SOLUTIONS to calendar

DECEMBER

Department editors
Margaret Coffey, *Margaret.Coffey@fcps.edu, Thomas Jefferson High School for Science and Technology, Alexandria, VA 22312; and* ***Art Kalish,*** *artkalish@verizon.net, Director of the Institute of MERIT at SUNY College at Old Westbury*

1. \$1.13. Factor 3277 into 29 • 113. Four options to consider: (1) 1 child paid the entire amount; (2) 29 children attended, and each paid \$1.13; (3) 113 children each paid 29¢; or (4) 3277 children attended, and each paid 1 penny. The first and last cases are not possible given the 6-coin restriction. No configurations of six coins attain 29¢. The only six-coin solution for \$1.13 consists of two half-dollars, one dime, and three pennies.

2. \$6.80. Walnuts cost the grocer \$150 for 30 lbs., or \$5/lb. Pecans cost \$140 for 40 lb., or \$3.50/lb. The ratio of pecans to walnuts is 2 : 1, so he needs to mix 2 lbs. of pecans for every 1 lb. of walnuts. Therefore, 3 lbs. of the mixed nuts will cost him \$5 + 2(\$3.50) = \$12, which is \$4/lb. Since the grocer wants to make a profit of 70%, he needs to charge 4(\$1.70) = \$6.80.

3. (16, 16), (16, –14), (–18, 16), (–18, –14), (9, 7), (9, –5), (–11, 7), (–11, –5), (7, 1), and (–9, 1). Recast the problem to suggest Pythagorean triples by moving the $(b-1)^2$ term to the right side of the equation. The triples must include a side length of 8, so use the 8-15-17 and the 6-8-10 triangles. Solve for every possible pair of values: $(a+1, b-1) = (\pm 17, \pm 15)$; $(a+1, b-1) = (\pm 10, \pm 6)$. In addition, $(a+1, b-1) = (\pm 8, 0)$.

4. 0. Simplify each expression separately:

$$81^{7/4} = \left(\sqrt[4]{81}\right)^7 = 3^7$$

and

$$\begin{aligned}\sqrt{3^{13} + 3 \cdot 9^6 + 3 \cdot 27^4} &= \sqrt{3^{13} + 3 \cdot \left(3^2\right)^6 + 3\left(3^3\right)^4} \\ &= \sqrt{3^{13} + 3^{13} + 3^{13}} \\ &= \sqrt{3^{14}} = 3^7.\end{aligned}$$

The difference between the two values is zero.

5. $7, 5\sqrt{2}, 3\pi - 2, 1 + \sqrt{42}, 3\sqrt{7}$. Without a calculator, we must use methods of approximation. Knowing that $\sqrt{2} \approx 1.41$, we get $5\sqrt{2} \approx 7.05$. We could also use the idea that $5\sqrt{2} = \sqrt{50}$, which would be slightly more than 7. Rewrite $3\sqrt{7} = \sqrt{63}$ to estimate a value close to (but less than) 8. Use $\pi \approx 3.14$ to find that $3\pi - 2 \approx 7.42$. Use the fact that 42 is the product of two consecutive integers 6 and 7 to approximate $\sqrt{42} \approx 6.5$. Thus, $1 + \sqrt{42} \approx 7.5$. Now order the five numbers: $7, 5\sqrt{2}, 3\pi - 2, 1 + \sqrt{42}, 3\sqrt{7}$.

6. 5. Since

$$\frac{2x^2 - (p+1)x + 3p}{2x - 3} = x - q + \frac{9}{2x - 3},$$

if we multiply both sides of the equation by $2x - 3$, we get $2x^2 - (p+1)x + 3p = 2x^2 - (2q+3)x + 3q + 9$. Therefore, $p + 1 = 2q + 3$, and $3p = 3q + 9$. The second equation reduces to $p = q + 3$; substituted into the first equation, it yields $q = 1$ and $p = 4$.

***Alternate solution*:** Use either long or synthetic division. The quotient is $x + (2 - p)/2$, and the remainder is $3p + 3(2 - p)/2$. The remainder equals 9, so $3p + 3 - 3p/2 = 9 \to p = 4$. Since the quotient is $x - q$, $q = -(2 - p)/2 = -(2 - 4)/2 = 1$. Finally, $p + q = 4 + 1 = 5$.

