POTM

Problem of the Month

Although the Problem of the Month department is being retired, *Mathematics Teacher* wishes to highlight submissions responding to previously published problems. As these solutions demonstrate, there is no one "best" way to approach a mathematics problem; persistence and precision determine one's success.

DECEMBER 2013 PROBLEM

The December 2013 Problem of the Month asked:

> Triangle *ABC* has sides of lengths 40, 51, and 77. Radius *r* of the circle inscribed in triangle *ABC* can be determined in several ways. Find at least one such way.

Figure 1 illustrates a solution submitted by Gayathri Ganesan and Jenson Lo, students of Frederick Deppe at Ithaca High School in Ithaca, New York. Gayathri and Jenson began their solution with a sketch of an arbitrary triangle *ABC* and constructed the inscribed circle, with center *I* and radius *r*. How could they relate *r* to the given triangle side lengths?

Heron's formula allows us to find the area of a triangle if we know all three side lengths as $A = \sqrt{s(s-a)(s-b)(s-c)}$, where *a, b,* and *c* are the side lengths, and $s = (1/2)(a + b + c)$ is the semiperimeter. Using 40, 51, and 77 as the side lengths yields an area of 924. How can we relate that to *r*? As shown in the diagram (see **fig. 1**), the area of triangle *ABC* equals the sum of the areas of triangles *AIC*, *CIB*, and *BIA*. Each of these triangles has altitude *r* and base *b, a,* and *c*, respectively, so the sum of their areas can be calculated readily: $(1/2) \cdot r \cdot (a + b + c) \rightarrow 84r$. By setting this equal to the area of 924 calculated earlier, Gayathri and Jenson were able to determine that $r = 924/84$, or 11 units. The students used geometric figures to illustrate and guide their algebraic solution.

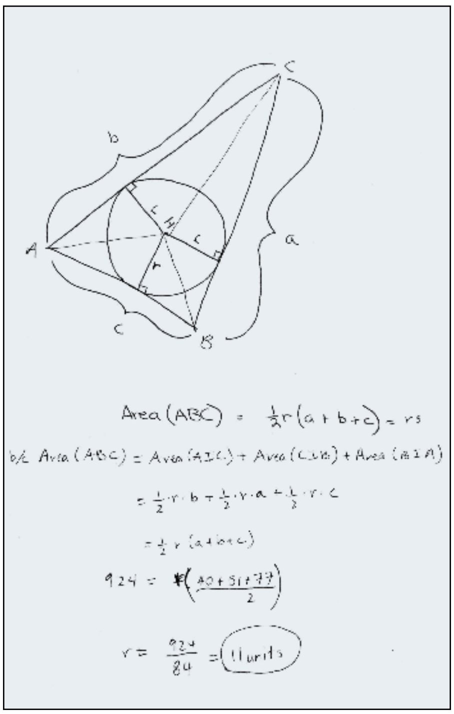


Fig. 1 Gayathri and Jenson's solution to the December 2013 problem

JANUARY 2014 PROBLEM

The January 2014 Problem of the Month was stated as follows:

> In the year 2013, two dates—4/16/13 and 6/18/13—have the special property that month^2 + day^2 + year^2 is a perfect square. For example, $4^2 + 16^2 + 13^2 = 21^2$. Which dates in 2014 have this same property?

We highlight three solutions to this problem.

Figure 2 shows the work of Brandon Allin, a student of Michelle Haberstock at Parkland College in Saskatchewan, Canada. Brandon noted that because the year is 14, in order for month^2 + day^2 + year^2 to be a perfect square, that perfect square had to be greater than or equal to 15^2. He built a table of perfect squares, beginning with 15^2, and subtracted 14^2 for the year, leaving a value for the month^2 + day^2. He then went through these values (29, 60, 93, etc.) and by inspection determined whether they could be expressed as the sum of two perfect squares. For example, he found that 29 can be expressed as $2^2 + 5^2$, so he knew that 2/5/14 and 5/2/14 had the desired property. Using this methodology, Brandon found nine dates in 2014 to satisfy the conditions of the problem.

