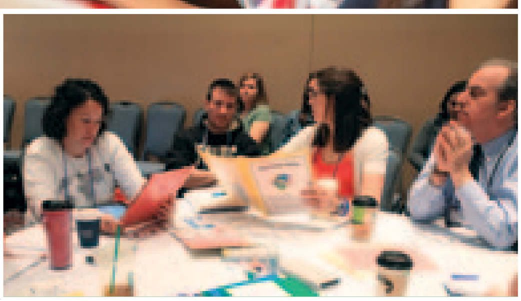
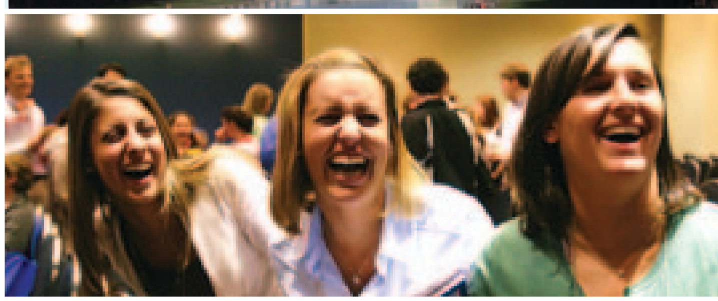
August 1–3, 2013 | Washington, D.C.

Engaging Students *in* Learning:
Mathematical Practices & Process Standards

AN NCTM INTERACTIVE INSTITUTE ON HIGH SCHOOL MATHEMATICS

PROGRAM WORKBOOK





NCTM | NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

www.nctm.org

Welcome!

On behalf of the NCTM High School Institute Advisory Group, we welcome you to this interactive institute focusing on Mathematical Practices and Process Standards. This Institute is one component of NCTM's ongoing initiative to help achieve the vision of reasoning and sense making as a part of the mathematics classroom every day. This professional learning experience has been designed to help you engage students in learning mathematics by examining Common Core Mathematical Practices and NCTM Process standards with guidance from recognized mathematics leaders as well as through thoughtful reflection and discussion with your peers.

One goal of the Institute is to work on using and creating tasks that stimulate reasoning and sense making and address the mathematical practices and processes. The two and a half days of the Institute, the networks you develop here, and the online component are all part of that goal. We encourage you to take full advantage of this Institute: participate in all plenary and breakout sessions, actively engage in task group work, and network with colleagues from throughout the United States and beyond. Then, at the end of the day, meet up with friends or family and enjoy the Washington area. After the Institute, plan to participate in the monthly online professional development sessions that extend your learning in the months that follow.

We wish to thank the staff at NCTM for helping us with the planning and logistics of the Institute, marketing, registration, and on-site work. We also thank all the presenters for agreeing to participate and share their expertise, views, and insights. Finally, we thank everyone in attendance, and we hope that you will find the Institute helpful as you work to make reasoning and sense making through mathematical practices and processes a daily classroom experience for all students.

High School Institute Advisory Group

Laurie Boswell
The Riverside School

Fred Dillon
Baldwin Wallace University

Ed Dickey
University of South Carolina

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National Council of Teachers of Mathematics, 1906 Association Drive, Reston, VA 20191-1502; Telephone (703) 620-9840; Fax (703) 476-2970; E-mail nctm@nctm.org; Web www.nctm.org

Range of Activities
All presentations are open to all Institute participants. Admission is on a first-come, first-served basis. Reserving spaces in line or saving seats is not permitted. The following activities are available:

Keynote Sessions (60–75 minutes)—Well-known leaders in math education will address crucial topics related to and supporting the mathematical practices and process standards.

Breakout Workshops (90 minutes)—Math education practitioners will engage participants in hands-on activities and strategies for implementing these activities in the classroom. Rooms are set with round tables for hands-on work. Choose from ten different workshops during each breakout workshop time slot.

Task Discussion Groups (90 minutes)—Participants will engage in guided activities and facilitated discussions that address the CCSSM mathematical practices. Discussions on the mathematical practices will be rooted in a content strand. Attend task groups according to the content strands you selected when registering.

Program Updates
Program updates, including speaker updates, will be available at the registration desk.

Materials Pickup & Information Desk
Located in the Grand Ballroom Prefunction area, the NCTM Information Desk is available to provide badge corrections and replacements and to offer general assistance to attendees. There will be no on-site registration for new attendees, and no payments will be collected on site at the meeting.

Materials Pickup & Information Desk Hours

Wednesday, July 31	4:00 p.m.–7:00 p.m.
Thursday, August 1	8:00 a.m.–5:00 p.m.
Friday, August 2	8:00 a.m.–4:30 p.m.

You must wear your badge to enter all presentations.

By registering for the NCTM Interactive Institute, participants grant NCTM the right to use, in promotional materials, their likeness or voice as recorded on, or transferred to, videotape, film, slides, audiotapes, or other media.

For Your Child's Safety
Because of the size and nature of the 2013 High School Institute, this event is not an appropriate setting for children under 16 years of age. Your hotel concierge can recommend activities for children while you are attending the Institute. We appreciate your understanding and cooperation.

Bookstore
The NCTM Bookstore, located in the Grand Ballroom Prefunction area, will have NCTM publications with a focus on grades 9–12 mathematics education. NCTM publications are available for purchase on site at a special 25% discount off the list price. Attendees can purchase books online as well and still save 25% using the conference discount code HSDC13; however, shipping fees will be an additional charge. Discount code applies to all NCTM publications and is not limited to those on display. Offer expires 9/30/2013.

Bookstore Hours

Wednesday, July 31	4:00 p.m.–7:00 p.m.
Thursday, August 1	8:00 a.m.–5:00 p.m.
Friday, August 2	8:00 a.m.–5:00 p.m.
Saturday, August 3	8:00 a.m.–11:30 a.m.

Lost-and-Found
Items for lost-and-found may be retrieved or turned in at the Registration Area located in the Grand Ballroom Prefunction area. At the end of the Institute, all lost-and-found items will be turned over to hotel security.

Boxed Lunches
Lunch tickets for Thursday and Friday will be distributed with Institute badges. You will need to present these tickets to receive lunch on both days.

Wireless Internet Access
You will be able to access the Internet via a wireless connection in all meeting rooms by using the log-in information below:

Network code to get online: **nctm2013**
Network name: **Renaissance_CONF**

Welcome Reception
Enjoy light hors d'oeuvres and a cash bar provided by the Renaissance Washington, D.C., in Congressional C.

Twitter
Use Twitter to follow the Institute! **#NCTMInstitutes**

Handouts
Handouts and PDFs of the presentations will be available online at the conclusion of the Institute at **http://nctm.org/hs13**

Wednesday, July 31

4:00 p.m.–7:00 p.m.	Registration, Materials Pickup

Thursday, August 1

8:00 a.m.–9:00 a.m.	Registration, Materials Pickup
9:00 a.m.–10:15 a.m.	Opening Session: Margaret (Peg) Smith
10:30 a.m.–12:30 p.m.	Task Discussion Groups
12:30 p.m.–1:30 p.m.	Networking Lunch
1:30 p.m.–3:00 p.m.	Breakout Workshops
3:15 p.m.–4:30 p.m.	Keynote Sessions
5:00 p.m.–6:00 p.m.	Welcome Reception in Congressional C

Friday, August 2

8:30 a.m.–9:45 a.m.	Keynote Session: Lee Stiff
10:00 a.m.–11:30 a.m.	Breakout Workshops
11:30 a.m.–12:30 p.m.	Networking Lunch
12:30 p.m.–2:00 p.m.	Breakout Workshops
2:15 p.m.–3:45 p.m.	Task Discussion Groups
4:00 p.m.–4:45 p.m.	Online Professional Development Course Orientation (Optional Session)

Saturday, August 3

8:30 a.m.–9:30 a.m.	Keynote Session: Dylan Wiliam
9:45 a.m.–11:15 a.m.	Task Discussion Groups
11:30 a.m.–12:30 p.m.	Closing Session: Cathy Seeley

9:00 A.M.–10:15 A.M.

Opening Session

Task, Tools, and Talk: A Framework for Enacting Mathematical Practices

Explore three key dimensions of classrooms crucial for giving students opportunities to engage in the Common Core State Standards mathematical practices: (1) the tasks in which students engage, (2) the tools available to support students' reasoning and sense making, and (3) the productive classroom talk that supports mathematical discourse.

Margaret (Peg) Smith
University of Pittsburgh, Pittsburgh, Pennsylvania

Grand Ballroom

10:30 A.M.–12:30 P.M.

Task Discussion Groups

Renaissance Ballroom East – Blue Circle Task Discussion Group 1 – Statistics

Congressional A – Yellow Circle Task Discussion Group 1 – Algebra 1

Congressional B – Green Circle Task Discussion Group 1 – Algebra 1

Congressional C – Orange Circle Task Discussion Group 1 – Algebra 1

Mount Vernon Square A – Black Circle Task Discussion Group 1 – Geometry

Mount Vernon Square B – Red Circle Task Discussion Group 1 – Geometry

Meeting Room 8/9 – Red Star Task Discussion Group 1 – Algebra 2

Meeting Room 12/13/14 – Blue Star Task Discussion Group 1 – Algebra 2

12:30 P.M.–1:30 P.M.

Networking Lunch

Grand Ballroom

1:30 P.M.–3:00 P.M.

Breakout Workshops

How Faithful Is Old Faithful? An Adventure in Data Detecting

Data explorations can lead to various ways of representing data, which can lead to interesting "notices and wonders" about the data and about the context they represent. Explore data from the Old Faithful geyser, use statistical analysis and data representations to make and defend a decision, and compare data sets for Old Faithful from several different periods.

J. Michael Shaughnessy
Portland State University, Portland, Oregon
NCTM Past President

Renaissance Ballroom East

Making Sense of Common Core's "Conditional Probability and the Rules for Probability"

Implementing the Common Core State Standards will require many teachers to teach probability for the first time. Students must learn probability in the spirit of the mathematical practices and be able to make sense of the underlying thought processes. Before exploring conditional probability and the rules of probability, we will start with activities that develop appropriate reasoning and understanding of probability concepts. Then we will make sense of the algebraic rules from the models we create.

David Spohn
Hudson High School, Hudson, Ohio

Congressional A

Geometry Concepts Tough to Teach and Tough to Learn

Assessment data show that students struggle with basic concepts such as area of geometric shapes or geometric relationships. By focusing on underlying geometric structures, regularities in repeated actions, and using dynamic interactive technology with the right questions, we can help students develop a better understanding of core geometric concepts.

Gail Burrill
Michigan State University, East Lansing
NCTM Past President

Congressional B

Developing Arguments about Congruence

Examine the process of developing arguments to justify a conjecture by participating in tasks that reveal hidden structures and require seeking patterns and relationships. Activities designed to address Common Core Content Standards related to congruence will include polygon angle sums, rotations, and area.

Alejandro Sorto
Texas State University, San Marcos

Congressional C

Using Recursion to Explore Linear and Exponential Functions

Use recursion to explore problems involving the amount of drugs in the human body, how long paying off a loan will take, and the level of pollution in the Great Lakes. Using technology, manipulatives, and algebra, predict the long-term behavior of these models and write closed-form functions for scenarios.

Maria Hernandez
North Carolina School of Science and Mathematics, Durham

Mount Vernon Square A

Systems of Linear Equations and Functions: Making Sense of Problems

Use real-life scenarios that incorporate systems of linear equations and linear and exponential functions to focus on the Common Core's first Standard for Mathematical Practice: Make sense of problems and persevere in solving them. We will discuss other Standards for Mathematical Practice, alignment with the Content Standards of the Common Core, and available resources.

Sarah Bush
Bellarmine University, Louisville, Kentucky

Mount Vernon Square B

Mathematical Relationships: Helping Students Reason Abstractly and Quantitatively

Explore strategies to help students make sense of quantities in problem situations. Analyze data and interpret slope, predict the effects (graphically and numerically) of changes in a variable, and explore a real-world problem where students develop and compare symbolic representations.

Elizabeth Gasque
Consultant, Charleston, South Carolina

Meeting Room 8/9

Improving Students' Perseverance through Mathematically Rich Tasks

A student's ability to persevere and the results of that effort contribute to his or her self-identity as a learner of mathematics. In this workshop, participants will take on the role of a student and work collaboratively on two familiar tasks with new twists. Students' understanding and solution methods to solving the classic billiards problem will be shared. In addition, participants will develop ways to solve arithmagons and extend the activity for various grade levels.

Darshan Jain

Kevin Gimre
Adlai E. Stevenson High School, Lincolnshire, Illinois

Meeting Room 12/13/14

3:15 P.M.–4:30 P.M.

Keynote Sessions

Mathematical Habits of Mind: From Processes to Practices

Merely teaching students mathematical procedures is not adequate; they need to develop the habits of mind necessary to understand the basis of those procedures, how they might be used, and what their outcomes might mean. Explore practical examples of how to build these habits of mind as the basis for procedural fluency.

W. Gary Martin
Auburn University, Auburn, Alabama

Renaissance Ballroom West

Developing Understanding: Leaving the Front of the Classroom Behind

Examine the vision of instruction that emerges not only from the depth of the Common Core State Standards but also from the very foundation of teaching for understanding. In this decade, "checking for understanding" during the class period has a whole new meaning. It affects our daily lesson design and instructional planning for lesson tasks, the management of those tasks, and the general rhythm of the high school class period. We will share two lesson design tools for analysis and use.

Timothy Kanold
Loyola University Chicago, Illinois

Grand Ballroom North Salon

Right Answer, but Wrong Thinking?

Explore ways to create opportunities for students who struggle to become more engaged in mathematical thinking. This session will share alternative instructional techniques to support algebraic and geometric concepts and skills.

Barbara J. Dougherty
University of Missouri–Columbia

Grand Ballroom Central/South Salon

5:00 P.M.–6:00 P.M.

Welcome Reception

Congressional C

Friday, August 2, 2013

8:30 A.M.–9:45 A.M.

Keynote Session

Rigor: The Foundation of Mathematical Practices

This session will define "rigor" for the mathematics classroom and examine its role in understanding the Common Core mathematical practices and the NCTM Process Standards. Explore mathematical content through the lens of the mathematical practices and Process Standards we promote. Learn instructional strategies that enable students to experience and develop the habits of mind of mathematical proficiency.

Lee V. Stiff
North Carolina State University, Raleigh
NCTM Past President

Grand Ballroom

10:00 A.M.–11:30 A.M.

Breakout Workshops

Using Informal Inference to Help Make Decisions under Uncertainty

Be prepared to put on your data detective hat. When making decisions from data, we often ask, "Could this result have happened by chance alone, or could other factors have played a part?" Investigate data from a real case involving possible job discrimination, and use techniques from informal inference to decide whether probability alone explains the data or whether other factors may have been involved.

J. Michael Shaughnessy
Portland State University, Portland, Oregon
NCTM Past President

Renaissance Ballroom East

Equity and Engagement with Culturally Relevant Mathematics: Reflections and Resources

Explore content with culturally rich contexts from the presenter's teaching experiences and work with organizations focused on equity (e.g., TODOS, North American Study Group on Ethnomathematics). Subjects will span geometry (e.g., symmetry, networks, pi), algebra (e.g., slopes in architecture, Egyptian methods for linear equations), and probability (e.g., games such as Toma Todo and La Lotería). We will then generate and discuss examples in the context of guiding principles that will help you include culturally relevant mathematics in your teaching.

Larry Lesser
University of Texas at El Paso

Congressional A

Using GeoGebra to Connect Algebra and Geometry Ideas

Explore the mathematical practices of modeling and use of tools, and connect these to student misconceptions about representations of figures and images. Link ideas of algebra and geometry by using dynamic and algebraic components of GeoGebra. Scripts will help you develop your own sketches. Please load GeoGebra before the session.