7. $1/\sqrt{2}$, or $\sqrt{2}/2$. The parallel segments ensure that we have similar triangles. Thus, the ratio of the areas, A_1 and A_2, relates to the ratio of corresponding altitudes, h_1 and h_2, or to any corresponding side lengths or linear measures. So $h_1^2/h_2^2 = A_1/A_2 = 1/2 \to h_1/h_2 = 1/\sqrt{2} = \sqrt{2}/2$.

8. 33 and 99. Verify that $33^2 = 1089$ and $99^2 = 9801$. The smallest two-digit number that has a four-digit square is 32 (1024). The greatest two-digit number with a four-digit square is 99 (9801). Use a spreadsheet to generate the squares of the integers from 32 to 99, and then look at the list to find the pair.

Alternate solution: Use the greatest integer function to separate the squares into the individual digits, rebuild the number with the digits reversed, and then find the square root of the result. Using Excel, list the numbers from 32 to 99 in column A, starting in row 1. The following formulas, from row 1, can be copied to the other rows:

column B: =A1^2
column C: =INT(B1/1000)
column D: =INT((B1-1000*C1)/100)
column E: =INT((B1-(1000*C1+100*D1))/10)
column F: =B1-(1000*C1+100*D1+10*E1)
column G: =1000*F1+100*E1+10*D1+C1
column H: =SQRT(G1)

9. 14 5/7, or 103/7. The reciprocals of any arithmetic sequence form a harmonic sequence. The harmonic mean is the middle of three numbers in a harmonic sequence. For an arithmetic sequence beginning and ending with p and q, the harmonic mean HM can be found as $1/\text{HM} = (1/2)(1/p + 1/q)$ or by using the formula $\text{HM} = 2pq/(p + q)$. Thus, $b = 2(25)(25/49)/(25 + 25/49) = 1$. Since $b = 1$ and a is the arithmetic mean of 25 and 1, $a = (25 + 1)/2 = 13$. The geometric mean, c, for 1 and 25/49 equals $\sqrt{1 \cdot 25/49} = 5/7$. Then $a + b + c = 13 + 1 + 5/7 = 14\ 5/7$.

10. 27. Complete the square to convert the given equation into $y = (x - 1)^2 - 9$. The vertex is at $(1, -9)$, and the x-intercepts at -2 and 4 are found by replacing y with 0 so $(x - 1)^2 = 9$. The base of the triangle is $4 - (-2) = 6$, and the height is 9. Therefore, the area of the triangle is $6 \cdot 9/2 = 27$.

11. 0. Apply substitution to find that

$$\begin{aligned} g(f(x)) &= a(x+3)^2 + b(x+3) + c \\ &= ax^2 + (6a+b)x + (9a+3b+c). \end{aligned}$$

Equating coefficients, we obtain $a = 2$, $6a + b = 7$, and $9a + 3b + c = 6$. Solve for each letter to find that $a = 2$, $b = -5$ and $c = 3$, so $a + b + c = 2 - 5 + 3 = 0$.

Alternate solution 1: Since $f(x - 3) = x$, we see that $g(x) = g(f(x - 3)) = 2(x - 3)^2 + 7(x - 3) + 6 = 2x^2 - 12x + 18 + 7x - 21 + 6 = 2x^2 - 5x + 3$. Thus, $a = 2$, $b = -5$, and $c = 3$.

Alternate solution 2: Create a system of three equations with $g(f(0)) = 9a + 3b + c = 6$; $g(f(-3)) = 0a + 0b + c = 3$; and $g(f(-6)) = 9a - 3b + c = 36$. Solve to get $a = 2$, $b = -5$, and $c = 3$.

12. $2 + \sqrt{3}$. Without any loss of generality, let the legs of $\triangle ABC = 2$. Let $BC = x$ and apply the law of cosines to calculate $x^2 = 2^2 + 2^2 - 2(2)(2)\cos 30° = 8 - 4\sqrt{3}$. The area of $\triangle DCB = x^2/4 = 2 - \sqrt{3}$. The area of $\triangle ABC = (1/2)(2)(2)\sin 30° = 1$. Thus, the ratio of the requested areas is $1/(2 - \sqrt{3}) = 2 + \sqrt{3}$.