Neal Mydonick, another student of Michelle Haberstock at Parkland College, also used an exhaustive approach (see **fig. 3**), although he approached the problem slightly differently than Brandon. Neal started by building a list of the squares of the numbers 1–37, large enough to handle the biggest possible solution (12/31/14). He next built a list of all possible sums of day^2 + year^2, although he notes in his solution that this step took a fair amount of time. (One can imagine using technology to assist in this process, such as a spreadsheet or computer algebra system.) Stepping through this day-year list, he compared it with his original list of the squares of the numbers 1–37; then he determined whether the differences were perfect squares of numbers from 1–12 in order to find the month. For example, from his day-year list, he knew that $5^2 + 14^2 = 221$. Examining his 1–37 list, he found that 225 – 221 was a perfect square (2^2), so the date 2/5/14 was a solution. Examining all possible combinations of the two lists yielded the same nine dates that Brandon found.

Ben Garrett, Jack Stefanski, and John Lucas, students of Dick Smith at the University of Dubuque in Iowa, submitted a solution that combines the exhaustive approach of the first two solutions with a clever factorization that dramatically decreases the amount of time necessary to identify all suitable month-day combinations. Ben, Jack, and John describe their work:

> We needed to find values for *d*, *m*, and *s* that solved the condition
>
> $$d^2 + m^2 + 14^2 = s^2 \quad (d < 31 \text{ and } m < 12)$$

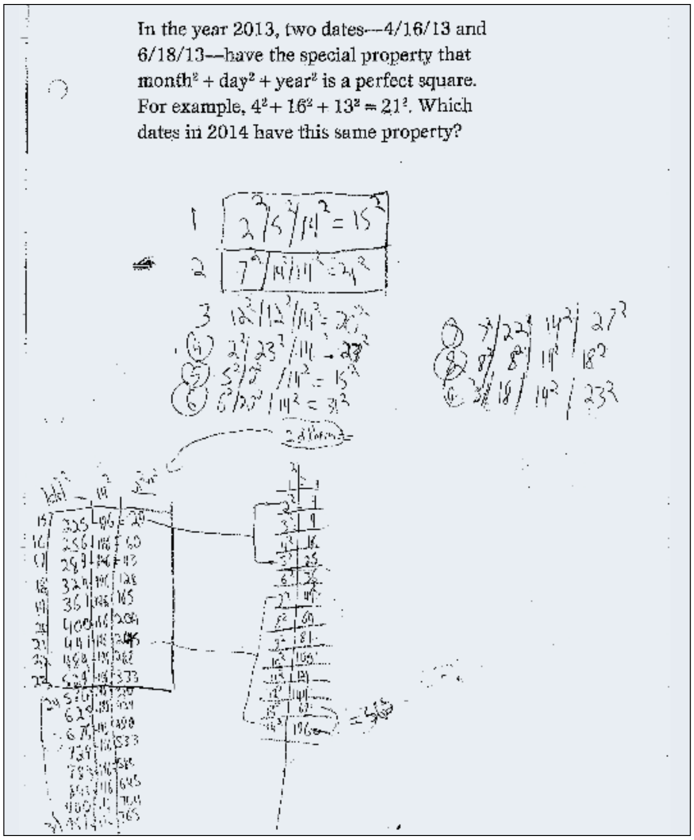


Fig. 2 Brandon's solution to the January 2014 problem

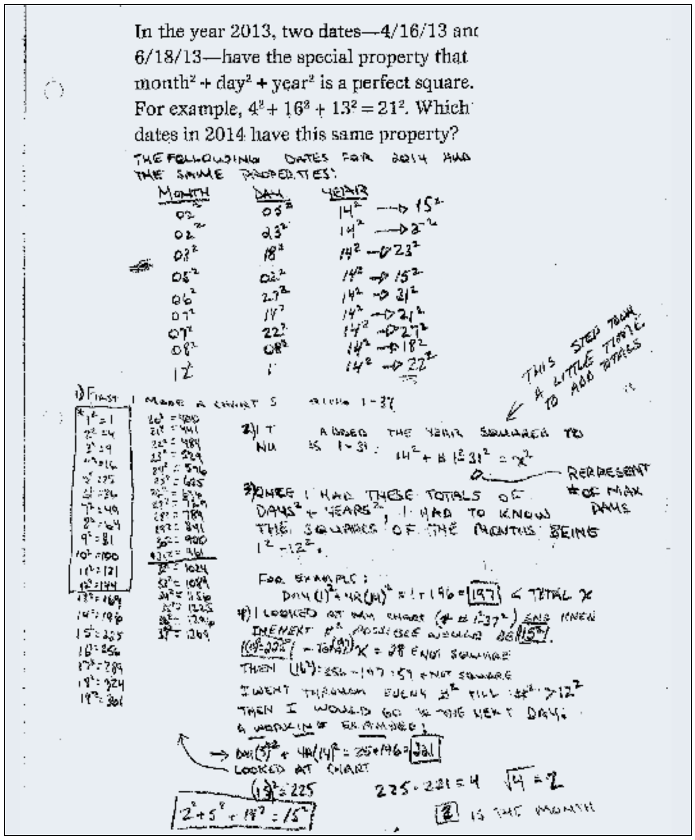


Fig. 3 Neal's solution to the January 2014 problem

where d represents the day of the month, m represents the month, and s is the sum of the squares. We decided that the two hardest numbers to find were d and s. We decided to find the values of $m^2 + 14^2$ for all twelve months:

