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Areas within Areas

Consider the following problem, which was the *MT* Calendar problem for December 3, 2006:

> A square is inscribed in a larger square. (That is, the four vertices of the inscribed square lie on the four sides of the larger square.) What is the smallest possible value of the ratio of the area of the inscribed square to that of the larger square? (2006/2007, p. 343)

This problem is reminiscent of the problem of exploring area ratios of related polygons, one of the e-examples from *Principles and Standards for School Mathematics* (NCTM 2000) on NCTM's Web page (http://www.nctm.org/standards/content.aspx?id=26786):

Delving Deeper offers a forum for classroom teachers to share the mathematics from their own work with the journal's readership; it appears in every issue of *Mathematics Teacher*. Manuscripts for the department should be submitted via http://mt.msubmit.net. For more background information on the department and guidelines for submitting a manuscript, visit http://www.nctm.org/publications/content.aspx?id=10440#delving.

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This applet, along with the Calendar problem, spurred us to look further into the general problem of area ratios. We will first share different ways to represent and solve the Calendar problem and then attempt to generalize and calculate the exact minimum area ratio for any regular polygon. The focus here is on regular polygons. As stated in the e-example, the minimum area ratios for nonregular polygons (other than squares) behave differently from their regular polygon counterparts.

SOLVING THE SQUARE

There are several ways to justify that the minimum area ratio of the two squares occurs when the interior square has vertices at the midpoints of the exterior square. One approach uses the parameterization shown in **figure 1**.

To find the minimum area of the inscribed square, we find the maximum area, A, of each triangle between the two squares (e.g., $\triangle AZT$):

$$A = \frac{x\left(w - x\right)}{2}$$

$$\frac{dA}{dx} = \frac{w}{2} - x$$

The critical point here is $x = w/2$. This point yields the maximum area for each triangle, which is also the minimum area for the interior square. This approach also suggests that we could demonstrate the minimum ratio by using the symmetry present in the figure and the similarity (congruencies) of the four triangles between the two squares.

Another approach that proves quite useful in better visualizing and understanding the problem is a dynamic graphing approach. This approach builds on NCTM's e-example by using a dynamic plot linked to a dynamic diagram. **Figure 2a** shows the plot of the ordered pair of the length AT and the corresponding ratio of areas of the squares. The dynamic diagram allows us to look at the path of the vertices of the interior square along the edges of the exterior square. The plot shows a graph of a quadratic with a minimum value when AT is half AB.

LOOKING AT OTHER REGULAR POLYGONS

The same dynamic approach can be used to investigate the minimum area ratio for equilateral triangles (see **fig. 2b**). The diagram shows that the minimum area ratio occurs when the vertices of the interior equilateral triangle are at the midpoints of the edges of the exterior equilateral triangle. This dynamic sketch shows the area ratio at this point to be approximately 0.25. We say "approximately" because without some other verification the exact value may be very close to (but not exactly) 0.25.

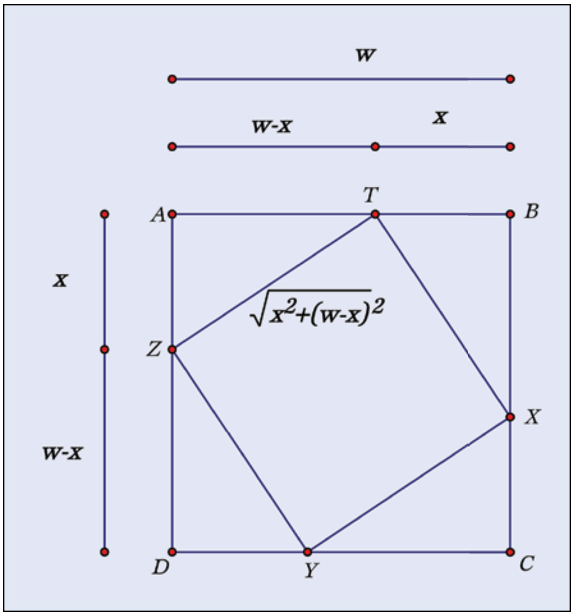

Fig. 1 Setting up a parameterization is one way to find the minimum ratio.