Robert Ronau
University of Louisville, Kentucky

Congressional B

Common Core State Standards: Making Sense of Congruence and Similarity

Why are transformations important in understanding congruence and similarity? Can we use transformations to prove the Pythagorean theorem or that all parabolas are similar? How should geometry change if we focus more on transformations? And how do we engage students in meaningful problem solving that requires persistence, reasoning, and sense making?

Gail Burrill
Michigan State University, East Lansing
NCTM Past President

Congressional C

Deducing Formulas from Sequences

Explore how multiple representations can support student understanding in representing patterns in arithmetic and geometric sequences. Activities will explore connecting representations, the value of different representations for different uses, and applying sequences in real-world situations.

Ed Nolan
Montgomery County Public Schools, Rockville, Maryland

Mount Vernon Square A

Algebraic Thinking, Reasoning, and the Common Core Performance Standards: Data

We live in an increasingly data-driven world. Use dot plots, histograms, and box plots to explore single-variable data numerically and graphically. The box plot is a powerful visual tool for comparing the overall shape, spread, and center of data distributions. Variability is a key element to understanding data, and we will explore one measure of variability: the MAD statistic. This workshop is appropriate for novice or experienced users of the TI-nspire.

Ron Armontrout
Hotchkiss School, Lakeville, Connecticut

Mount Vernon Square B

Mathematical Modeling: Engaging Students with Real Problems

Explore activities that give students opportunities to develop and interpret mathematical relationships in real contexts. Collect and graph data, fit models, and discuss strategies to help students interpret their results. We will emphasize multiple representations as well as proportional and nonproportional relationships.

Elizabeth Gasque
Consultant, Charleston, South Carolina

Meeting Room 8/9

Promoting Students' Algebraic Thinking with Hands-On Activities

In this workshop, participants will work collaboratively through two classroom-tested activities that introduce students to algebraic thinking by promoting reasoning and sense making. Activities will focus on developing expressions to model a pattern built from connecting blocks. In addition, we will work through a knot-tying activity that strengthens students' understanding of linear expressions through multiple representations.

Darshan Jain

Brad Habel
Adlai E. Stevenson High School, Lincolnshire, Illinois

Meeting Room 12/13/14

11:30 A.M.–12:30 P.M.

Networking Lunch

Grand Ballroom

12:30 P.M.–2:00 P.M.

Breakout Workshops

Making Sense of Common Core's "Using Probability to Make Decisions"

Using probability to make decisions will have meaning only when students understand how these decisions are made. Use simulation and modeling to develop the ability to make informed choices consistent with the mathematical practices and the NCTM focus on reasoning and sense making. Investigate the question of when to use expected value and when to use probability in decision making. Use simulation and modeling to evaluate decisions and understand the meaning of fair game.

David Spohn
Hudson High School, Hudson, Ohio

Renaissance Ballroom East

Equity as a Vehicle for Meaningful Mathematical Exploration

A founding editor of the TODOS journal Teaching for Excellence and Equity in Mathematics shares various real-world equity issues that can motivate meaningful and memorable mathematics, spanning geometry (e.g., redistricting, urban density), algebra (function classification), and probability/statistics (e.g., racial profiling, drug testing, inequality). Explore guiding principles, pedagogical tips, and further resources that will help you include equity and social justice in your teaching.

Larry Lesser
University of Texas at El Paso

Congressional A

Using Sketchup to Explore Geometry in 3-D

Google Sketchup is a free three-dimensional modeling tool. Readily construct cylinders, prisms, and pyramids that you can measure, rotate, and view by cross section. You will have the opportunity to construct your own models. Please download Sketchup before the session.

Robert Ronau
University of Louisville, Kentucky

Congressional B

Making Sense of Similarity

By exploring, discussing, and developing mathematical arguments related to similarity, we will connect content areas while making sense of a problem situation. Activities will model teaching that builds learner reasoning and sense making about similarity and also will make connections to measurement, data analysis, and linear functions.

Alejandro Sorto
Texas State University, San Marcos

Congressional C

Using Structure and Repeated Reasoning to Explore Functions

Explore how structure and repeated reasoning can support student understanding of many functions, including linear, exponential, quadratic, logarithmic, and trigonometric.

Ed Nolan
Montgomery County Public Schools, Rockville, Maryland

Mount Vernon Square A

Algebraic Thinking, Reasoning, and the Common Core Performance Standards: Conjectures

Teach students to make conjectures, defend them with arguments, and communicate ideas: Use a TI-nspire or TI-84 and a motion detector to graph distance versus time. Walk linear and piecewise functions and fit mathematical models to the data. Model the walks with point–slope and slope–intercept functions as well as systems of linear equations. Apply the learned concepts to walk a V and fit a piecewise function to the data. Finally, fit a transformed absolute-value function to model the walk.

Ron Armontrout
Hotchkiss School, Lakeville, CT

Mount Vernon Square B

Using Videos to Capture Data

Explore how to collect data from videos. Create models to represent the data by using transformations of functions and reexpression techniques. Explore water-flow and ball-bounce data, as well as the motion of a swing over time. Learn to collect the data with LoggerPro and view the data in various formats.

Maria Hernandez
North Carolina School of Science and Mathematics, Durham

Meeting Room 8/9

Slope, Expressions and Equations, and Systems: Justifying, Critiquing, and Precision

Engage in activities that incorporate proportional reasoning, slope, graphing, expressions and equations, and systems using real-life contexts. Focus on Common Core Standards for Mathematical Practice 3 and 6. We will discuss other Standards for Mathematical Practice, alignment with the Content Standards of the Common Core, and available resources.

Sarah Bush
Bellarmine University, Louisville, Kentucky

Meeting Room 12/13/14

2:15 P.M.–3:45 P.M.

Task Discussion Groups

Renaissance Ballroom East – Blue Circle Task Discussion Group 2 – Statistics

Congressional A – Yellow Circle Task Discussion Group 2 – Algebra 1

Congressional B – Green Circle Task Discussion Group 2 – Algebra 1

Congressional C – Orange Circle Task Discussion Group 2 – Algebra 1

Mount Vernon Square A – Black Circle Task Discussion Group 2 – Geometry

Mount Vernon Square B – Red Circle Task Discussion Group 2 – Geometry

Meeting Room 8/9 – Red Star Task Discussion Group 2 – Algebra 2

Meeting Room 12/13/14 – Blue Star Task Discussion Group 2 – Algebra 2

4:00 P.M.–4:45 P.M.

Online Professional Development Course Orientation (Optional Session)

This session will provide information about the online course, Engaging Students in Learning: Mathematical Practices and Process Standards. Participation in the online course was chosen as an option for an additional fee during your initial registration to the High School Institute. This 12-week online course is intended to serve as a supplement to the High School Interactive Institute and is offered twice throughout the 2013–2014 school year. Expectations for non-credit and credit receiving participants, as well as a demonstration on how to participate in the online course will be discussed.Staff and volunteers will be available to answer questions throughout the session. If you have a laptop or tablet, you may find it helpful to bring it with you to the session.

NCTM Staff

Renaissance Ballroom West

8:30 A.M.–9:30 A.M.

Keynote Session

Formative Assessment: The Bridge between Teaching and Learning in High School Mathematics

Students do not learn what we teach. This is rarely acknowledged but is the daily reality that all high school mathematics teachers face. No matter how well we plan instruction, predicting what students will learn is, in general, impossible. That is why assessment is the key process in effective instruction. Only through assessing students can we determine whether what we taught was, in fact, learned. Explore five key strategies and one big idea of formative assessment as well as many examples of practical techniques to increase student engagement and make your instruction more responsive to students' needs.

Dylan Wiliam
Institute of Education, University of London, London, UK

Grand Ballroom

9:45 A.M.–11:15 A.M.

Task Discussion Groups

Renaissance Ballroom East – Blue Circle Task Discussion Group 3 – Statistics

Congressional A – Yellow Circle Task Discussion Group 3 – Algebra 1

Congressional B – Green Circle Task Discussion Group 3 – Algebra 1

Congressional C – Orange Circle Task Discussion Group 3 – Algebra 1

Mount Vernon Square A – Black Circle Task Discussion Group 3 – Geometry

Mount Vernon Square B – Red Circle Task Discussion Group 3 – Geometry

Meeting Room 8/9 – Red Star Task Discussion Group 3 – Algebra 2

Meeting Room 12/13/14 – Blue Star Task Discussion Group 3 – Algebra 2

11:30 A.M.–12:30 P.M.

Closing Session

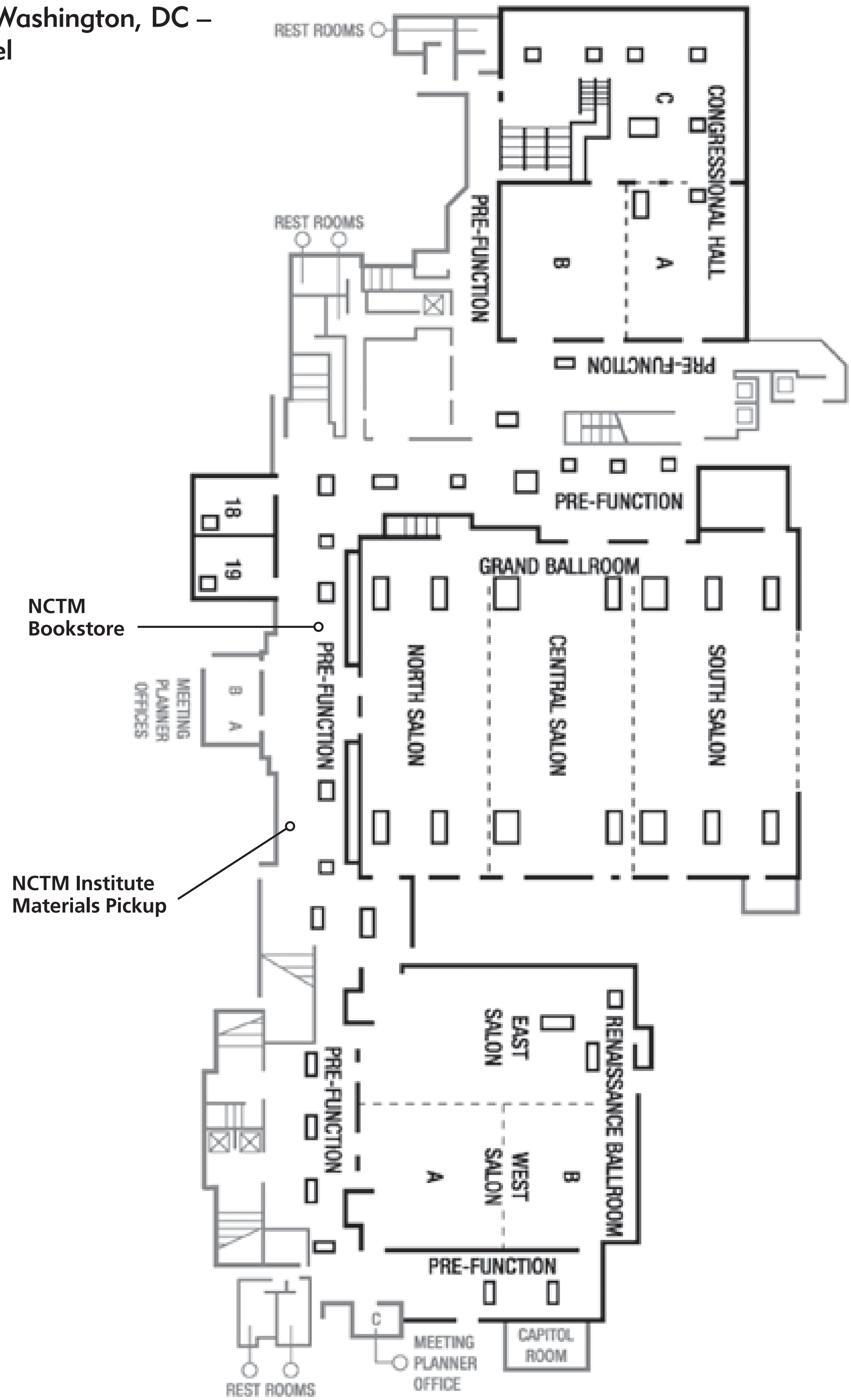
Reflecting on Student Engagement

We will look back over the experiences of the institute and consider the opportunities and challenges for the coming year. How can each person transform the classroom to become an ever-richer environment for students to engage in mathematics and become powerful mathematical thinkers?

Cathy Seeley
Charles A. Dana Center at the University of Texas at Austin (retired)
NCTM Past President

Grand Ballroom

Renaissance Washington, DC – Ballroom Level



Renaissance Washington, DC—Meeting Room Level

Task 1.1

(*a*) Multiply each pair of factors. How are the terms in the resulting polynomial related to the terms in the factors? Make at least two conjectures.

Set 1: $(x + 3)(x + 4)$; $(x + 3)(x - 4)$; $(x - 3)(x + 4)$; $(x - 3)(x - 4)$

Set 2: $(x + 1)(x + 5)$; $(x + 1)(x - 5)$; $(x - 1)(x + 5)$; $(x - 1)(x - 5)$

Set 3: $(x + 3)(x + 3)$; $(x + 3)(x - 3)$; $(x - 3)(x + 3)$; $(x - 3)(x - 3)$

Set 4: $(x + 10)(x + 8)$; $(x + 10)(x - 8)$; $(x - 10)(x + 8)$; $(x - 10)(x - 8)$

Set 5: $(2x + 3)(x + 5)$; $(2x + 3)(x - 5)$; $(2x - 3)(x + 5)$; $(2x - 3)(x - 5)$

Task 1.2

Using algebra tiles, form a rectangle to represent each of the following expressions, and then answer the questions that follow.

(1) $x^2 + 7x + 10$

(2) $x^2 + 11x + 10$

(3) $x^2 + 7x + 12$

(4) $x^2 + 7x + 16$

(5) $x^2 + 11x + 5$

Gurl, T. J., Artzt, A. F., & Sultan, A. (2012). *Implementing the Common Core State Standards through Mathematical Problem Solving*. Reston, VA: National Council of Teachers of Mathematics.

High School Vignette: Completing the Square

This final vignette features a high school teacher inspired by Vinogradova's (2007) description of a lesson introducing students to "completing the square." When completing the square is introduced to students, it is most often explained from a procedural perspective because many students will not use this method in actual practice when solving quadratic equations unless they are asked to. The completing the square method (CTS) becomes important as students learn to solve quadratic equations and use translation methods to graph functions (and conics). Additionally, CTS is used in integral calculus.

This activity, however, supports students in understanding the underlying concepts through making connections between the algebraic procedure of completing the square and its geometric meaning. The following vignette involves students in an algebra II class; they have just finished sections on graphing quadratic equations from tables of values. At this point, the students know how x-intercepts connect to solutions of quadratic equations and have some knowledge of squares and experience with algebra tiles.

To begin the activity, students are asked to use algebra tiles to square binomials, for example $(x + 3)^2$. The common error students make is to square only the x and the 3, yielding an incorrect answer of $x^2 + 9$. Since this activity has been done before, students working in groups are able to build the correct representation of $x^2 + 6x + 9$ with their tiles. Note: the algebra tiles consist of x-by-x squares, 1-by-x rods, and 1-by-1 small squares (unit squares). The teacher poses a question to the class.

Koestler, C., Felton, M. D., Bieda, K. N, & Otten, S. (2013). *Connecting the NCTM Process Standards & the CCSSM Practices*. Reston, VA: National Council of Teachers of Mathematics.

Ms. Liston: Would you be able to work "backward"? By that I mean, if you started with the trinomial $x^2 + 6x + 9$, would you be able to write the binomial you would square to get that trinomial?

Sam: Sure, $(x + 3)^2$

Ms. Liston: Right. Would you be willing to try another one?

[*Sam nods.*]

Ms. Liston: How about $x^2 + 8x + 16$? I'd like all the groups to work on this while Sam is thinking about his answer. Sam, feel free to talk with your group.

Sam: I got $(x + 4)^2$.