January:	$1 + 196 = 197$
February:	$4 + 196 = 200$
March:	$9 + 196 = 205$
April:	$16 + 196 = 212$
May:	$25 + 196 = 221$
June:	$36 + 196 = 232$
July:	$49 + 196 = 245$
August:	$64 + 196 = 260$
September:	$81 + 196 = 275$
October:	$100 + 196 = 296$
November:	$121 + 196 = 317$
December:	$144 + 196 = 340$

We then rewrote our equation as

$$m^2 + 196 = s^2 - d^2 = (s+d)(s-d) = ef.$$

We could factor the sums in the list above to find d less than 31. This required the difference $(e - f)$ be divisible by 2 and less than 62. We found that this was true twice in February because $200 = (10)(20) = (4)(50)$. From the factors 10 and 20, we get $s + d = 20$ and $s - d = 10$. Therefore, $2s = 30$, $s = 15$, and $d = 5$. So $2^2 + 5^2 + 14^2 = 15^2$. Similarly, from the factors 4 and 50, we get $s + d = 50$ and $s - d = 4$. Therefore, $2s = 54$, $s = 27$, and $d = 23$. So $2^2 + 23^2 + 14^2 = 27^2$.

Using similar reasoning, the three students found the other seven solutions. As you can see, they began with a approach similar to Brandon's and Neal's but limited the scope of the potential month-day combinations they would have to check as a result of their factorization of $s^2 - d^2$. After finding all the solutions, they also offered the following observation: "All of these numbers have a geometric interpretation in addition to the date interpretation. The variable s is the oblique diagonal of a right rectangular prism of dimension d by m by 14."

FEBRUARY 2014 PROBLEM

The February 2014 Problem of the Month was stated as follows:

> Which event is more likely to occur—exactly 1 head in 2 flips of a coin or exactly 5 heads in 10 flips of a coin?

As is true of many probability problems, February's challenge is very easy to state but can be confusing to solve. In particular, beginning probability students are likely to have trouble distinguishing between "exactly 1 head" and "at least 1 head." As is also true of many probability problems, a variety of techniques can be used to find the solution.

Juliana Valencia, a student of Justine Aten and Lindsey Koch at Lincoln Park High School in Chicago, used several approaches (see **fig. 4**). She started out by using a tree diagram to enumerate all possible outcomes in the sample space. Her tree on the top left of the diagram shows that there are 4 outcomes for 2 flips: HH, HT, TH, and TT. She could have used this diagram directly to find the probability of exactly 1 head in 2 flips as 2/4, or 50%, because 2 possibilities (HT and TH) out of the total 4 possibilities meet the criteria. However, Juliana instead computed the probability of exactly 2 tails to be 25% and knew that the probability of the complement, at least 1 head, would be 100% – 25%, or 75%. Although correct, this calculation has no bearing on the problem at

