speed.

OTHER AREAS TO EXPLORE

Knowing the general formula for student 2's rate can lead to other questions. For example, after how many seconds should student 2 arrive if she is walking at the same speed as student 1? How much later can student 2 arrive and still be walking faster than student 1? Can this formula be generalized in terms of rate, time, and distance? **Figure 6** shows, pictorially and algebraically, student 2 walking at various rates compared with student 1 walking at 1/2 ft./sec.

CONCLUSION

Let's revisit the equation

$$\frac{1}{2} \cdot |x - 15| = 5$$

using the model from the discussion above. The term 1/2 demonstrates that a student is walking at a rate of 1/2 ft./sec. to a classroom that takes 15 seconds to reach traveling at that speed. The student is currently 5 feet from the classroom, and solving for x tells us how many seconds he or she has walked to be 5 feet away from the classroom. Solving the equation, we now know that the person could have walked for either 5 seconds or 25 seconds.

In this activity, the concept of absolute value is more than an algorithm. It has meaning to students' daily lives.

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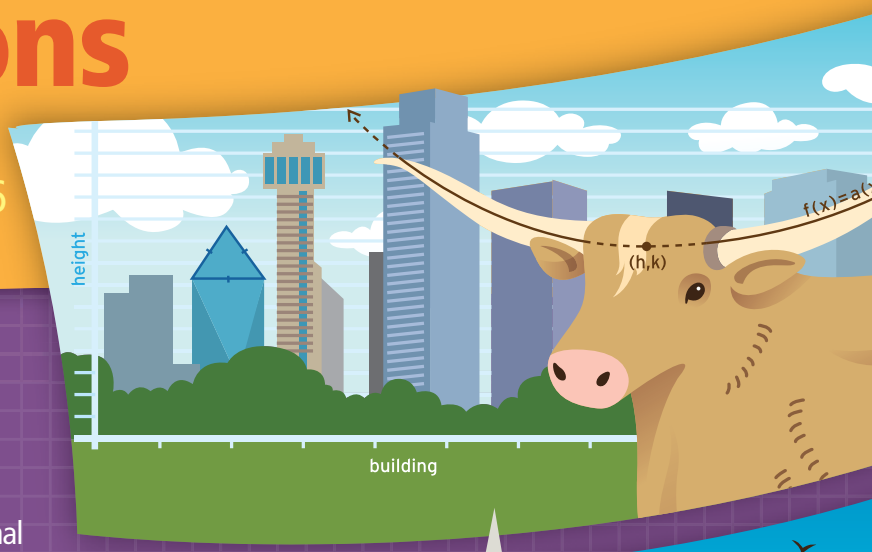

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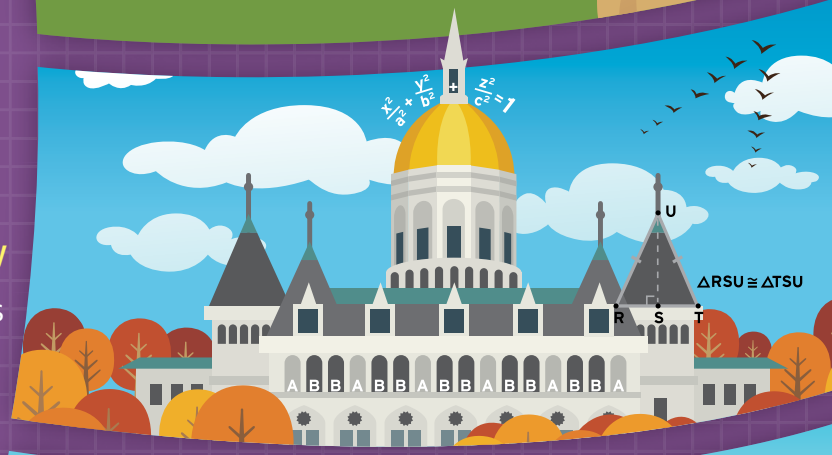

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