Zoe: I got $(x + 8)^2$.

Ms. Liston: Zoe, tell me about that. What was your thinking?

Zoe: Well, you need to get 16, and 2 times 8 is 16.

Sam: No, you get the 16 from 4 times 4 because the last number is a square number.

Zoe: Oh, yeah, you're right.

Ms. Liston: Great, what do others think? [*The class agrees with Sam and Zoe that $(x + 4)^2$ is correct.*] OK, let's move on to a new idea that will connect to what we've been doing. [*The lesson begins with familiarizing students with an arithmetic-geometry perspective.*]

Ms. Liston: Use your unit tiles to build a 3-by-5 rectangle [*see fig. 1.2*]. How many square units make up the area of this rectangle?

Zoe: 15.

Ms. Liston: How did you get 15?

Zoe: I just did 3 times 5.

Fig. 1.2. A 3-by-5 rectangle (re-created from Vinogradova [2007, p. 404])

Ms. Liston: OK, very nice. Here's what I would like you to try to do: I would like to see if you can make a square out of this rectangle of 15 square units.

Sam: You can't.

Ms. Liston: Tell me, why do you say that?

Sam: Because 15 is not a square number.

Ms. Liston: But 15 is close to 16.

Sam: Yeah, it's one less.

Ms. Liston: Yes. Can everyone show me by rearranging the tiles how our rectangle of 15 unit squares could "almost" look like a square and record your ideas on paper?

[*The students draw and redraw and working together they come up with "squares" that have a missing piece. See fig. 1.3.*]

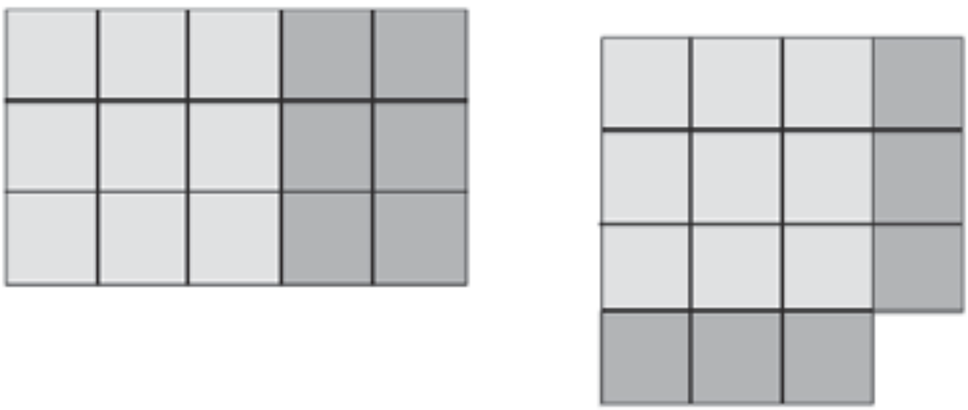


Fig. 1.3. Recomposing a rectangle into a "square" (re-created from Vinogradova [2007, p. 404])

Ms. Liston: Good job. One way to express our square with a missing piece is to write $15 = (3 + 1)^2 - 1$. Try making a square out of 22 square units. Remember, you might have some missing pieces. [*She lets the students work for a few minutes.*] Mateo, what did you get, and can you show us your drawing?

Mateo: I got $(4 + 1)^2 - 3$.

[*Students then practice the same process of completing the square using algebraic tiles.*]

Ms. Liston: So far we have been working with a rectangle with set dimensions and then making that into something that is "almost" a square. But what if we don't know the exact dimensions of the rectangle?

Mateo: We'll have to use variables.

Ms. Liston: How would you like to do that?

Mateo: We could use x and y.

Ms. Liston: OK. Will you come up to the board and draw your rectangle for the class?

[*Mateo comes to the board and draws a rectangle with y being longer than x.*]

Ms. Liston: How can we decide how long y is in comparison to x?

Mateo: It's longer.

Ms. Liston: How much longer?

Mateo: Let's make it two longer.

Ms. Liston: Can one of your group members tell us how to use algebra to write that relationship?

Jessica: In place of y you could put $x + 2$.

[*Students nod, so Ms. Liston continues.*]

Ms. Liston: Okay, would everyone use the tile blocks and build Mateo's rectangle so that it is x on one dimension and $x + 2$ on the other? [*See fig. 1.4.*]

[*Ms. Liston walks around and checks on the groups.*]

Fig. 1.4. Building an x by $x + 2$ rectangle
(re-created from Vinogradova [2007, p. 404])

Ms. Liston: Your tile rectangles look good. What's the area of your rectangle?

Zoe: Mine is $x(x + 2)$.

Sam: I got $x^2 + 2$.

Zoe: Well, you could do that, but you have to have $x^2 + 2x$.

Sam: That's what I meant to say.

Ms. Liston: Why are both answers, $x(x + 2)$ and $x^2 + 2x$ correct?

Sam: Because you distribute the x.

Ms. Liston: Great. What can we do to try to make a square out of our rectangle?

Sam: Well, you could move one of the x-rods like this. [*Shows moving x-rod as in fig. 1.5.*]

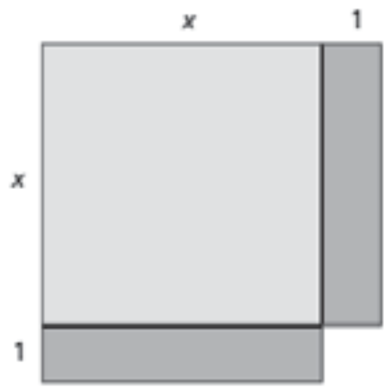


Fig. 1.5. Making a "square" out of the rectangle (re-created from Vinogradova [2007, p. 404])

Ms. Liston: Very nice. This is very close to a square now. How much is missing?

Sam: Only one unit-square.

Ms. Liston: I would like everyone to think about how to write the area of your rectangle, remembering that it is almost a square; it just has one unit-square missing.

Sam: The big square is $(x + 1)^2$. Then you have to subtract the 1, so $(x + 1)^2 - 1$.

Ms. Liston: That looks different from $x^2 + 2x$. Can you convince me that they are the same?

[*Students distribute $(x + 1)^2$, subtract 1, and find that the expressions are equivalent.*]

What is interesting about this introduction to completing the square is that with these examples, students begin to realize (by manipulation of the algebra tiles) that rectangles can be viewed as squares with missing pieces. In addition, the number of 1-by-x rods must always be divided into two parts. As the idea of CTS continues, students can be asked, what number of square units will complete the square? After the connection is made by students that "what's missing from the square" is what is needed to complete the square, the generalization in figure 1.6 can be made.

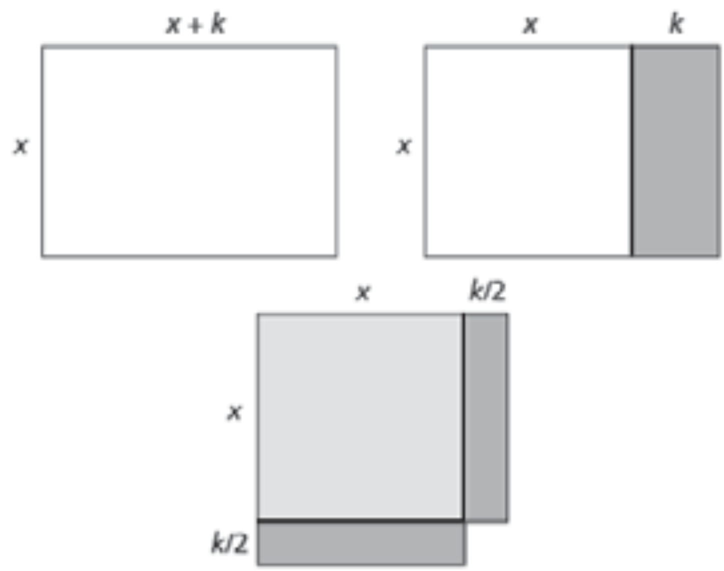


Fig. 1.6. Recomposing a rectangle into a "square"
(re-created from Vinogradova [2007, p. 405])

These three vignettes show how students can be engaged in problems by making sense of them and persevere in solving them by using strategies that make sense to them and others. Students are able to develop and communicate their strategies and to modify their strategies if needed. In doing so, they construct a new understanding of the mathematics concepts involved, as well as strengthening their knowledge of related concepts.

Task 1.8

Solve each of the following by completing the square:

(*a*) $2x^2 + 6x + 1 = 0$

(*b*) $2x^2 + 6x + c = 0$

(*c*) $2x^2 + bx + c = 0$

(*d*) $ax^2 + bx + c = 0$

What do you notice about each of the solutions? Do any patterns emerge? How can the solution to part (*d*) help us?

Gurl, T. J., Artzt, A. F., & Sultan, A. (2012). *Implementing the Common Core State Standards through Mathematical Problem Solving*. Reston, VA: National Council of Teachers of Mathematics.

Counting Trains

Adapted from the *Trains of Thought* task found in Ways to Think About Mathematics by Benson, et al., published jointly by Corwin Press and Education Development Center.

Cuisenaire rods can be used to build trains of a given length. Below are examples of "trains of length 4".

1	2	1

3	1

1	1	2

4

Notice the 1-2-1 train and the 1-1-2 train are made up of the same rods. However, since the order of the rods in each are different, they are considered two different trains.

1. Use the Cuisenaire rods to make all of the trains of length 1, 2, 3, 4, and 5. Make a list of all the trains using any notation that works for you. *As you build the trains think about and write down how you know you have made them all and how you know you didn't make the same train more than once.*

2. How many trains are there of length 1, length 2, length 3, length 4, length 5?

3. Write a rule for determining the number of trains of any length. How do you know your rule will work for any length train?

4. How many trains of length 12 have 3 cars?

5. How could you determine the number of trains of a given length with a given number of cars?

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Retrieved from: http://threeacts.mrmeyer.com/25billionapps/

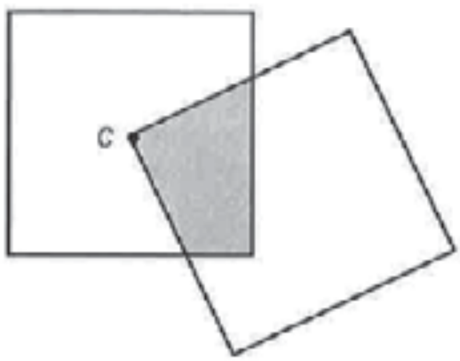
Rotating Squares

Adapted from *Focus in High School Mathematics: Reasoning and Sense Making in Geometry*, NCTM, 2010, pp 5-14, and NCTM Online Workshop, *Reasoning and Sense Making In Geometry*, February, 2013.

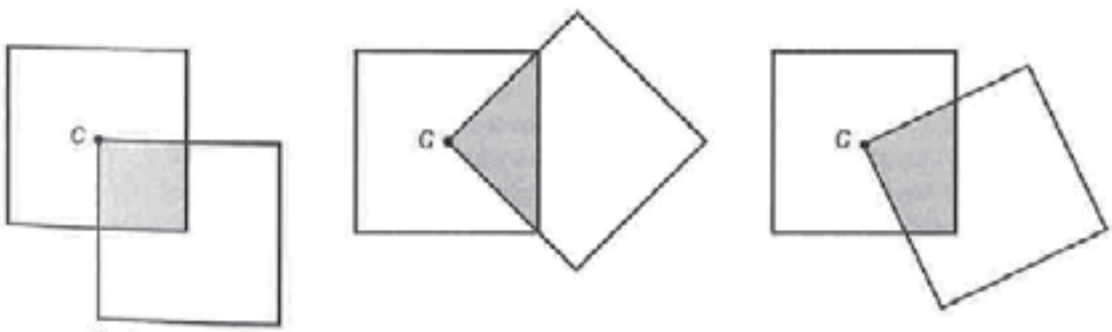
Core Mathematical Idea "The concepts of congruence, similarity, and symmetry can be understood from the perspective of geometric transformation. Fundamental are the rigid motions: translations, rotations, reflections, and combinations of these are the keys. Similarity can be approached with dilations and contractions." –*Common Core State Standards In Mathematics,* page 74
Related geometric ideas: polygon angle sums, rotations, areas
Core Pedagogical Focus Orienting learners to others' mathematical ideas promotes sense making and precise mathematical communication when *reasoning abstractly and quantitatively* and *constructing viable arguments* (MP2 & MP3)

1) What mathematical questions arise when making sense about this figure?

Two congruent squares (n units by n units) overlap as shown in the figure. Vertex C of one square is at the center of the other square.



2) If the square with vertex C is allowed to rotate about the center, C, of the other square, what do you notice about the polygons formed by the overlapping regions?



(a) Square (b) Triangle (c) General quadrilateral

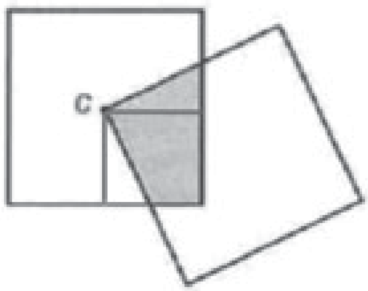
3) What is true about the largest possible value of the overlapping shaded area? Justify your reasoning.

Rotating Squares

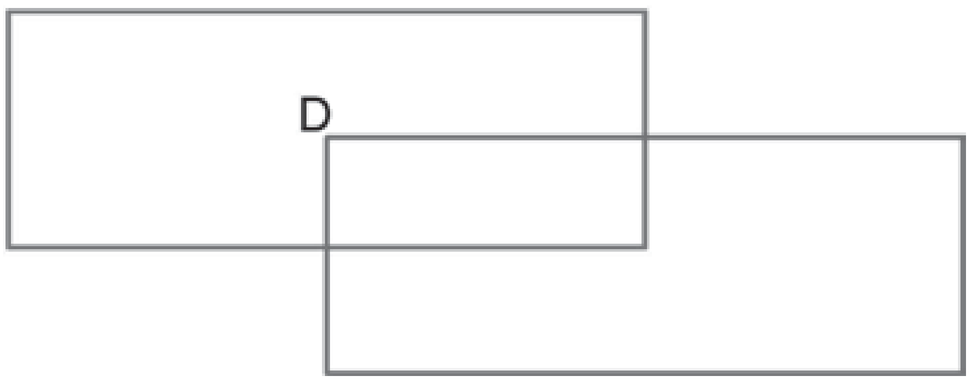
Adapted from *Focus in High School Mathematics: Reasoning and Sense Making in Geometry*, NCTM, 2010, pp 5-14, and NCTM Online Workshop, *Reasoning and Sense Making In Geometry*, February, 2013.

4) Let's examine a student's conjecture.
"*The area of the shaded region is always ¼ of the area of the non-rotating square with center C.*"

Use what you understand about congruence and similarity to convince others that this conjecture is true.



5) Is the conjecture true for all pairs of squares? … other quadrilaterals? … for other regular polygons? … for circles?



6) How can you help students communicate their reasoning about this activity? What questions and instructional moves are most mathematically productive?

PROBLEM SOLVING

Mathematics Assessment Project

CLASSROOM CHALLENGES

A Formative Assessment Lesson

Solving Geometry Problems: Floodlights

Mathematics Assessment Resource Service
University of Nottingham & UC Berkeley
Beta Version

Solving Geometry Problems: *Floodlights*

MATHEMATICAL GOALS

This lesson unit is intended to help you assess how well students are able to identify and use geometrical knowledge to solve a problem. In particular, this unit aims to identify and help students who have difficulty in:

- Making a mathematical model of a geometrical situation.
- Drawing diagrams to help with solving a problem.
- Identifying similar triangles and using their properties to solve problems.
- Tracking and reviewing strategic decisions when problem-solving.

COMMON CORE STATE STANDARDS

This lesson relates to the following *Mathematical Practices* in the *Common Core State Standards for Mathematics*:

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
4. Model with mathematics.
5. Use appropriate tools strategically.
7. Look for and make use of structure.