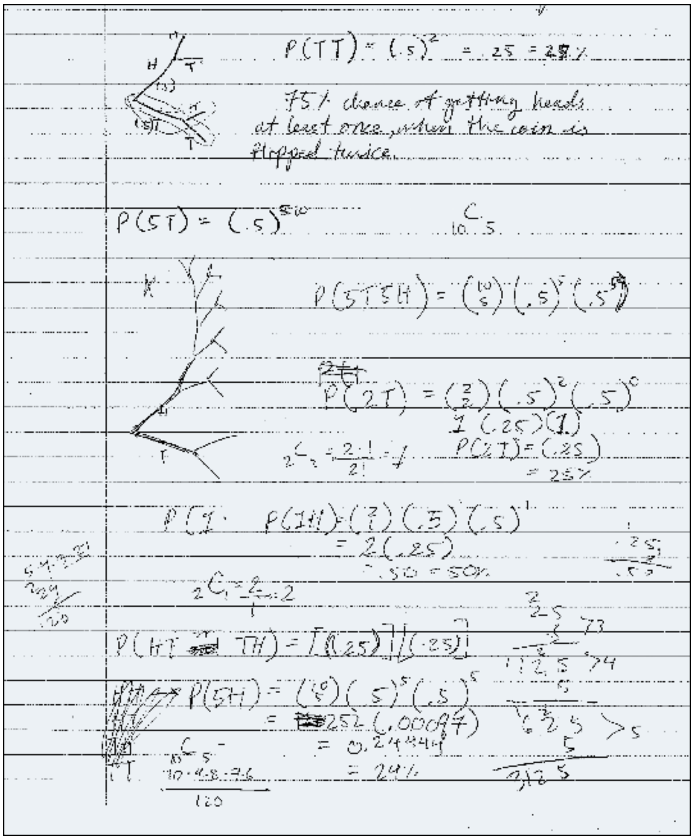


Fig. 4 Juliana's solution to the February 2014 problem

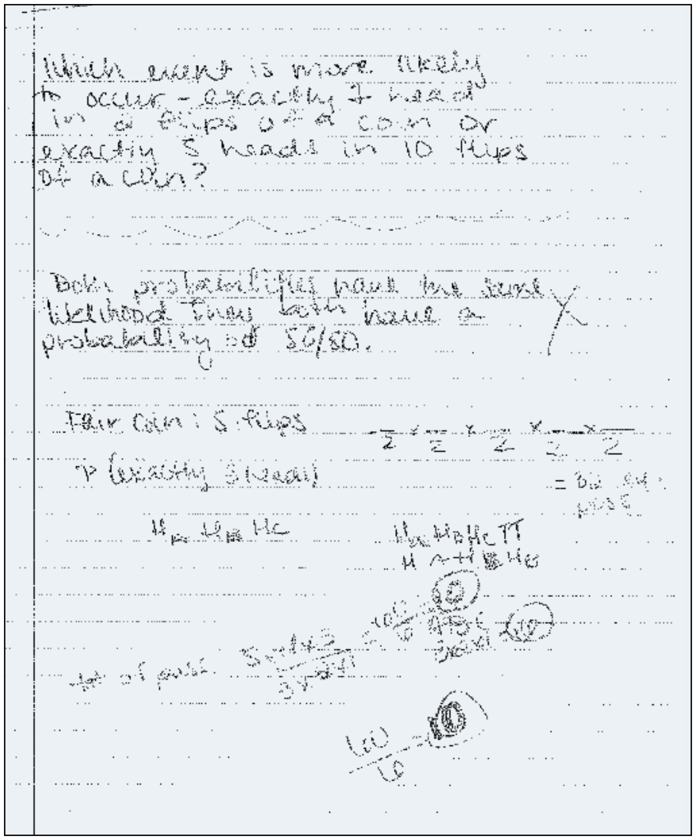


Fig. 5 Andrea's solution to the February 2014 problem

hand. Juliana next started drawing a tree diagram for the 10-flip situation but abandoned the effort as being too cumbersome for such a large sample space. Instead, she used a formula for the binomial distribution to determine that

$$P(5T5H) = \binom{10}{5}(.5)^5(.5)^5.$$

She did not immediately evaluate this result, instead using the binomial distribution to find the probability of 2 tails in 2 flips:

$$P(2T) = \binom{2}{2}(.5)^2(.5)^0.$$

She evaluated this result and confirmed her earlier result of 25%. However, she then realized that this was not what the problem called for. She then used the binomial distribution to compute

$$P(1H) = \binom{2}{1}(.5)^1(.5)^1,$$

which she then evaluated as 50%. Next, she returned to evaluating the expression for $P(5T5H)$ and found a value of between 24% and 25%. Since 50% is greater, it is more likely to achieve exactly 1 head in 2 flips than exactly 5 heads in 10 flips.

Andrea Beltran, another student of Justine Aten and Lindsey Koch at Lincoln Park High School in Chicago, used a similar strategy (see **fig. 5**). Instead of drawing a tree diagram, she enumerated all the possible outcomes of 2 coin flips to find a probability of 2 in 4, or 50%, for exactly 1 head in 2 flips. She verified this with the binomial distribution formula and then directly used the formula for the 10-flip case, finding a probability of approximately 25%.