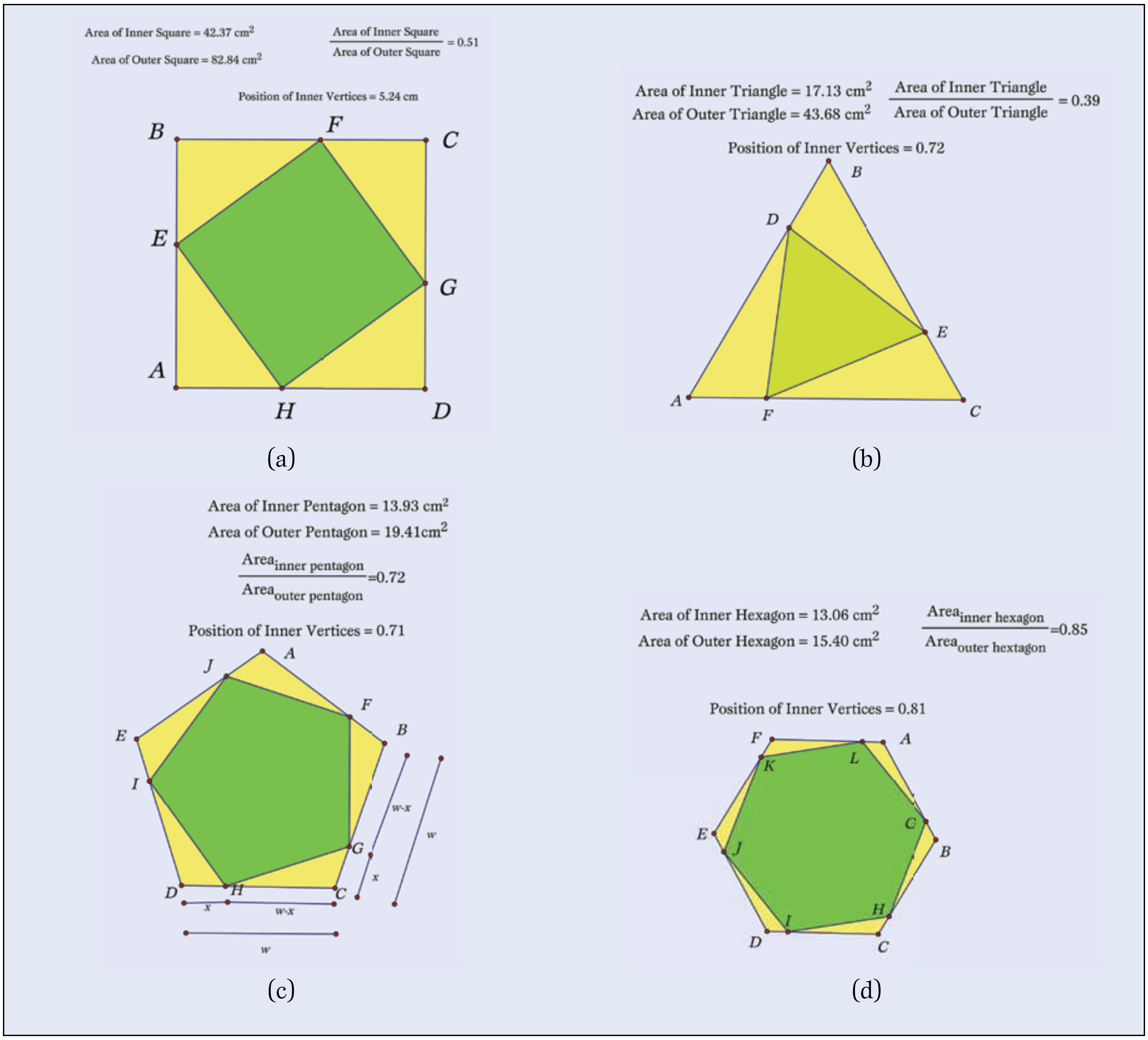

Fig. 2 Plotting the length versus the ratio of the areas produces the curves shown for regular quadrilaterals (**a**), triangles (**b**), pentagons (**c**), and hexagons (**d**).