This lesson gives students the opportunity to apply their knowledge of the following *Standards for Mathematical Content* in the *Common Core State Standards for Mathematics*:

G-CO: Prove geometric theorems
G-SRT: Prove theorems involving similarity

INTRODUCTION

The lesson is structured in the following way:

- Before the lesson, students attempt an assessment task individually. You review their work, and formulate questions to help them improve their solution.
- During the lesson, students first work individually, using your questions, to improve their solutions.
- They then work collaboratively in pairs or threes on the same task. They justify and explain their decisions to peers. Working in the same small groups, they critique examples of other students' work on the task.
- In a whole-class discussion, students explain and compare the alternative approaches they have seen and used. Finally, they again work individually to reflect on their solutions to the task.

MATERIALS REQUIRED

- Each student will need a copy of the task sheet *Floodlights,* a copy of the questionnaire *How did you work?,* and a sheet of squared paper.
- Each small group of students will need a large sheet of paper, and copies of the *Sample Responses to Discuss*.
- Throughout the lesson provide squared and plain paper, rulers, pencils, protractors, and calculators for students to choose from. There are some projector resources provided to support whole-class discussions.

TIME NEEDED

30 minutes before the lesson, a 60-minute lesson, and 10-15 minutes in a follow-up lesson. All timings are approximate. Exact timings will depend on the needs of the students.

Floodlights

Eliot is playing football.

He is 6 feet tall.

He stands exactly half way between two floodlights.

The floodlights are 12 yards high and 50 yards apart.

The floodlights give Eliot two shadows, falling in opposite directions.

1. Draw a diagram to represent this situation.

   Label your diagram with the measures.

2. Find the total length of Eliot's shadows.

   Explain your reasoning in detail.

3. Suppose Eliot walks in a straight line towards one of the floodlights.

   Figure out what happens to the total length of Eliot's shadows.

   Explain your reasoning in detail.

The Volume of a Pyramid:
Low-Tech and High-Tech Approaches

Masha Albrecht

This lesson came about spontaneously during a geometry unit on volume. I had used the lesson shown here in **activity sheet 1,** in which students use cubic blocks to rediscover the formulas for volumes of right prisms, that is, $V = Bh$ and $V = lwh$. This lesson was a simple review for my tenth-grade class, and they completed it easily before the end of the period. With the wooden cubes still on their desks, most of them used the remaining time to build towers and other objects. I noticed that many students piled the cubes into bumpy pyramidal shapes. Because the next day's lesson involved studying the volume of pyramids, I wondered whether these bumpy shapes could be useful for discovering the volume of a real pyramid with smooth sides. Students could compare the volumes of these "pyramids of cubes" with the volumes of corresponding right prisms and perhaps discover the ratio 1/3 to obtain the formula for the volume of a pyramid, $V = (1/3)Bh$. As it turns out, the ratio of 1/3 does not become evident right away. To my students' delight, we found that using a spreadsheet is an excellent way to investigate this problem. My geometry classes had not used spreadsheets before, and the students enjoyed the experience of using the efficiency of technology to compare hundreds—and even thousands—of shapes with ease.

Prerequisites: Students with only very basic mathematical knowledge can benefit from this lesson. Students should have some skill at describing a pattern with an algebraic equation and some familiarity with a spreadsheet. However, I used this lesson with students who had no previous spreadsheet experience.

Grade levels: Although I originally used this lesson with a regular tenth-grade geometry class, the lesson is appropriate for students at different levels and with different abilities. A prealgebra class could do the low-tech part of the lesson, in which students find patterns by using blocks, but they would need help with the formulas for the spreadsheet. Eleventh-grade or twelfth-grade students with more advanced algebra skills could be left on their own to find the spreadsheet formulas and could be given the difficult challenge of finding the closed formula for the volume in the "pyramid of cubes" column on **activity sheet 2.** A calculus class could find the limit of the ratio column as n goes to infinity before they check this limit on the spreadsheet.

Materials: The entire lesson works well in a two-hour block or in two successive fifty-minute lessons, with the low-tech lesson in the first hour and the high-tech spreadsheet lesson in the second. Cubic blocks are needed for the low-tech lesson. Because approximately forty blocks are needed for each group of four students, large classes will need many blocks. If you do not have enough blocks, groups can share. Simple wooden blocks work best; plastic linking cubes do not work as well, because their extruding joints can get in the way when students build the pyramids.

Spreadsheet software is needed for the high-tech lesson. If you are using a separate computer lab, sign out the lab for the second hour of this activity.

For the low-tech extension lesson, the following additional materials are needed: a hollow pyramid and prism with congruent bases and heights, as well as water, sand, rice, or small pasta.

TEACHING SUGGESTIONS

Sheet 1: Using cubic blocks—volume of prisms

This activity sheet is elementary, and more advanced students can skip it. Have students work in groups, with one set of blocks per group. Often one student quickly sees the answers without needing manipulatives, but the other group members are too shy to admit that they need to build the shapes. Require that each group build most of the solids, even if students protest that this activity seems easy.

Sheet 2: Using cubic blocks—volume of pyramids

Students may initially have difficulty understanding what the "pyramids of cubes" look like. Make sure that they build the one with side length 3 correctly. After using the blocks to build a few of the shapes, students recognize the patterns and start filling in the table without using the blocks. Calculating decimal answers for the last column of ratios instead of leaving answers in fraction form helps students look for patterns. Have a whole-class discussion about questions 4, 5, and 6 after students have had a chance to answer these questions in smaller groups, but do not reveal the answers to these questions. Students discover the answers when they continue the table on the spreadsheet.

The last row of the table, where students generalize the results for side length n, is optional. On the spreadsheet, students do not need the difficult closed formula for the second column. They can instead use the recursive formula, which is easier and more intuitive. The solutions include more explanation.

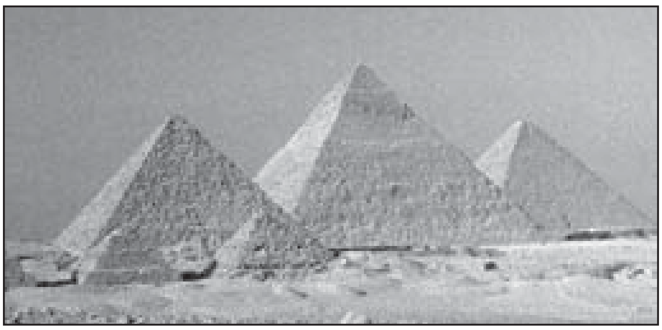
Sheet 3: Using a spreadsheet—volume of pyramids

This activity sheet is designed for students who have some spreadsheet knowledge. Having one pair of students work at each computer is useful if at least one student in each pair knows how to use computers and spreadsheets. For students who have no experience with spreadsheets, you can use this activity sheet as the basis for a whole-class discussion while demonstrating the process on an overhead-projection device. Do not bother photocopying **activity sheet 3** for students who are familiar with spreadsheets. Instead ask them to continue the table from **activity sheet 2,** and give them verbal directions as needed.

POSSIBLE EXTENSIONS

My students enjoyed moving away from the computers for this low-tech finale. If you have a hollow pyramid-and-prism set that has congruent bases and congruent heights, have students use the pyramid as a measuring device to fill the prism with water, sand, rice, or pasta. They should find that three pyramids of water or sand fill the prism exactly to the brim.

I ended the lesson by giving students a picture of some Egyptian pyramids from a book on architecture. The caption to the picture includes measurements, so students can calculate the volume of one of the actual pyramids.



The pyramid of Cheops, the biggest of the three pyramids at Giza, measures 230.5 meters (756 feet) at its base and is 146 meters high. The slope is 51° 52'. At the center is the pyramid of Chephren. Although it is 215 meters (705 feet) at its base and 143 meters (470 feet) high, it appears higher because of its steeper slope (52° 20'). The pyramid of Mycerinus, in the foreground, is the smallest of the three. It measures 208 meters (354 feet) at its base and 62 meters (203 feet) in height, with a slope of 51°.

REFERENCE

Norwich, John Julius. *World Atlas of Architecture.* New York: Crescent Books, 1984.

Using Cubic Blocks—*Volume of Prisms* Sheet 1

Part 1: Volume of a rectangular box

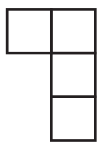
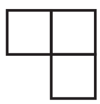
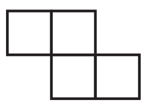
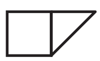
1. Construct each solid with your cubic blocks, and complete the chart. Use your imagination for the last answer.

Length	Width	Height	Volume
2 units	2 unit	4 units	
1 unit	2 unit	3 units	
2 units	2 unit		8 cubic units
0.5 units	2 unit	2 units	

2. Write a formula for the volume of a rectangular box. ______________________________

Part 2: Volume of a right prism

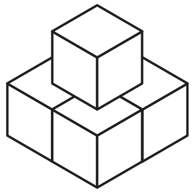
1. Construct each solid with your cubic blocks, and complete the chart. Use your imagination for the last answer.

Base	Area of the Base	Height	Volume
		2 units	
		3 units	
			12 cubic units
		4 units	

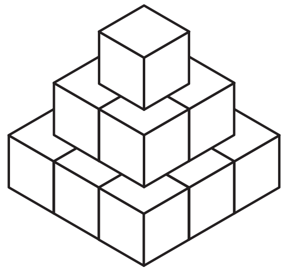
2. Write a formula for the volume of any right prism. ______________________________

Using Cubic Blocks—*Volume of Pyramids* Sheet 2

Although we cannot build exact pyramids with cubes, we can approximate them by building "pyramids of cubes" such as the two pictured below. You will compare the volume of a "pyramid of cubes" with the volume of the prism having the same base and height.



"Pyramid of cubes" with a square base of side length 2 and height of 2



"Pyramid of cubes" with a square base of side length 3 and height of 3

1. Find the volume of a cubic solid with a side of length 2. ______________________________

2. Find the volume of the "pyramid of cubes" with a square base of side length 2 and a height of 2 (pictured above). ______________________________

3. Complete the chart below. In the last column, compute the ratio of the number in the third column divided by the number in the second column.

Length of Side	Volume of Cubic Solid	Volume of "Pyramid of Cubes"	Volume of "Pyramid" Divided by Volume of Cubic Solid
1			
2			
3			
4			
5			
6			
7			
8			
9			
10			
n (if you can)			

4. What happens to the ratio in the last column as your solids become larger?

5. Why do you think that you obtain this result?

6. Does the ratio in the last column ever become 0?

Using a Spreadsheet—*Volume of Pyramids* Sheet 3

As you can tell, finding the pattern in the last column of your table is difficult unless you continue the table. You can create a spreadsheet to do the work for you instead of doing the work by hand.

1. In a spreadsheet, type the headings for the four columns of your table, as shown. You may want to abbreviate the headings.

Length of Side	Volume of Cubic Solid	Volume of "Pyramid of Cubes"	Volume of "Pyramid" Divided by Volume of Cubic Solid

2. Enter the values for the first two rows into your spreadsheet. Do not enter numbers for the last column, because you will use a formula to cause the spreadsheet to calculate these values.

Length of Side	Volume of Cubic Solid	Volume of "Pyramid of Cubes"	Volume of "Pyramid" Divided by Volume of Cubic Solid
1	1	1	
2	8	5	

3. Enter a formula for ratio into the first empty cell in the last column. Remember that the formulas in a spreadsheet begin with an "=." Do not just type in the number 1.

4. Copy the ratio formula that you just wrote into the cell below it. Your spreadsheet should look something like the following:

Length of Side	Volume of Cubic Solid	Volume of "Pyramid of Cubes"	Volume of "Pyramid" Divided by Volume of Cubic Solid
1	1	1	1
2	8	5	0.625

5. The next row of your spreadsheet will contain only formulas. Enter all four appropriate formulas for the next row. For help, use the patterns that you noticed when you built the shapes with blocks. You can also work with other students.

6. Select the row of formulas that you just created, and copy them into the next row. Continue to copy down into more and more rows. Use any shortcut that your software allows, such as Fill Down, until your table is long enough that you are sure of a pattern in the last column.

7. Use the graphing feature of your spreadsheet to make a graph of the ratio numbers in the last column.

Use your spreadsheet to answer the following questions. Some of them are repeated from sheet 2.

8. What happens to the ratio in the last column as the solids become larger?

9. Will the ratio in this column ever be 0? Why or why not?

10. You can use your experience with the "bumpy" pyramids that you made with blocks to generalize the outcome for any pyramid. If a pyramid has a base of area B and a height of h, write a formula for its volume.

Over the Hill

Prepared by Jason Slowbe, San Marcos High School, San Marcos, California

Purpose

Students determine locations on a hillside for a cell phone tower erected to provide a signal to people on the other side of the hill. They identify necessary information, represent the problem with a scale model, and answer questions in context. This task is appropriate for students who have had experience in determining equations of linear functions through two points and in solving systems of linear equations.

Task Overview

A cell phone tower is to be built somewhere on the west side of a hill, as pictured in the diagram on the activity sheet. Find the lowest point on the west side of the hill above which the tower can be based and still provide a signal to anyone east of a lake lying at the foot of the east side of the hill. *An activity sheet that gives students the complete task is included.*

Focus on Reasoning and Sense Making

Reasoning Habits

Focus in High School Mathematics: Reasoning and Sense Making

Analyzing a problem—identifying relevant concepts, procedures, or representations; making preliminary deductions and conjectures

Implementing a strategy—making purposeful use of procedures; monitoring progress toward a solution

Reflecting on a solution—interpreting a solution; revisiting initial assumptions

Process Standards

Principles and Standards for School Mathematics

Problem Solving—monitor and reflect on the process of mathematical problem solving

Connections—recognize and apply mathematics in contexts outside of mathematics

Representation—use representations to model and interpret physical, social, and mathematical phenomena

Standards for Mathematical Practice

Common Core State Standards for Mathematics

1. Make sense of problems and persevere in solving them.
4. Model with mathematics.
5. Use appropriate tools strategically.
7. Look for and make use of structure.

Focus on Mathematical Content

Key Elements

Focus in High School Mathematics: Reasoning and Sense Making

Reasoning with algebraic symbols—connecting algebra with geometry

Reasoning with functions—using multiple representations of functions

Reasoning with geometry—geometric connections and modeling

Standards for Mathematical Content

Common Core State Standards for Mathematics

A-CED.2. Create equations in two or more variables to represent relationships between quantities.

A-REI.6. Solve systems of linear equations exactly and approximately, focusing on pairs of linear equations in two variables.

F-BF.1b. Combine standard function types using arithmetic operations.

Materials and Technology

- Over the Hill activity sheet
- Straightedge or ruler
- Online applet (optional) at www.MathRSM.net/applets/hill

Use in the Classroom

Working with an applet available at www.MathRSM.net/applets/hill might help students to make sense of the problem before solving it, to monitor their progress toward a solution, or to check the reasonableness of their solutions after solving the problem.

The teacher supports students in the appropriate use of models to represent and understand the problem.

Distribute part 1 of the activity sheet, which presents the task and a not-to-scale diagram of the hillside. Students can work on the activity individually or in small groups. Allow students ample time to think about the problem and what might be a good entry point into it. Question 1 asks students to identify information needed to solve the problem, and question 2 asks them to think about ways to mathematize the problem by using geometry or algebra.

Allowing sufficient wait time supports students in analyzing and making sense of a problem.

Once students have completed these preliminary tasks, you might lead a class discussion based on their responses. Focus students' thinking on which pieces of information are important to know and which are unimportant, as well as on possibilities for quantifying particular measurements. Don't be concerned if students fail to recognize the need for a particular measurement right away—as they monitor their progress toward a solution, they will discover that more information is necessary. Support students as they grapple with decisions about what information is important, since this process is central to the formulation stage of mathematical modeling.

Discerning relevant information and ways to quantify and represent measurements helps students formulate a mathematical model as well as select and implement an appropriate strategy.