Jack Stefanski, a student of Dick Smith at the University of Dubuque, also relied on the binomial distribution, but instead of using the formula that Juliana and Andrea used, he interpreted the values in Pascal's triangle to find the solution. Each row n of Pascal's triangle, which starts at $n = 0$, consists of $(n + 1)$ integers. The kth integer in row n represents the number of ways we can get exactly k heads on n flips of a coin. Therefore, the sum of all the integers in a row is the total number of outcomes of the n flips. Jack used this strategy to find the relevant probabilities, as he describes in his solution:

I created Pascal's triangle up to row 10:

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 25 56 70 56 28 8 1
1 9 36 84 126 126 84 36 9 1
1 10 45 120 21 252 210 120 45 10 1

Using Pascal's triangle for 2 flips, I found that the sum of all of the "second" row numbers equals 4. I then found that there are 2 ways to get exactly 1 head. This would be head first and tail second or tail first and head second. This yields a (2/4), or 50%, chance.

Using Pascal's triangle for 10 flips, I found the sum of all the "tenth" row numbers equals 1024. I then found from Pascal's triangle that there are 252 ways to get

exactly 5 heads. This yields a (252/1024), or 24.6%, chance. So getting exactly 1 head in 2 flips of a coin is almost *twice* as likely a possibility as exactly 5 heads in 10 flips of a coin.

Self-described math enthusiast Sal Culosi used a different form of the binomial distribution formula relying on the fact that the likelihood of heads and the likelihood of tails on any single flip are both equal to 1/2. He computed the two desired probabilities as follows:

Probability of exactly one head in two flips of a coin is

$$\binom{2}{1}\left(\frac{1}{2}\right)^2 = 2\left(\frac{1}{4}\right) = \left(\frac{1}{2}\right).$$

Or, by enumeration: (HH), (TT), (HT), (TH), 2 out of 4 = 1/2.

Probability of getting exactly 5 heads in 10 flips of a coin is

$$\binom{10}{5}\left(\frac{1}{2}\right)^{10} = \frac{10\bullet 9\bullet 8\bullet 7\bullet 6}{5\bullet 4\bullet 3\bullet 2\bullet 1}\left(\frac{1}{2^{10}}\right) = \frac{63}{256} < \frac{1}{2}.$$

Therefore, there is a higher probability of getting exactly 1 head in 2 flips of a coin than getting exactly 5 heads out of 10 flips of a coin.

APRIL 2014 PROBLEM

The April 2014 Problem of the Month was stated as follows:

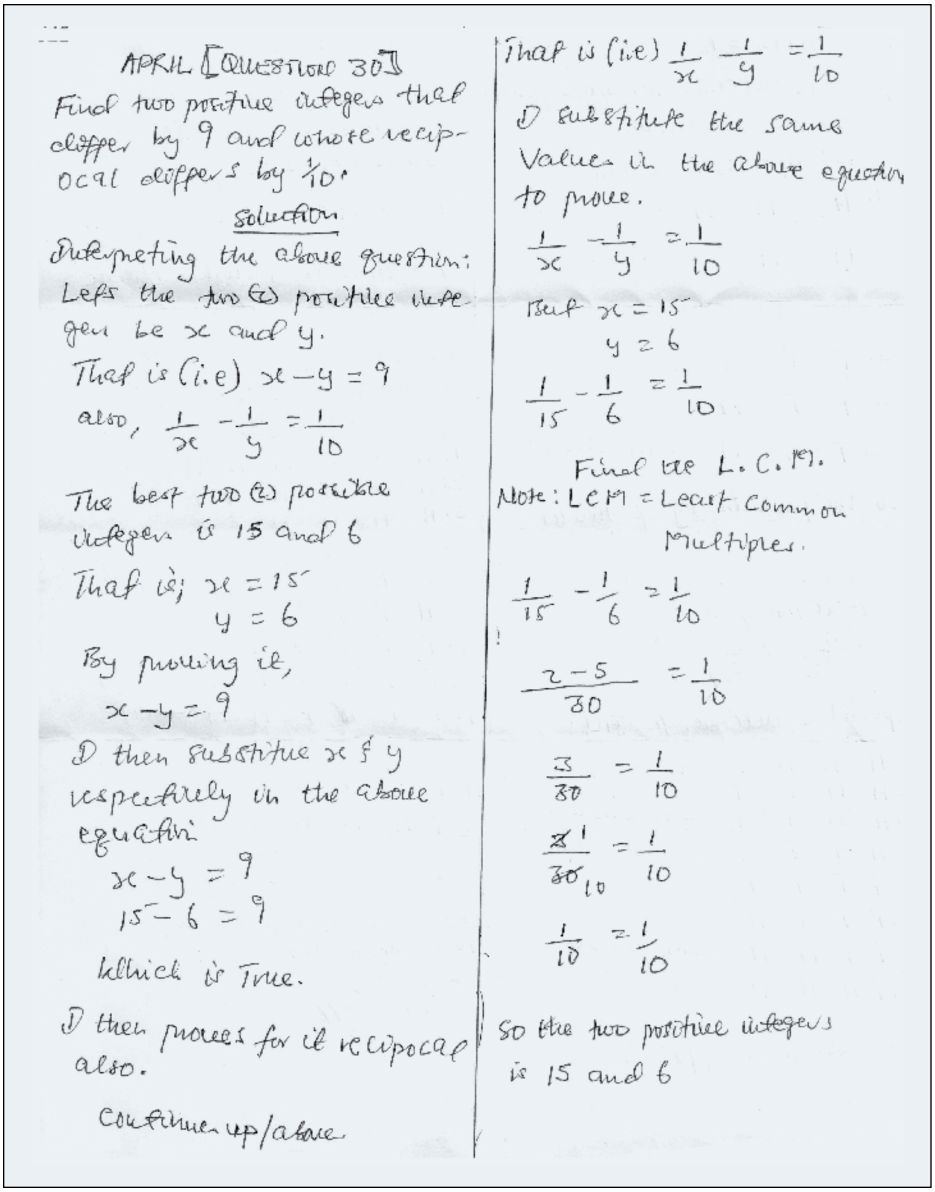
Find two positive integers that differ by 9 and whose reciprocals differ by 1/10.