13. 12 times. For each month in a given century, only one day and one year will compose a date consisting of three consecutive numbers.

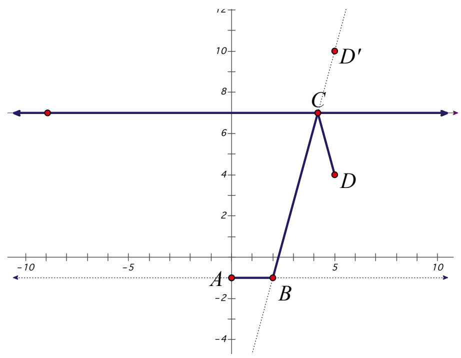
14. 13 : 35. The trapezoids are not similar, but the three triangles are similar. That is, $\triangle ADE \sim \triangle AFG \sim \triangle ABC$. Since corresponding sides are in a ratio of 4 : 9 : 16, corresponding areas are $4^2 : 9^2 : 16^2 = 16 : 81 : 256$. The ratio of the area of $DEGF$ to the area of $FGCB = (81 - 16) : (256 - 81) = 65 : 175 = 13 : 35$.

15. (b) $(x, y) \to (x^2, y)$. The chart indicates the coordinates of the points after each transformation. Determine the slopes between points to show collinearity. The slopes in cases a, c, and d are all 1.

16. 15/7 hr. The slower flow rate is 1/5 vat/hr., and the faster flow rate is 1/3 vat/hr. In t hours, the amount of liquid in the first, slower-emptying vat is $1 - t/5$, and the amount in the second vat is $1 - t/3$. The first volume is twice the second, so solve the equation $1 - t/5 = 2(1 - t/3)$ to get $t = 15/7$.

17. 117°. In $\triangle ABC$, since $m\angle ABC = 90°$ and $m\angle C = 36°$, $m\angle A$ must be 54°. Triangle ADB is isosceles with $AD = AB$, so $\angle ADB \cong \angle ABD$. Then $m\angle ADB = (180° - 54°)/2 = 126°/2 = 63°$. Thus, $m\angle CDB = 180° - 63° = 117°$.

18. $2 + \sqrt{130} \approx 13.40175$. Since the shortest distance from A to B is along a horizontal line, y must be -1, and $AB = 2$. Interestingly, we never need to calculate x specifically. Reflect point D over the line $y = 7$ to locate the point D' at $(5, 10)$. Place point C at the intersection of segment BD' and the line $y = 7$. The distance from C to D' is the same as the distance from C to D, so the length of the shortest route from B to C to D equals BD'. Use the distance formula or the Pythagorean theorem to find $\sqrt{130}$. Therefore, the minimal total distance is $2 + \sqrt{130}$.



19. 17 cm. Let the edges have lengths a, b, and c. Then $abc = 864$, $ab = 72$, and $bc = 96$. Divide the first equation by each of the other two equations to get $(abc)/(ab) = 864/72 \to c = 12$ and $(abc)/(bc) = 864/96 \to a = 9$. It follows that $b = 8$. The space diagonal can be found using the extended Pythagorean theorem or

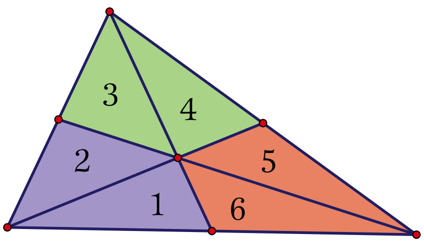
$$\begin{aligned} d &= \sqrt{a^2 + b^2 + c^2} \\ &= \sqrt{9^2 + 8^2 + 12^2} \\ &= \sqrt{289} = 17. \end{aligned}$$