Distribute part 2 of the activity sheet. As students create their scale models in question 3, you might discuss how coordinates are useful in representing their solutions. You might ask, "How can we describe locations in the diagram mathematically?" or, "Are some places better than others to draw x- and y-axes to create a useful coordinate system? Why?"

In question 4, students predict how far up the west side of the hill the tower must be based to send a signal to every point on the east side of the lake. Once they have determined their solution, you might ask students how they could describe that point on the hillside to others. You might also ask students to describe to one another the key pieces of information that they considered in making their predictions and how those considerations guided their thinking.

Making a prediction helps students analyze a problem and provides a basis for monitoring their progress toward a solution.

As students discuss solution methods, either as a whole class or in small groups, you could ask questions to focus their attention on auxiliary lines as a way to mathematize the problem further. If students used coordinate or synthetic geometric approaches, you could have them compare one another's approaches for similarities and differences.

Students often need support in recognizing and using structure.

In question 5, students reflect on their predictions and the reasonableness of their solutions. To support students in understanding their solution point as a lower bound that must be exceeded in the placement of the tower, you might ask the question, "If you were standing exactly at the edge of the lake, would you actually get a signal?" You might pose additional reflection questions, such as, "Which other methods can you use to solve this problem?" and, "How might you check the reasonableness of your solution?"

Through questioning, the teacher prompts students to interpret their solution in context and interpret its reasonableness.

Focus on Student Thinking

At first, in the absence of any numerical measurements, students might struggle to make sense of the problem—particularly in question 1 as they attempt to determine which pieces of information are necessary, might be helpful, or are not necessary to solve the problem. The following questions can start a class discussion about these issues:

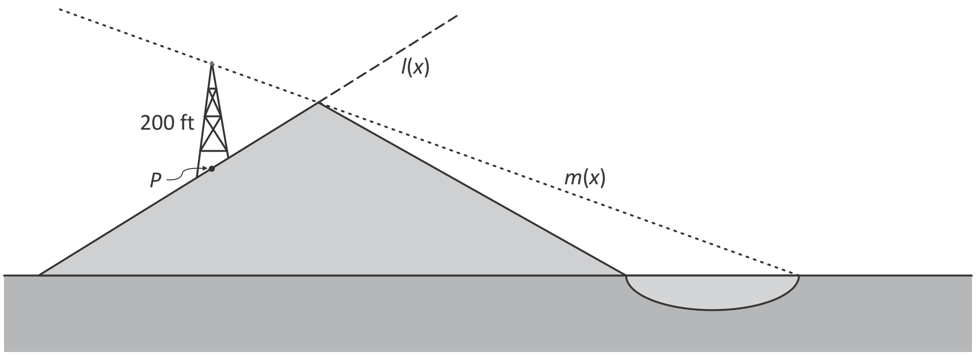
- Is it important to know the units of measure?
- Will using different units change the solution?
- Is it necessary to know a particular piece of information, or can we solve the problem another way without that information?

In question 3, students might use the grid but not automatically construct coordinate axes. To prompt students to think about this issue, you might ask questions such as the following:

- Would constructing axes help us describe locations in the diagram?
- Are some places better than others to place the origin?
- How would the location of the tower on the hillside change if we placed the axes somewhere else?

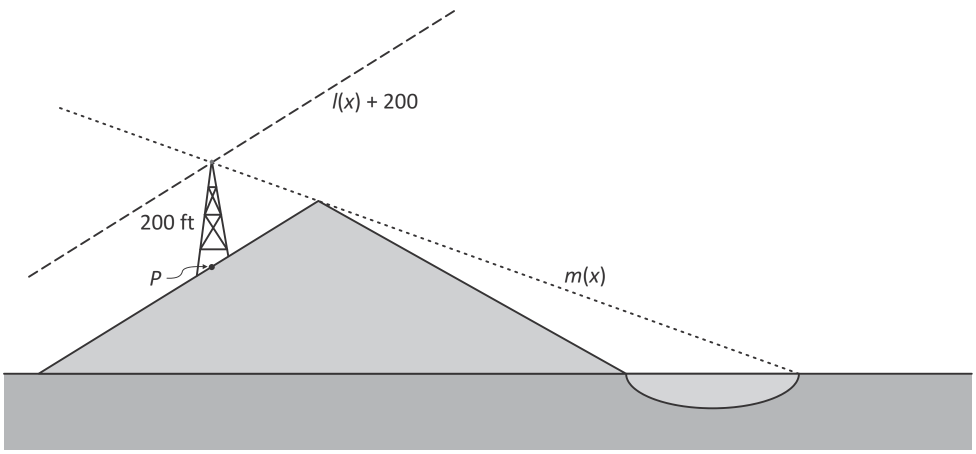
If students struggle to find an entry point into the problem, you might ask them to mark two points on the west side of the hill—one point where the tower's signal can definitely reach everyone on the other side of the lake, and one point where the tower's signal definitely cannot reach everyone. Asking students to explain why the tower's signal can or cannot reach every point on the eastern side of the lake can help students gain access to the problem and engage in solving it.

Some students might determine the linear functions representing the line up the west side of the hill, $l(x)$, and the line from the eastern edge of the lake through the vertex of the hill, $m(x)$. Then they might reason that the difference of these two functions gives the vertical distance between the lines (see fig. A). Solving the equation $m(x) - l(x) = 200$ gives the x-coordinate of point P—the point at the base of the cell tower on the hill in the diagram below. Other students might translate line $l(x)$ vertically 200 feet for the height of the tower and then solve for the intersection point of $l(x) + 200$ and $m(x)$ (see fig. B). Students who finish early might solve the problem by using both of these methods and then might demonstrate that the two methods are algebraically equivalent.



$m(x) - l(x) = 200$

Fig. A



$l(x) + 200 = m(x)$

Fig. B

Focus on Student Thinking—*Continued*

Students might also use similar triangles or trigonometry to find the horizontal distance from point *P* to the eastern edge of the lake, as pictured in figure C.

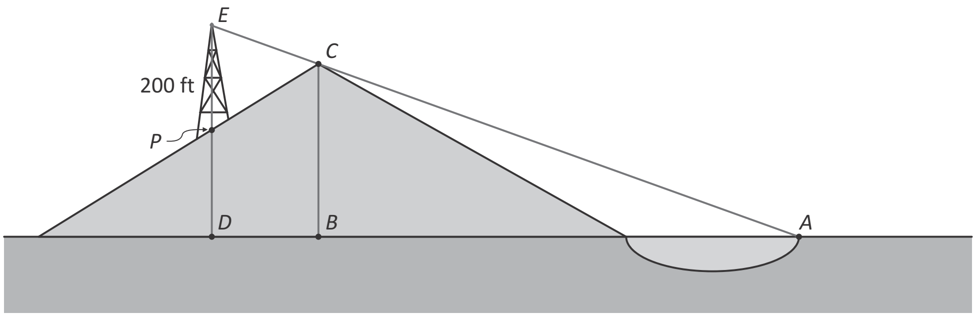


Fig. C

In question 4, some students might reason backward to verify their solutions. If students imagine themselves standing at the eastern edge of the lake, they can determine whether they could look back over the hill and see the top of the tower. Students can make sense of the problem by realizing that if the tower were visible to them, then, equivalently, the tower's signal would be able to reach them.

Assessment

Students should summarize their approaches to solving this problem, including their thought processes throughout the development of their solution.

Invite students to change one of the measurements in this problem (height of the tower, width of the lake, width or height of the hill, or the slope of one or both sides of the hill) and then solve the new problem. Students should explain how changing each measurement affects how far up or down the hillside the tower must be shifted.

Resources

National Council of Teachers of Mathematics (NCTM). *Principles and Standards for School Mathematics.* Reston, Va.: NCTM, 2000.

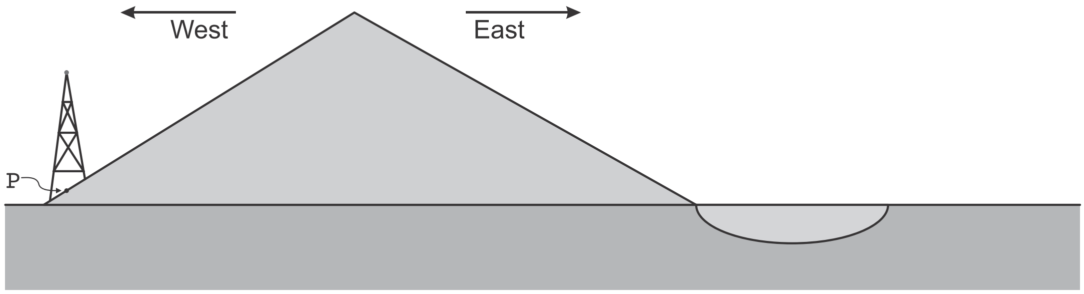
———. *Focus in High School Mathematics: Reasoning and Sense Making.* Reston, Va.: NCTM, 2009.

National Governors Association Center for Best Practices and Council of Chief State School Officers (NGA Center and CCSSO). *Common Core State Standards for Mathematics. Common Core State Standards (College- and Career-Readiness Standards and K–12 Standards in English Language Arts and Math)*. Washington, D.C.: NGA Center and CCSSO, 2010. http://www.corestandards.org.

Over the Hill

Student Activity Sheet

A cell phone tower will be built somewhere on the west side of the hill pictured in figure 1. How far up the hill must the tower be placed to provide a signal to anyone on the east side of the lake?



(*Not drawn to scale*)

Fig. 1

Part 1: Preliminary Probing

1. What information is needed to solve the problem? What information is not important to know?

2. Thinking algebraically or geometrically, how can you mathematize the problem?

Over the Hill

Student Activity Sheet (Continued)

Part 2: Down to Details

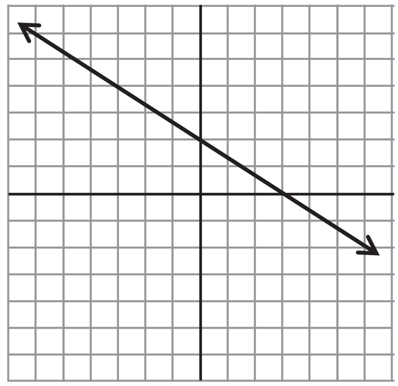
3. Use the following information to draw an accurate model:
    - The cell tower is 200 feet tall.
    - The hill is 800 feet tall and 2800 feet wide from west to east at the base.
    - The hill has vertical symmetry.
    - The lake starts at the base of the hill and is 600 feet wide.

4. First, predict how far up the hillside the tower must be built so that the tower can provide a signal to all points on the other side of the lake. Then determine the exact location on the hillside for the base of the tower. Explain your thinking completely.

5. How close was your prediction to your actual solution? Is your solution reasonable? Explain.

Building Polynomial Functions

NAME ______________________________



1. What is the equation of the following linear function?

2. How did you find it?

3. The slope-y-intercept form of a linear function is $y = mx + b$. If you've written the equation in another form, re-write your equation in slope-intercept form. ____________________

4. Now, factor out the slope, and re-write the function as $y = m(x + \frac{b}{m})$

5. Choose a second linear function and write it in slope-y-intercept form: ____________________

6. Graph the function on the axis above and be sure to label it.

7. Re-write your second function with the slope factored out (as in #4): ____________________

8. For each function, what does $\frac{b}{m}$ represent on the graph?

If you let $c = -\frac{b}{m}$, then the form $y = m(x - c)$ could be called the slope-x-intercept form of a linear equation, where c is the x-intercept. The factor theorem states that if c is a root (x-intercept) of a polynomial function, then $(x - c)$ must be a factor of that polynomial function. Note that $(x - c)$ is a factor of the expression. The only other factor is the slope m.

9. From their slope-y-intercept form, multiple the two functions together:

10. Graph the resulting function on the axis above.

11. What kind of function did you get?

12. What relationship do you see between the graph from #10 and the lines?

- x-intercepts?
- y-intercepts?

13. Identify the left-most x-intercept on the graph. With a straight-edge, cover everything to the right of that point. What connections do you see relating the signs of the y-values?

14. Identify the right-most intercept on the graph. With a straight-edge, cover up everything to the left of that point. What connections do you see relating to the signs of the y-values?

Complete the following sentences:

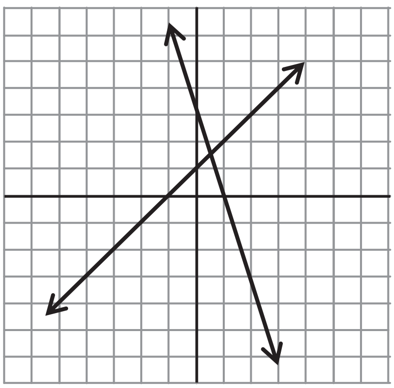
15. When both lines are above the x-axis, the y-values are _______ and the parabola _________________________.

16. When both lines are below the x-axis, the y-values are _______ and the parabola _________________________.

17. When one line is above, and the other below the x-axis, the parabola __________________.

Y-VALUE OF L_1	Y-VALUE OF L_2	PARABOLA IS ABOVE/BELOW THE X-AXIS
+	+	
+	-	
-	+	
-	-	

18. From just the patterns above, draw a sketch of the quadratic function that would be obtained from the linear expressions of these lines.

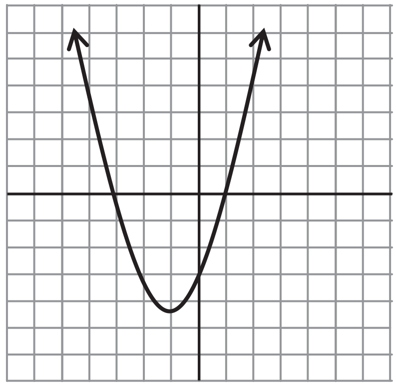


19. Write the equation for each line: ___________________ and ________________________

20. Multiply the expressions together and graph the resulting function to check your work for #18.

Working Backwards

NAME ______________________________



To the right, a parabola is given. Working backwards, we want to find two lines that could represent the linear factors.

1. What is the left-most x-intercept of the parabola?

2. To the left of that point, what do we know about the two lines that represent the linear factors?

3. What is the right-most x-intercept of the parabola?

4. To the right of that point, what do we know about the two lines that represent the linear factors?

5. What do we know about the two lines between the two linear factors?

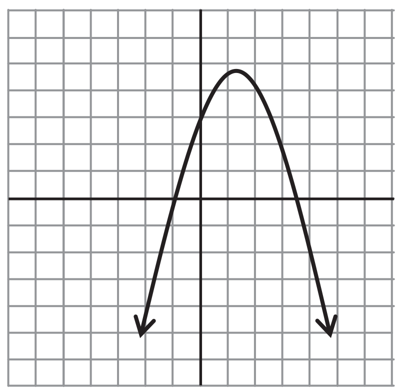
6. On graph sketch two lines that could represent the linear factors.

7. Write the equations for the lines you sketched.

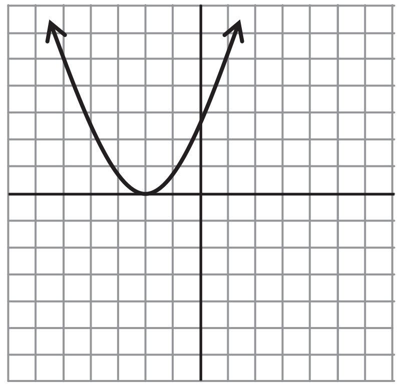
8. Multipty the two expressions together.

9. Graph the resulting parabola. How does it compare to the graph given?

10. Follow the same steps for the parabola to the right to determine a possible equation. Below, describe how you got the equation you've found.



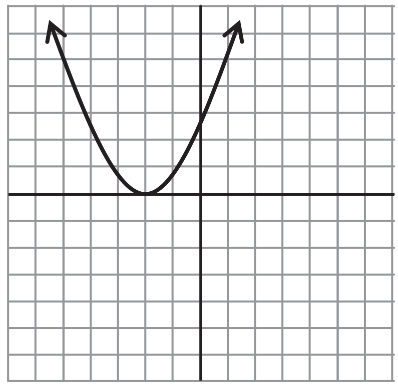
11. What is different about the x-intercept(s) of the graph to the right?