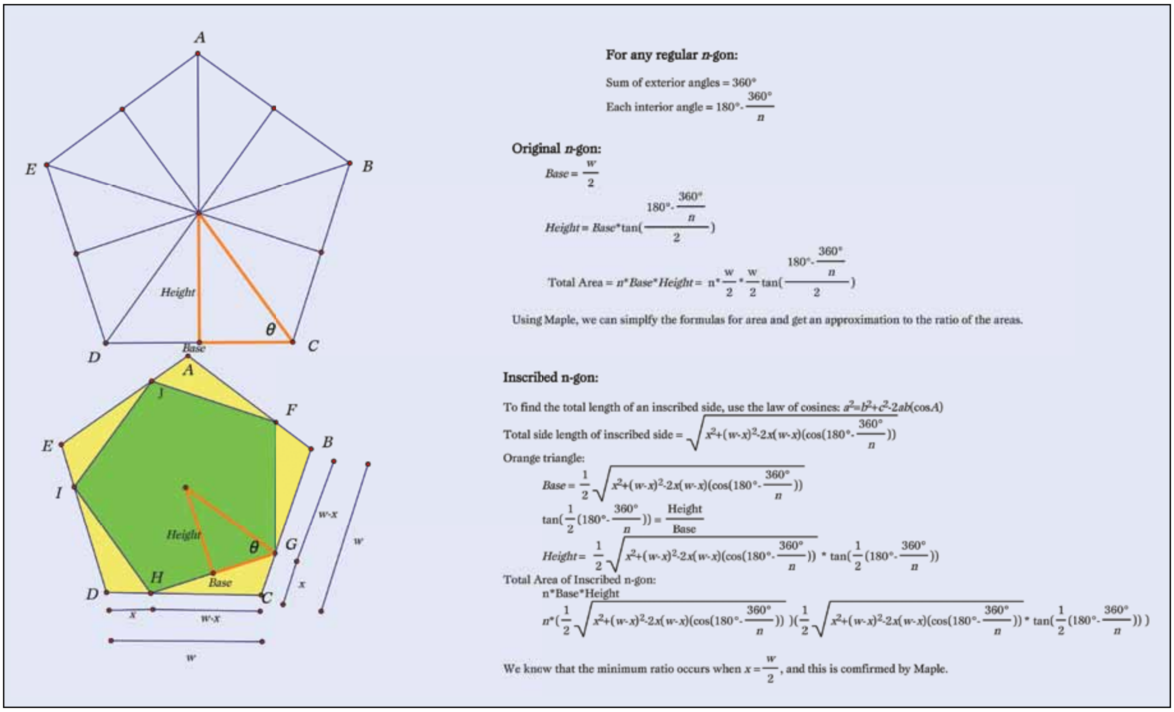

Fig. 3 The indicated triangle is the key to an analytic solution.

Figure 2c shows the minimum area ratio for a regular pentagon to be approximately 0.65, which is close to 2/3; once again, however, further work is needed to know for sure. Finally, **figure 2d** shows the minimum area ratio for a regular hexagon to be approximately 0.75.

This approach to finding the area ratios is unsatisfying because it relies on interpreting coordinates from a graph. Also, because of the limitations of technology, we cannot immediately conclude that the shown area ratio is the true ratio value. For example, several points close to the vertex on the graph in **figure 2d** show the ratio as 0.75 for the regular hexagon.

FURTHER ANALYSIS

To find exact solutions for the area ratios of regular pentagons, we use a standard congruent triangle method to calculate the area of the pentagons; one such triangle is indicated in **figure 3**. Knowing that each interior angle of a regular pentagon is 108°, we use the law of cosines to calculate the length of side GH (labeled *Base*). The angle adjacent to the base (labeled θ) is 54° because the side of the small triangle bisects the 108° angle of the inscribed pentagon. This means that the base, b, can be written as

$$b = \sqrt{x^2 + (w-x)^2 - 2x(w-x)\cos 108°}.$$

To find the height, we use $h = b \cdot (\tan 54°)/2$. With this information, we can calculate the area of the inscribed pentagon as follows:

$$\begin{aligned} A &= \frac{5}{2}bh \\ &= 5\left(\frac{1}{2}\right)\left(\frac{1}{2}b \cdot \tan 54°\right) \cdot b = \left(\frac{5}{4}\tan 54°\right)b^2 \\ &= \frac{5}{4}\tan 54°\left(x^2 + (w-x)^2 - 2x(w-x)\cos 108°\right) \end{aligned}$$

Taking the derivative, we see that the only critical number occurs at $x = w/2$, or when the position of the inscribed vertex is at the midpoint of the original side. This critical number is confirmed to be a minimum by the second derivative test. The value of the inscribed area at this value is

$$\begin{aligned} &\frac{5}{4}\tan 54\ \left(x^2 + (w-x)^2 - 2x(w-x)\cos 108\ \right) \\ &\approx 1.12606707w^2. \end{aligned}$$