Augustine Mulai spent the 2011–12 school year at Greeley West High School in Greeley, Colorado, as an exchange student from Sierra Leone. He was a student of Seán Madden, one of the Problem of the Month editors. Augustine has since returned to Sierra Leone and is pursuing opportunities for higher education. He has limited access to technology and communications and had to walk to an Internet cafe several miles from his home to submit his solution. Augustine developed two equations, $x - y = 9$ and $1/x - 1/y = 1/10$, that he believed must be satisfied by the two integers and then used inspection to find the pair of integers that satisfied the equations (see **fig. 6**).

Note that although Augustine found the correct solution, his methodology is problematic. First, his two equations are inconsistent in the use of variables x and y. If $x - y = 9$, then x is the larger of the two integers, so the second equation should be $1/y - 1/x = 1/10$. Augustine recognizes this in his work, switching the order of the 15 and the 6 when substituting into his second equation. The second issue is that although Augustine found a solution, he has not demonstrated that this is the only solution. Substituting $y + 9$ for x in the modified second equation yields

$$\frac{1}{y} - \frac{1}{y+9} = \frac{1}{10} \rightarrow y^2 + 9y - 90 = 0.$$

The solution to this quadratic equation is $y = -15$ or $y = 6$. Since the problem required all solutions to be positive integers, we are left with exactly one solution, as shown in Augustine's work.


APRIL [QUESTION 30]
Find two positive integers that differ by 9 and whose reciprocal differs by 1/10.
Solution
Interpreting the above question:
Lets the two (2) positive integers be x and y.
That is (i.e) $x - y = 9$
also, $\frac{1}{x} - \frac{1}{y} = \frac{1}{10}$
The best two (2) possible integers is 15 and 6
That is; $x = 15$
$y = 6$
By proving it,
$x - y = 9$
I then substitute x & y respectively in the above equation
$x - y = 9$
$15 - 6 = 9$
Which is True.
I then proves for it reciprocal also.
Continue up/above

That is (i.e) $\frac{1}{x} - \frac{1}{y} = \frac{1}{10}$
I substitute the same values in the above equation to prove.
$\frac{1}{x} - \frac{1}{y} = \frac{1}{10}$
But $x = 15$
$y = 6$
$\frac{1}{15} - \frac{1}{6} = \frac{1}{10}$
Find the L.C.M.
Note: LCM = Least Common Multiples.
$\frac{1}{15} - \frac{1}{6} = \frac{1}{10}$
$\frac{2-5}{30} = \frac{1}{10}$
$\frac{3}{30} = \frac{1}{10}$
$\frac{\cancel{3}\,1}{\cancel{30}\,10} = \frac{1}{10}$
$\frac{1}{10} = \frac{1}{10}$
So the two positive integers is 15 and 6

Fig. 6 Augustine's solution to the April 2014 problem