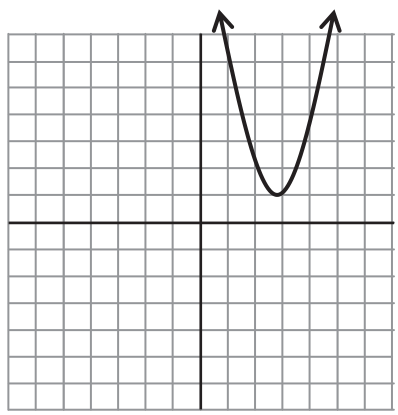
12. Sketch the lines that could represent the factors.

13. Describe the process of how you determined your answer for Question 12.

14. To the right is another graph just like the one above. Find an alternative pair of lines that could also satisfy the requirements for representing the factors. Graph them over the parabola to the right.



15. For the graph to the right, sketch the lines that could represent the factors.



16. Describe the process of how you determined your answer for Question 15.

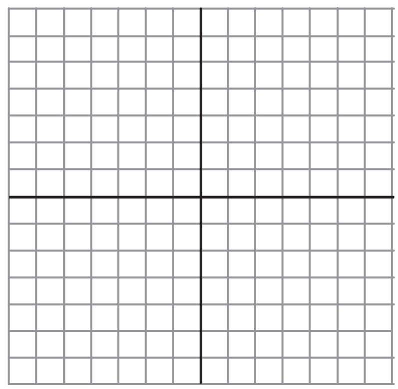
Building Polynomial Functions
Polynomials of Degree 3

NAME ____________________________

1. Graph each of the following three linear functions on the axis provided.

$$y_1 = \frac{1}{2}x - 2 \qquad y_2 = x + 3 \qquad y_3 = -x + 1$$

2. Multiply the three functions together.

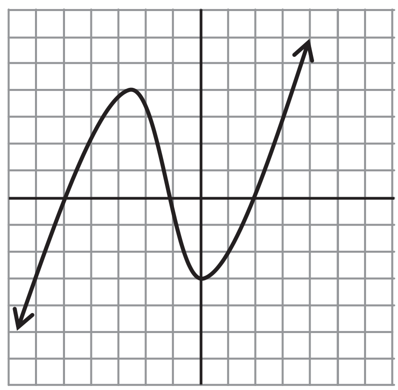


3. What type of function is the result?

4. Using what you know about multiplying signed numbers and the graphs of the lines, sketch your prediction for the graph of the product of the three linear functions.

5. Describe in your own words how you chose your graph.

6. Work-backwards: Find three lines that could be the components for the cubic represented by the graph to the right.



7. Describe in your own words how you chose your lines.

Late Shift at the Orange Bowl

Name ______________________________________

The last bus of the day from the Old Town Line in Miami arrives at the Orange Bowl between 11:00 and 11:20 p.m. The last bus from the New Town Line arrives between 11:10 and 11:25 p.m. The exact arrival times of each bus are randomly and uniformly distributed over its time interval. Tony and Rhonda both work the late shift at the Orange Bowl. Tony always arrives on the last Old Town bus, and Rhonda always arrives on the last New Town bus. Today is Rhonda's birthday, and Tony wants to surprise her with flowers as she climbs off her bus. What is the probability that Tony's bus will arrive before Rhonda's?

1. *a*. Design a simulation of the arrival of the two buses and describe how to use your simulation to make an estimate of the probability that Tony arrives before Rhonda.

   b. What assumptions have you made in designing your simulation?

2. *a*. As a class, discuss the designs of several simulations. Decide on one design that everyone will use independently to estimate the probability that Tony will arrive before Rhonda. On your own, run this simulation to generate 100 trials. Record the number of times that Tony's simulated arrival time falls ahead of Rhonda's simulated arrival time.

   Number of times (in 100 trials) that Tony arrives first __________________

   b. On the basis of the results of your simulation, calculate an experimental probability for the event "Tony arrives at the Orange Bowl before Rhonda."

   Experimental probability (based on the results of 100 trials) __________________

3. *a*. Each of your classmates also conducted the simulation. Were all the experimental probabilities the same? __________________ Why, or why not?

   b. Combine the results of all your classmates' simulations to make an estimate of the theoretical probability. Also make an estimate of the uncertainty of your prediction.

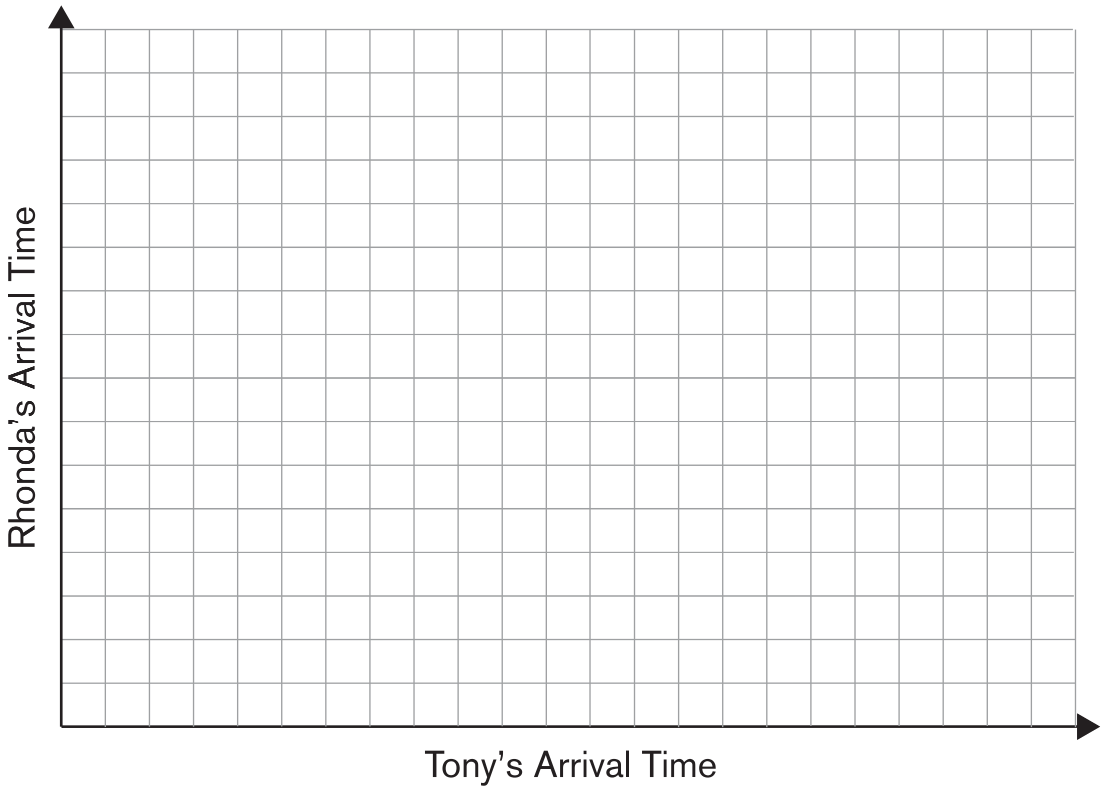
   Estimate of the theoretical probability that Tony arrives first __________________

   Estimate of the uncertainty of the probability estimate above __________________

4. *a*. To calculate the theoretical probability of Tony arriving before Rhonda, you can think of each possible outcome as an ordered pair (Tony's arrival time, Rhonda's arrival time) and use geometry to represent

Name ______________________________

the set of all possible outcomes. Time is a continuum of moments, so listing all the pairs of arrival times that could occur is impossible. However, using the grid below, draw and label a region representing the set of all the possible pairs of arrival times of the two buses.



b. What is the area of the region that you have drawn?

Area of the region representing all the possible outcomes ______________

5. *a*. In the region that you constructed in step 4, shade the portion that corresponds to the outcomes in which Tony arrives before Rhonda. What is the area of this shaded region?

   Area of the shaded region ______________

   b. Calculate the theoretical probability that Tony arrives before Rhonda, and explain any assumptions that you are making in your calculation.

   Theoretical probability that Tony arrives first ______________

Late Shift at the Orange Bowl (continued)

Name ______________________________

6. Write a paragraph in which you compare or contrast the theoretical result that you obtained in step 5 with the estimate that you made in step 3 on the basis of the pooled results. How accurate were the simulations?

7. Create another probability question involving the arrival times of Tony and Rhonda. Your problem should be one that you can solve by using the geometric method described in step 4. Show the solution to the problem.

Will Women Run Faster than Men in the Olympics?

Shaughnessy, J. M., Chance, B., & Kranendonk, H. (2009). *Focus in high school mathematics: Reasoning and sense making in statistics and probability*. Reston, VA: NCTM

The gold-medal times for the Olympic 200-meter dash are listed in the table below and represent the modern-day Olympic winning times from 1900 through 2008.

Use this data to help answer the questions: Can you use this data to predict the future times of men and women in this event? Do you think women will ever run faster than men, and if yes, when? Construct an argument to support your answer to these questions by using data.

Times for the Olympic 200-Meter Dash

Year	Male	Time (in seconds)	Female	Time (in seconds)
1900	Walter Tewksbury, USA	22.2		
1904	Archie Hahn, USA	21.6		
1908	Robert Kerr, CAN	22.6		
1912	Ralph Craig, USA	21.7		
1920	Allan Woodring, USA	22.0		
1924	Jackson Scholz, USA	21.6		
1928	Percy Williams, CAN	21.8		
1932	Eddie Tolan, USA	21.12		
1936	Jesse Owens, USA	20.70		
1948	Mel Patton, USA	21.10	Fanny Blankers-Koen, NED	24.40
1952	Andy Stanfield, USA	20.81	Marjorie Jackson, AUS	23.89
1956	Bobby Morrow, USA	20.75	Betty Cuthbert, AUS	23.55
1960	Livio Berruti, ITA	20.62	Wilma Ruldolph, USA	24.13
1964	Henry Carr, USA	20.36	Edith McGuire, USA	23.05
1968	Tommie Smith, USA	19.83	Irena Szewinska, POL	22.58
1972	Valeriy Borzov, USSR	20.00	Renate Stecher, GDR	22.40
1976	Don Quarrie, JAM	20.23	Barbel Eckert, GDR	22.37
1980	Pietro Mennea, ITA	20.19	Barbel Wockel (Eckert), GDR	22.03
1984	Carl Lewis, USA	19.80	Valerie Brisco-Hooks, USA	21.81
1988	Joe DeLoach, USA	19.75	Florence Griffith-Joyner, USA	21.34
1992	Mike Marsh, USA	20.01	Gwen Torrence, USA	21.81
1996	Michael Johnson, USA	19.32	Marie-Jose Perec, FRA	22.12
2000	Konstantinos Kenteris, GRE	20.09	Marion Jones, USA	21.84
2004	Shawn Crawford, USA	19.79	Veronica Campbell, JAM	22.05
2008	Usain Bolt, JAM	19.30	Veronica Campbell-Brown, JAM	21.74

Consider the gap times between men and women gold medalists for each of the Olympic years that included men and women.

Gap Times between Men and Women Victors for the Olympic 200-Meter Dash

Year	**1948**	**1952**	**1956**	**1960**	**1964**	**1968**	**1972**	**1976**
Difference in times (in seconds)	3.30							

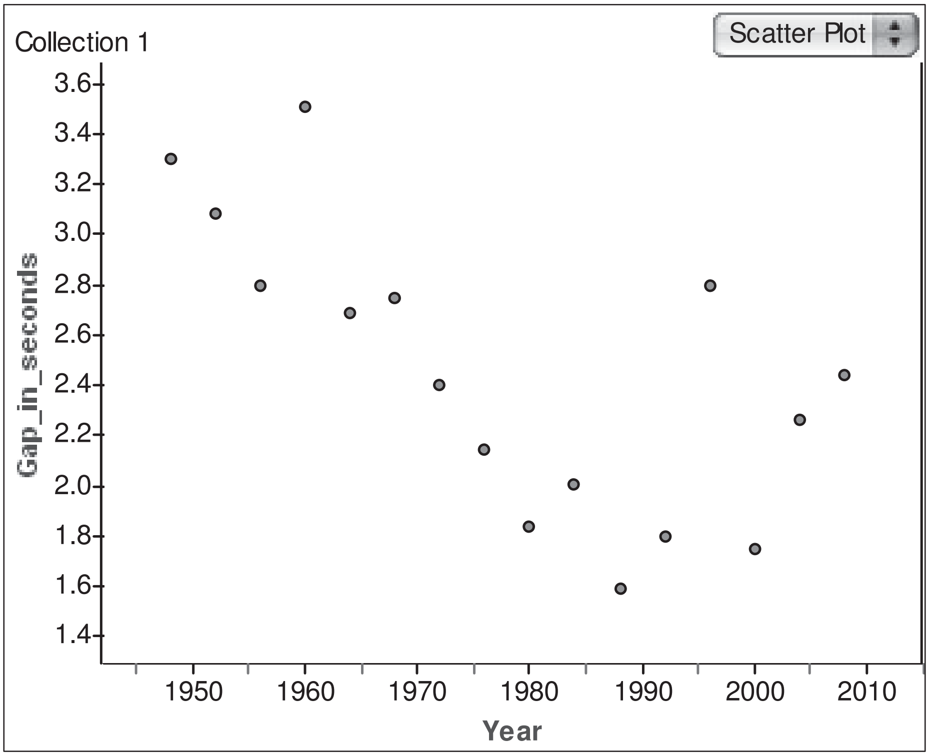
Year	**1980**	**1984**	**1988**	**1992**	**1996**	**2000**	**2004**	**2008**
Difference in times (in seconds)								

Why are all of the differences positive?

Is the gap between men and women closing? Why?

Looking at the table, what predictions or conclusions can you make?

Now graph the data in your table to see what type of relationship exists.



Now that you have your data plotted, are you surprised by the result? Explain.

What is meant by the phrase "line of best fit"?

Estimate a line of best fit by placing a piece of spaghetti (or a pipe cleaner) on your graph and then tracing it.

Is there a positive, a negative, or no relationship between the year and the difference in times? Explain.

What would **no relationship** look like?

How do you know if your line is the BEST line of fit?

Remember, the goal is to answer the questions: Can you use this data to predict the future times of men and women in this event? Do you think women will ever run faster than men, and if yes, when?

Now that you have an approximate line of best fit, how can you write an equation to represent this line?

Using a method of your choice, write an equation for a line of best fit.

Compare your equation for a line of best fit with another student's equation. What do you notice?

What does the slope mean in the context of this problem?

What is the independent variable?

What is the dependent variable?

How do you know which variable is which?

Use your line of best fit to answer the question: Do you think women will ever run faster than men, and if yes, when?

When will men and women have the same times?

When will women have a faster time then men?

How can you use the graph to answer these questions?

How can you use your equations to answer these questions?

Use your model to predict the difference in winning times for the men's and women's 200-meter dash in 2012.

What other questions can you ask about the times for men and women in the 200-meter dash and how can you use your line to answer those questions?

Extension

You used the method of graphing the difference in times for men and women to make future predictions. Is there another way you could have used the data to make predictions?

If you plotted the men's times and the women's times on the same graph, how would you know when they ran the same time?

What would the *x*-intercepts represent in that situation?

Could there be a different, non-linear model that would better predict men's and women's times? Explain.

Will I Have a Seat for My Flight?

Euclid Airlines has a small plane it uses for commuter flights between Park City and Aspen. The plane holds 15 passengers. From many years of experience, the company knows that 90% of ticketed passengers usually show up. The company decides to sell 15 tickets plus another 10% to ensure the plane is always at capacity. (90% of 17 is a little more than 15.) If your flight is sold out, what is the probability you may be bumped?