Using the same method, we find the area of the original pentagon. The base of one triangle is w, so the area of that triangle is $w^2 \tan 54°/4$, and the area of the entire original pentagon is $5\,w^2 \tan 54°/4 \approx 1.72047740125w^2$. The value of the area ratio is

$$\frac{Area_{inscribed}}{Area_{original}} \approx \frac{1.126067078w^2}{1.72047740125w^2} \approx 0.6545.$$

In general, for any regular convex n-gon, the ratio of areas can be calculated in a similar manner. Because the sum of the exterior angles of a polygon is 360°, each interior angle measures $180° - 360°/n$ where n is the number of sides of the regular polygon.

Table 1 Ratios of Embedded Polygons

Regular Polygon	Ratio
Triangle	$\frac{1}{4}$
Square	$\frac{1}{2}$
Pentagon	$\frac{\sqrt{5}+3}{8}$
Hexagon	$\frac{3}{4}$

Using a parameterization similar to the pentagon parameterization shown in **figure 3**, the length of a side of the inscribed polygon would be

$$\sqrt{x^2+(w-x)^2-2x(w-x)\cos\left(180-\frac{360}{n}\right)^\circ}.$$

We find the area of the interior regular polygon by using the congruent triangle method used on the pentagon. The base of one of the triangles will be *a* (we labeled this *Base* earlier, but here we label it *a* for simplicity), and the height will be

$$\frac{a\tan\left(90-\frac{180}{n}\right)^\circ}{2}.$$

The total area of the original regular polygon will be

$$\frac{w}{2}\left(\frac{w}{2}\tan\left(90-\frac{180}{n}\right)^\circ\right)n.$$

We compute the area ratio as follows:

$$\frac{Area_{inscribed}}{Area_{original}}=\frac{\frac{1}{2}ahn}{\frac{w^2}{4}n\tan\left(90-\frac{180}{n}\right)^\circ}$$

$$=\frac{\tan\left(90-\frac{180}{n}\right)^\circ}{\frac{w^2}{4}n\tan\left(90-\frac{180}{n}\right)^\circ}\cdot\frac{n}{4}\cdot$$

$$\left(\sqrt{x^2+\left(w-x\right)^2-2x(w-x)\cos\left(180-\frac{360}{n}\right)^\circ}\right)^2$$

$$=\frac{x^2+\left(w-x\right)^2-2x(w-x)\cos\left(180-\frac{360}{n}\right)^\circ}{w^2}$$

Substituting $x = w/2$ (because the minimum ratio is at the midpoint) into this result yields

$$\frac{\left(\frac{w}{2}\right)^2+\left(w-\frac{w}{2}\right)^2-w\left(w-\frac{w}{2}\right)\cos\left(180-\frac{360}{n}\right)^\circ}{w^2}=$$

$$\frac{1-\cos(180-\frac{360}{n})^\circ}{2}.$$

This final result allows us to calculate area ratios for any regular polygon.

This approach as well as the end result is similar (but not equivalent) to a formula developed in Wanko (2006). In that investigation, Wanko's students determined a way to compute individual infinite geometric series terms represented by Baravelle spirals. Joining midpoints of sides of regular polygons to obtain new polygons forms the basis of these spirals. The areas of specific triangles in between the embedded polygons are then examined.

Using the result, we find that the triangle, the square, and the hexagon are the exact values that we conjectured (see **table 1**). However, the exact value for the pentagon is revealed to be $(\sqrt{5}+3)/8$, which does not appear to fit any pattern that might have been suggested by the other figures.

Further, our formula will not give all ratios an exact numeric value. For example, the heptagon ratio is

$$\frac{\sin\left(\frac{3}{14}\pi\right)+1}{2}\approx 0.8117.$$

The reason why is another great avenue for investigation.

REFERENCES

Calendar, problem 3. 2006/2007. *Mathematics Teacher* 100 (5): 343.

National Council of Teachers of Mathematics (NCTM). 2000. *Principles and Standards for School Mathematics*. Reston, VA: NCTM.

———."7.3 Understanding Ratios of Areas of Inscribed Figures Using Interactive Diagrams." http://standards.nctm.org/document/eexamples/chap7/7.3/index.htm.

Wanko, Jeffrey J. 2006. "Discovering Relationships Involving Baravelle Spirals." *Mathematics Teacher* 99 (6): 394–400.


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