1. To simulate this situation, assign a probability of "showing up" to 17 students and decide whether or not this flight was oversold.

2. Can you do this problem theoretically, **easily**? Explain.

3. How can you create a single trial using technology that can be easily repeated?

4. Run your simulation many times to get an idea of the probability of being bumped. Record your number of simulations and your total number of times you would be bumped.

5. How many times do you think you need to run the simulation to get an idea of the "true" probability of being bumped? Explain.

6. Combine the results of your class to make an estimate of the theoretical probability. Are you surprised by these results? Explain.

Mathematics Assessment Project

CLASSROOM CHALLENGES

A Formative Assessment Lesson

Modeling Conditional Probabilities: 2

Mathematics Assessment Resource Service
University of Nottingham & UC Berkeley

Beta Version

A Fair Game

Dominic has made up a simple game.
Inside a bag he places nine balls. These balls are either black or white.
He then shakes the bag.

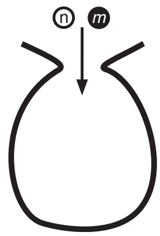
He asks Amy to take two balls from the bag without looking.

Dominic

If the two balls are the same color then you win.
If they are different colors then I win.

OK.
That sounds fair to me.

Amy



Amy is right, Dominic has made the game fair.

How many white balls (n) and how many black balls (m) has Dominic put in the bag?

Fully explain your answer.

Hints: You could think about likely combinations of black and white balls and then check to see if they would make the game fair.

Or you could use algebra to work out the number of black and white balls in the bag.

Make It Fair

A.

1 black ball.

How many white balls for a fair game?

B.

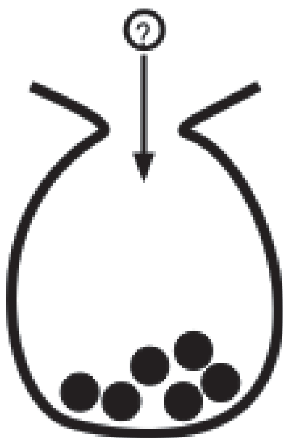
3 black balls.

How many white balls for a fair game?

C.

6 black balls.

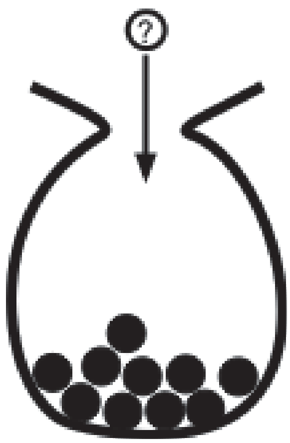
How many white balls for a fair game?



D.

10 black balls.

How many white balls for a fair game?



Make It Fair: Extension Task

E.

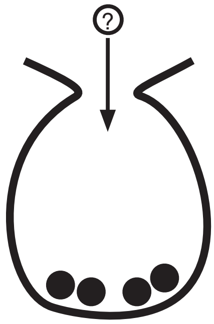
2 black balls.

How many white balls for a fair game?

F.

4 black balls.

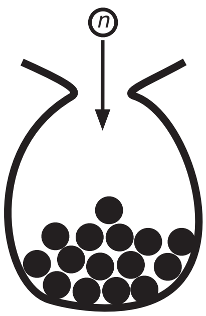
How many white balls for a fair game?



G.

15 black balls.

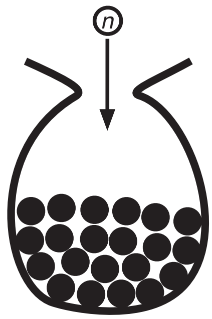
How many white balls for a fair game?



H.

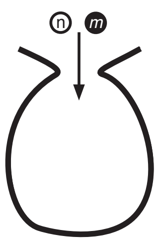
21 black balls.

How many white balls for a fair game?



I.

***m* black balls and *n* white balls.**
If the game is fair, what formulae can you say about *m* and *n*?



NOTES

REFLECTION: TASK GROUPS, SESSION 1

Strand:__________________ Task:____

- Describe the mathematical content students should learn from doing this task.

- What did you think about doing mathematics through a task?

- What did you learn?

- What did the presenter do to facilitate learning? What kinds of questions emerged?

- What habits of reasoning were evident during this task?

- From a student perspective, how does a task approach to reasoning and sense making (RSM) differ from other types of RSM activities (e.g., class discussion, homework problems, problem solving, and projects)?

- How might students solve the task or have misconceptions about the content?

REFLECTION: TASK GROUPS, SESSION 2

Strand:__________________ Task:____

- Describe the mathematical content students should learn from doing this task. What opportunities to learn mathematics did the task afford?

- How did this experience differ from the session 1 task experience? How was it the same?

- What is the value in using reasoning and sense-making tasks with your students?

- What type of planning would a teacher need to do to create or adapt a task like this for use in the classroom?
    - What type of questions would the teacher ask students?

    - What would the teacher do to motivate and engage students in this type of task?

- How might you modify this task to fit the needs of your students?

REFLECTION: TASK GROUPS, SESSION 3

Strand:__________________ Task:____

- What makes a good reasoning and sense-making task?

- What do you still need to learn to adapt or create good tasks that align with your curriculum?

- What are some rich sources of mathematical content that you can adapt to reasoning and sense-making tasks?

- What steps will you take to create or modify a reasoning and sense-making task for your students?

- How can you assess the desired *mathematical* outcomes of the tasks that you use with your students?

- How can you assess the desired *reasoning* outcomes of the tasks that you use with your students?

TASK TEMPLATE

TITLE—*Should be a catchy title that gives a hint of the context of the problem.*

Source—*Include statement of source if this is a direct adaptation of a task from another source.*

Adapted/Prepared by ____________________________—*Include name and affiliation of author(s).*

Purpose	*This should be a two- or three-sentence broad-brush summary of the task meant to hook further interest. Mention where this may fit into the curriculum, including prerequisites, as well as major reasoning habits or mathematical practices (see below). Some indication of the context may also be useful.*	
Focus on Reasoning and Sense Making	***Focus in High School Mathematics (FHSM): Reasoning and Sense Making*** **Reasoning Habits** *List the major reasoning habits (see pp. 9–10 of* ***FHSM: Reasoning and Sense Making****). For each, give the general category, followed by a hyphen and then the specific habit. Use semicolons to separate multiple habits from a category. With "Seeking and using connections," you might pick a particular item from the list following the stem.* ***Principles and Standards for School Mathematics (PSSM) Process Standards*** *List the major process standards from PSSM. For each, give the standard name, followed by a hyphen and the area of emphasis within the standard. Use semicolons to separate multiple areas of emphasis from the same standard.*	***Common Core State Standards (CCSS)*** **Mathematical Practices** *List the standards from the CCSS Mathematical Practices (see pp. 7–8 of CCSSM) by standard number.*
Focus on Mathematical Content	***FHSM: Reasoning and Sense Making*** **Key Elements** *List the key elements from* ***FHSM: Reasoning and Sense Making*** *(see chapters 4–7). For each, give the content strand, followed by a hyphen and the key element. Use semicolons to separate multiple key elements from the same strand.*	**CCSS Content Standards** *Given the high school standards addressed. For each, give the conceptual area and domain (as abbreviated in CCSS), along with the standard number, all separated by hyphens (e.g., F-BF-3). Then give the full statement of the standard.*
Materials and Technology	*Include a bulleted list of any handouts provided, other materials, and technological resources that a teacher might need to implement the task.*	

In the Classroom	
Task **State the task. Refer to any handouts distributed.**	
Use in the Classroom	
This section should give teaching suggestions showing how the task might be used to promote student reasoning and sense making. *The focus of this discussion should be on what students will actually be doing and how the teaching will help this to happen.*	*Explain how the teaching suggestions support the reasoning habits and mathematical practices. These comments should line up with the teaching suggestions discussed.* *In general, one of the comments should explicitly address all reasoning habits and mathematical practices listed above.*
Focus on Student Thinking	
The body of this section should reflect student thinking related to the implementation of the ideas in the "In the Classroom" section. The range of student thinking should contain examples of student thinking, both successful and unsuccessful (e.g., both their insights and their misconceptions).	
Assessment	
This section should give suggestions for how a teacher will know whether the students are achieving the intended outcomes. This section might include classroom observations, classwork to be turned in, homework assignments, associated projects or other write-ups.	
Source(s)	
Give the complete sources of any resources from which the task was drawn.	

BLANK TASK TEMPLATE

Purpose		
Focus on Reasoning and Sense Making	*Focus in High School Mathematics (FHSM): Reasoning and Sense Making* Reasoning Habits *Principles and Standards for School Mathematics (PSSM)* Process Standards	*Common Core State Standards* Mathematical Practices
Focus on Mathematical Content	*FHSM: Reasoning and Sense Making* Key Elements	*Common Core State Standards* Content Standards
Materials and Technology		

BLANK TASK TEMPLATE

In the Classroom	
Task	
Use in the Classroom	
Focus on Student Thinking	
Assessment	
Source	

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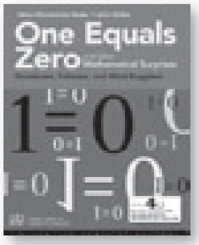

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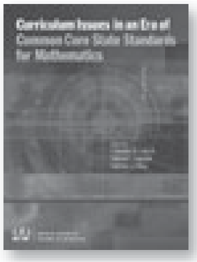

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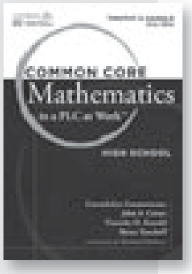

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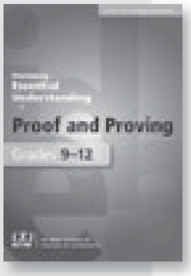

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An NCTM Interactive Institute for Grades 9–12

Descriptive Information

Course title: Engaging Students in Learning: Mathematical Practices and Process Standards

Course number: University of San Diego—X784C

Description: An online interactive professional learning experience that enables participants to effectively address the Common Core mathematical practices and NCTM process standards.

Course credit: 2 credit hours

Intended audience: Grade 9–12 Teachers

Course facilitator: TBD

Dates: August 19, 2013–November 17, 2013, Wednesdays, 7:00 p.m., eastern time
or January 20, 2014–April 20, 2014, Thursdays, 4:00 p.m., eastern time

A certificate of participation will be issued at the end of the course to participants who are not registered for course credit.

Required Text

- *Principles and Standards for School Mathematics*, National Council of Teachers of Mathematics, 2000. Available online at http://www.nctm.org/standards/content.aspx?id=16909
- *Common Core State Standards for Mathematics*, 2010. Available online at http://www.corestandards.org/the-standards/mathematics
- Additional readings will be made available through Moodle, the online course-management system.

Required Technology

- Moodle Course Management – http://onlinecoursesupport.com/nctm/kb/article/id/398
- Adobe Connect Online Meeting Software – http://admin.adobeconnect.com/common/help/en/support/meeting_test.htm
- Speakers and a computer microphone; a headset; or a phone line

Course Goals And Objectives

Goal:
To provide an interactive professional learning experience that enables high school mathematics teachers to effectively address the Common Core mathematical practices and NCTM process standards.

Objectives:

- Understand that the Common Core mathematical practices and NCTM process standards are integral to teaching.
- Learn instructional strategies that enable students to experience and to develop the habits of mind of a mathematically proficient student.
- Examine mathematical content through the lens of the mathematical practices and process standards.
- Engage in activities related to task selection, development and implementation both during and after the Institute.

Course Expectations

The following is a list and brief description of responsibilities and assignments that will help you meet the goals and objectives of this course. You will receive further information about each of these assignments during an introductory online session, as well as through Moodle. If you participate in the face-to-face summer institute, an additional training in online course participation may be provided.

The course facilitator will work with students, monitoring their progress and providing feedback on student assignments and progress throughout the course.

Online Course Participation

Meaningful interaction in the online environment is a right and responsibility of each student in an online course. Since the course takes place entirely online, regular synchronous and asynchronous participation is key to building the classroom environment. **If you must miss an online session, notify your course facilitator in advance**.

Description of online sessions:

e-Keynotes: e-Keynotes are online presentations from preeminent experts in their respective areas of mathematics education. These 75-minute sessions may include a concluding question and answer session, but are mostly presented lecture-style. A total of four online e-Keynotes will be presented during the 12-week semester course. Participants are expected to attend all e-Keynotes.

e-Workshops: e-Workshops are working class sessions. These 75-minute sessions will include some new mathematical or pedagogical information. Participants will work alone or in online breakout groups to understand and apply the new ideas. An e-Workshop does not require preparation before the class session, but participants are expected to contribute both verbally and via the chat box during the session. Participants are expected to attend all e-Workshops.

e-Shares: e-Shares are interactive online discussions. These 75-minute sessions require prior preparation as well as active participation in the online event. E-Shares will be used primarily for presentations and participants should be prepared to speak on the work they have done. Participants are expected to attend all e-Shares.

Requirements

Adapted Task and Written Summary:

1. Participants will adapt a task using sample textbook problems or problems that have been used previously. Participants will choose at least one NCTM process standard AND at least one Common Core mathematical practice to emphasize as they implement the task and reflect on the practice used.
2. Participants will implement the task in their classroom.
3. Participants will join and contribute in an e-Share reflecting on the process of modifying the task and implementing it in their classroom.
4. Participants will write a 1–2 page (double spaced) summary on the implementation and reflection of the adapted task. The following must be included in the summary: why you chose the task; which process standard and common core practice you have emphasized in the task; how you implemented the task; the overall outcome; and how you will change the task for next time.
5. Participants will submit their work on a forum in Moodle and post at least one substantive comment on the work of TWO other classmates. When posting their work, students must include: a.) a copy of the original task (including source information), b.) a copy of the adapted task, c.) the written summary.

Teacher Observation:

1. Participants will observe a high school math teacher for at least 30 minutes.

During your observation:

- Bring the list of the NCTM process standards and Common Core practices (on Moodle).
- Use your list of the practices and standards to identify some of the NCTM process standards and Common Core practices that the teacher demonstrates. Be sure to note the specific examples of what the teacher did.
- Document the specific actions that went well during the lesson and actions that did not go well.

Reflection:

- Choose one of the practices or standards demonstrated by the teacher that you felt was implemented well and explain in detail what made the action effective.
- Choose another one of the practices or standards and explain in detail how you might teach that standard or practice differently.

Forum:

- In the designated forum on Moodle, describe an overview of the lesson you observed. Attach your completed list and reflection to this forum post.
- Comment meaningfully on at least TWO other forum posts. Ask questions, make comments and give advice when appropriate. Additionally, be sure to respond to each person that comments on your initial post.

Weekly Forums:

Weekly forums are the online equivalent of class discussions and are key to building online communities. The topic of the weekly forums will be based on the corresponding e-Keynote, e-Workshop or e-Share for that week. Participants must post at least ONE substantive comment and then comment on the posts of at least TWO other classmates.

Asynchronous Task Forums:

During weeks when no online sessions are scheduled, participants will engage in asynchronous discussion forums. These forums will focus on a specific topic and may require completing small tasks such as reading and responding to given articles; collectively revising student assessments or activities; or engaging in other relevant tasks. Participants will be expected to post comments and respond to others about the task throughout the week.

Evaluation

Assignments	**Points**
Live attendance and participation of all online sessions x9 (10 pts ea.)	90
Weekly Forums x9 (10 pts ea.)	90
Asynchronous Task Forums x3 (20 pts ea.)	60
Adapted Task Assignment	50
Teacher Observation Assignment	50
	340

Final Grade

Final course grades will be based on the percentage of possible points earned.

A 90% or better

B+ 88%–89.5%

B 80%–87.5%

C+ 78%–79.5%

FALL 2013 SCHEDULE

All times are Eastern Time.

Week	Date	Time	Type of Session	Class Session
Week 1	8/19–8/25	Wednesday Aug. 21, 2013 7:00 p.m.– 8:15 p.m.		Introduction to Online Learning
Week 2	8/26–9/1	Wednesday Aug. 28, 2013 7:00 p.m.– 8:15 p.m.		TOPIC: Implementing NCTM Process Standards and Common Core Math Practices w/ William McCallum
Week 3	9/2–9/8	Wednesday Sep. 4, 2013 7:00 p.m.– 8:15 p.m.		TOPIC: Adapting a Task
Week 4	9/9–9/15	**Asynchronous**		Task Forum #1
Week 5	9/16–9/22	Wednesday Sep. 18, 2013 7:00 p.m.– 8:15 p.m.		TOPIC: Engaging Tasks w/ Gary Martin
Week 6	9/23–9/29	Wednesday Sep. 25, 2013 7:00 p.m.– 8:15 p.m.		Presentations: Adapted Task
Week 7	9/30–10/6	Wednesday Oct. 2, 2013 7:00 p.m.– 8:15 p.m.		TOPIC: Teacher Observations w/Jon Wray
Week 8	10/7–10/13	Wednesday Oct. 9, 2013 7:00 p.m.– 8:15 p.m.		TOPIC: Teacher Observations
	10/14–10/20			**Fall Break**
Week 9	10/21–10/27	**Asynchronous**		Task Forum #2
Week 10	10/28–11/3	Wednesday Oct. 30, 2013 7:00 p.m.– 8:15 p.m.		Presentations: Teacher Observations
Week 11	11/4–11/10	**Asynchronous**		Task Forum #3
Week 12	11/11–11/17	Wednesday Nov. 13, 2013 7:00 p.m.– 8:15 p.m.		TOPIC: (CONCLUSION) Engaging Students in Learning w/ Fred Dillon

All times are Eastern Time.

Week	Date	Time	Type of Session	Class Session
Week 1	1/20–1/26	Thursday Jan. 23, 2014 4:00 p.m.– 5:15 p.m.		Introduction to Online Learning
Week 2	1/27–2/2	Thursday Jan. 30, 2014 4:00 p.m.– 5:15 p.m.		TOPIC: Implementing NCTM Process Standards and Common Core Math Practices w/ William McCallum
Week 3	2/3–2/9	**Asynchronous**		Task Forum #1
Week 4	2/10–2/16	Thursday Feb. 13, 2014 4:00 p.m.– 5:15 p.m.		TOPIC: Adapting a Task
Week 5	2/17–2/23	Thursday Feb. 20, 2014 4:00 p.m.– 5:15 p.m.		TOPIC: Engaging Tasks w/ Gary Martin
Week 6	2/24–3/2	Thursday Feb. 27, 2014 4:00 p.m.– 5:15 p.m.		Presentations: Adapted Task
Week 7	3/3–3/9	Thursday Mar. 6, 2014 4:00 p.m.– 5:15 p.m.		TOPIC: Teacher Observations w/Jon Wray
Week 8	3/10–3/16	Thursday Mar. 13, 2014 4:00 p.m.– 5:15 p.m.		TOPIC: Teacher Observations
	3/17–3/23			**Spring Break**
Week 9	3/24–3/30	**Asynchronous**		Task Forum #2
Week 10	3/31–4/6	Thursday Apr. 3, 2014 4:00 p.m.– 5:15 p.m.		Presentations: Teacher Observations
Week 11	4/7–4/13	**Asynchronous**		Task Forum #3
Week 12	4/14–4/20	Thursday Apr. 17, 2014 4:00 p.m.– 5:15 p.m.		TOPIC: (CONCLUSION) Engaging Students in Learning w/ Fred Dillon

HS Institute 2013-2014: Online Keynote Sessions

From Process to Practice: Developing Ways Students Think about Mathematics

The NCTM Process standards and the Common Core Standards for Mathematical Practice describe the characteristics of a mathematically proficient student as that student carries out mathematical work. These process and practice standards come to life when linked to specific mathematical content. The speaker will talk about how the Illustrative Mathematics Project is developing illustrations for these standards.

William McCallum, University of Arizona

It Starts with Your Tasks: Meeting the Challenge of Promoting Mathematical Processes/Practices

Selecting useful tasks is the first step in engaging students in the mathematical reasoning habits as described in the Common Core practices and NCTM Standards process standards. This session will explore qualities of effective tasks that achieve that goal and where to find those tasks. Finally, the session will consider how to get the most of tasks that are designed to promote mathematical processes and practices.

W. Gary Martin, Auburn University

Student Learning and the Standards for Mathematical Practice in the Common Core State Standards

The Mathematical Practices are the underlying foundation of helping students learn. The speaker will share examples of lessons that employ the Mathematical Practices while enabling students to learn mathematics with understanding and by actively building new knowledge from experience and prior knowledge. Students can best learn math by conjecturing, learning to critique mathematical arguments, and developing mathematical reasoning skills.

Fred Dillon, Baldwin Wallace University

Helping Students Grow in the Mathematical Practices

The Standards for Mathematical Practice describe the expertise required of mathematically proficient students. But how can teachers best help ALL of their students to develop these proficiencies? Further, how can student growth in these behaviors be monitored/measured? This session will address these issues and provide practical tools to assist teachers in supporting student growth in the Mathematical Practices.

Jonathan Wray, Howard County Public Schools and NCTM Board of Directors

The teacher of mathematics should pose tasks that are based on—

- sound and significant mathematics;
- knowledge of students' understandings, interests, and experiences; and
- knowledge of the range of ways that diverse students learn mathematics—

and that

- engage students' intellect;
- develop students' mathematical understandings and skills;
- stimulate students to make connections and develop a coherent framework for mathematical ideas;
- call for problem formulation, problem solving, and mathematical reasoning;
- promote communication about mathematics;
- represent mathematics as an ongoing human activity;
- display sensitivity to, and draw on, students' diverse background experiences and dispositions; and
- promote the development of all students' dispositions to do mathematics.

National Council of Teachers of Mathematics (NCTM). *Professional Teaching Standards.* Reston, Va.: NCTM, 1991

NCTM PROCESS STANDARDS

Problem Solving
Instructional programs from prekindergarten through grade 12 should enable all students to—

- Build new mathematical knowledge through problem solving
- Solve problems that arise in mathematics and in other contexts
- Apply and adapt a variety of appropriate strategies to solve problems
- Monitor and reflect on the process of mathematical problem solving

Reasoning and Proof
Instructional programs from prekindergarten through grade 12 should enable all students to—

- Recognize reasoning and proof as fundamental aspects of mathematics
- Make and investigate mathematical conjectures
- Develop and evaluate mathematical arguments and proofs
- Select and use various types of reasoning and methods of proof

Communication
Instructional programs from prekindergarten through grade 12 should enable all students to—

- Organize and consolidate their mathematical thinking through communication
- Communicate their mathematical thinking coherently and clearly to peers, teachers, and others
- Analyze and evaluate the mathematical thinking and strategies of others
- Use the language of mathematics to express mathematical ideas precisely

Connections
Instructional programs from prekindergarten through grade 12 should enable all students to—

- Recognize and use connections among mathematical ideas
- Understand how mathematical ideas interconnect and build on one another to produce a coherent whole
- Recognize and apply mathematics in contexts outside of mathematics

Representation
Instructional programs from prekindergarten through grade 12 should enable all students to—

- Create and use representations to organize, record, and communicate mathematical ideas
- Select, apply, and translate among mathematical representations to solve problems
- Use representations to model and interpret physical, social, and mathematical phenomena

National Council of Teachers of Mathematics (NCTM). *Principles and Standards for School Mathematics.* Reston, Va.: NCTM, 2000

Mathematics | Standards for Mathematical Practice

The Standards for Mathematical Practice describe varieties of expertise that mathematics educators at all levels should seek to develop in their students. These practices rest on important "processes and proficiencies" with longstanding importance in mathematics education. The first of these are the NCTM process standards of problem solving, reasoning and proof, communication, representation, and connections. The second are the strands of mathematical proficiency specified in the National Research Council's report *Adding It Up*: adaptive reasoning, strategic competence, conceptual understanding (comprehension of mathematical concepts, operations and relations), procedural fluency (skill in carrying out procedures flexibly, accurately, efficiently and appropriately), and productive disposition (habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy).

1 Make sense of problems and persevere in solving them.
Mathematically proficient students start by explaining to themselves the meaning of a problem and looking for entry points to its solution. They analyze givens, constraints, relationships, and goals. They make conjectures about the form and meaning of the solution and plan a solution pathway rather than simply jumping into a solution attempt. They consider analogous problems, and try special cases and simpler forms of the original problem in order to gain insight into its solution. They monitor and evaluate their progress and change course if necessary. Older students might, depending on the context of the problem, transform algebraic expressions or change the viewing window on their graphing calculator to get the information they need. Mathematically proficient students can explain correspondences between equations, verbal descriptions, tables, and graphs or draw diagrams of important features and relationships, graph data, and search for regularity or trends. Younger students might rely on using concrete objects or pictures to help conceptualize and solve a problem. Mathematically proficient students check their answers to problems using a different method, and they continually ask themselves, "Does this make sense?" They can understand the approaches of others to solving complex problems and identify correspondences between different approaches.

2 Reason abstractly and quantitatively.
Mathematically proficient students make sense of quantities and their relationships in problem situations. They bring two complementary abilities to bear on problems involving quantitative relationships: the ability to *decontextualize*—to abstract a given situation and represent it symbolically and manipulate the representing symbols as if they have a life of their own, without necessarily attending to their referents—and the ability to *contextualize*, to pause as needed during the manipulation process in order to probe into the referents for the symbols involved. Quantitative reasoning entails habits of creating a coherent representation of the problem at hand; considering the units involved; attending to the meaning of quantities, not just how to compute them; and knowing and flexibly using different properties of operations and objects.

3 Construct viable arguments and critique the reasoning of others.
Mathematically proficient students understand and use stated assumptions, definitions, and previously established results in constructing arguments. They make conjectures and build a logical progression of statements to explore the truth of their conjectures. They are able to analyze situations by breaking them into cases, and can recognize and use counterexamples. They justify their conclusions,

communicate them to others, and respond to the arguments of others. They reason inductively about data, making plausible arguments that take into account the context from which the data arose. Mathematically proficient students are also able to compare the effectiveness of two plausible arguments, distinguish correct logic or reasoning from that which is flawed, and—if there is a flaw in an argument—explain what it is. Elementary students can construct arguments using concrete referents such as objects, drawings, diagrams, and actions. Such arguments can make sense and be correct, even though they are not generalized or made formal until later grades. Later, students learn to determine domains to which an argument applies. Students at all grades can listen or read the arguments of others, decide whether they make sense, and ask useful questions to clarify or improve the arguments.

4 Model with mathematics.
Mathematically proficient students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace. In early grades, this might be as simple as writing an addition equation to describe a situation. In middle grades, a student might apply proportional reasoning to plan a school event or analyze a problem in the community. By high school, a student might use geometry to solve a design problem or use a function to describe how one quantity of interest depends on another. Mathematically proficient students who can apply what they know are comfortable making assumptions and approximations to simplify a complicated situation, realizing that these may need revision later. They are able to identify important quantities in a practical situation and map their relationships using such tools as diagrams, two-way tables, graphs, flowcharts and formulas. They can analyze those relationships mathematically to draw conclusions. They routinely interpret their mathematical results in the context of the situation and reflect on whether the results make sense, possibly improving the model if it has not served its purpose.

5 Use appropriate tools strategically.
Mathematically proficient students consider the available tools when solving a mathematical problem. These tools might include pencil and paper, concrete models, a ruler, a protractor, a calculator, a spreadsheet, a computer algebra system, a statistical package, or dynamic geometry software. Proficient students are sufficiently familiar with tools appropriate for their grade or course to make sound decisions about when each of these tools might be helpful, recognizing both the insight to be gained and their limitations. For example, mathematically proficient high school students analyze graphs of functions and solutions generated using a graphing calculator. They detect possible errors by strategically using estimation and other mathematical knowledge. When making mathematical models, they know that technology can enable them to visualize the results of varying assumptions, explore consequences, and compare predictions with data. Mathematically proficient students at various grade levels are able to identify relevant external mathematical resources, such as digital content located on a website, and use them to pose or solve problems. They are able to use technological tools to explore and deepen their understanding of concepts.

6 Attend to precision.
Mathematically proficient students try to communicate precisely to others. They try to use clear definitions in discussion with others and in their own reasoning. They state the meaning of the symbols they choose, including using the equal sign consistently and appropriately. They are careful about specifying units of measure, and labeling axes to clarify the correspondence with quantities in a problem. They calculate accurately and efficiently, express numerical answers with a degree of precision appropriate for the problem context. In the elementary grades, students give carefully formulated explanations to each other. By the time they reach high school they have learned to examine claims and make explicit use of definitions.

7 Look for and make use of structure.
Mathematically proficient students look closely to discern a pattern or structure. Young students, for example, might notice that three and seven more is the same amount as seven and three more, or they may sort a collection of shapes according to how many sides the shapes have. Later, students will see 7 × 8 equals the well remembered 7 × 5 + 7 × 3, in preparation for learning about the distributive property. In the expression $x^2 + 9x + 14$, older students can see the 14 as 2 × 7 and the 9 as 2 + 7. They recognize the significance of an existing line in a geometric figure and can use the strategy of drawing an auxiliary line for solving problems. They also can step back for an overview and shift perspective. They can see complicated things, such as some algebraic expressions, as single objects or as being composed of several objects. For example, they can see $5 - 3(x - y)^2$ as 5 minus a positive number times a square and use that to realize that its value cannot be more than 5 for any real numbers x and y.

8 Look for and express regularity in repeated reasoning.
Mathematically proficient students notice if calculations are repeated, and look both for general methods and for shortcuts. Upper elementary students might notice when dividing 25 by 11 that they are repeating the same calculations over and over again, and conclude they have a repeating decimal. By paying attention to the calculation of slope as they repeatedly check whether points are on the line through (1, 2) with slope 3, middle school students might abstract the equation $(y - 2)/(x - 1) = 3$. Noticing the regularity in the way terms cancel when expanding $(x - 1)(x + 1)$, $(x - 1)(x^2 + x + 1)$, and $(x - 1)(x^3 + x^2 + x + 1)$ might lead them to the general formula for the sum of a geometric series. As they work to solve a problem, mathematically proficient students maintain oversight of the process, while attending to the details. They continually evaluate the reasonableness of their intermediate results.

Connecting the Standards for Mathematical Practice to the Standards for Mathematical Content
The Standards for Mathematical Practice describe ways in which developing student practitioners of the discipline of mathematics increasingly ought to engage with the subject matter as they grow in mathematical maturity and expertise throughout the elementary, middle and high school years. Designers of curricula, assessments, and professional development should all attend to the need to connect the mathematical practices to mathematical content in mathematics instruction.

The Standards for Mathematical Content are a balanced combination of procedure and understanding. Expectations that begin with the word "understand" are often especially good opportunities to connect the practices to the content. Students who lack understanding of a topic may rely on procedures too heavily. Without a flexible base from which to work, they may be less likely to consider analogous problems, represent problems coherently, justify conclusions, apply the mathematics to practical situations, use technology mindfully to work with the mathematics, explain the mathematics accurately to other students, step back for an overview, or deviate from a known procedure to find a shortcut. In short, a lack of understanding effectively prevents a student from engaging in the mathematical practices.

In this respect, those content standards which set an expectation of understanding are potential "points of intersection" between the Standards for Mathematical Content and the Standards for Mathematical Practice. These points of intersection are intended to be weighted toward central and generative concepts in the school mathematics curriculum that most merit the time, resources, innovative energies, and focus necessary to qualitatively improve the curriculum, instruction, assessment, professional development, and student achievement in mathematics.

Common Core State Standards Initiative (CCSSI). *Common Core State Standards for Mathematics*. Washington, D.C.: National Governors Association Center for Best Practices and the Council of Chief State School Officers, 2010. http://www.corestandards.org

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