Solution 15

Original Point	(a) $\left(x, \sqrt{y-1}\right)$	(b) $\left(x^2, y\right)$	(c) $\left((x+1)^2, y-1\right)$	(d) $\left(x^2+2x, y-2\right)$
(–1, 1)	(–1, 0)	(1, 1)	(0, 0)	(–1, –1)
(0, 2)	(0, 1)	(0, 2)	(1, 1)	(0, 0)
(1, 5)	(1, 2)	(1, 5)	(4, 4)	(3, 3)

20. $n = 0$, 1, 2, or 3. Division yields $(2n + 3)/(2n - 3) = 1 + 6/(2n - 3)$. The remainder will be an integer whenever $2n - 3$ is a factor of 6—that is, when $2n - 3 = \pm1, \pm2, \pm3$, or ±6. When $2n - 3$ is even, n is not an integer. The four odd-number options give $2n - 3 = 1 \rightarrow n = 2$; $2n - 3 = -1 \rightarrow n = 1$; $2n - 3 = 3 \rightarrow n = 3$, and $2n - 3 = -3 \rightarrow n = 0$.

21. The point is the intersection of the three medians of the triangle (i.e., the centroid). The centroid is the point at which the three medians meet. Each median divides the triangle into two triangles of equal area; drawing all three medians results in six triangles of equal area. Grouping them as shown demonstrates the three quadrilaterals of equal area.



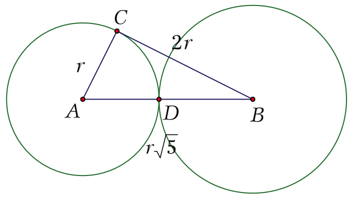
22. 1/8. Regroup to form $(\cos 20°) \cdot [(\cos 40°)(\cos 80°)]$ and write as $(\cos 20°)[(\cos 120° + \cos 40°)/2]$. Since $\cos 120° = -1/2$, we simplify the expression to $(-\cos 20°)/4 + (\cos 20°) \cdot (\cos 40°)/2$. Apply the identity to the second term to get $(\cos 20°)(\cos 40°) = (\cos 60° + \cos 20°)/2 = (1/2 + \cos 20°)/2$, so that $(\cos 20°)(\cos 40°)(\cos 80°) = (-\cos 20°)/4 + (1/4 + \cos 20°/2)/2 = 1/8$. Note that any two of the factors could have been grouped together first to produce the same result.

23. 3. Apply the rule $\log a - \log b = \log(a/b)$ to write $\log(\log 64) - \log(\log 4) = \log((\log 64)/(\log 4)) = \log x$. Exponentiate each side so $(\log 64)/(\log 4) = x$. From the rule $\log(b^a) = a \cdot \log b$, we obtain $\log 64 = \log 4^3 = 3 \log 4$, so that we have $(3 \log 4)/(\log 4) = 3 = x$.

24. 35/2. The two greatest one-digit primes are 5 and 7. The positive difference of their reciprocals is $1/5 - 1/7 = 2/35$. The reciprocal of 2/35 is 35/2.

25. $(3 + \sqrt{5})/8$. Let the radius of circle A be r. Then $BC = 2r$, and, by the Pythagorean theorem, $AB = r\sqrt{5}$. The radius of circle B is $DB = r\sqrt{5} - r$. The ratio of the areas of circles A and B is the ratio of the squares of the respective radii, or

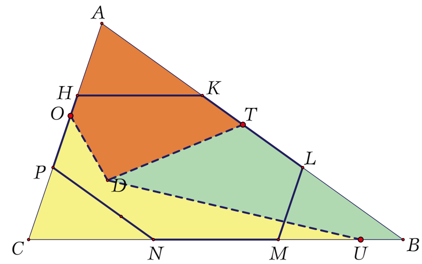
$$\begin{aligned} r^2/\left(r\sqrt{5}-r\right)^2 &= 1/\left(\sqrt{5}-1\right)^2 \\ &= 1/\left(6-2\sqrt{5}\right) \\ &= \left(3+\sqrt{5}\right)/8. \end{aligned}$$



26. The only possibility has one die with all odd integers and the other with three odd (and three even) numbers. To show that there is no other possibility, consider $1/2 = 18/36 = 6/6 \cdot 3/6$, representing the probability of rolling one of the six odd numbers on one die and one of the three odd numbers on the other die. The only factorizations of 18 are $1 \cdot 18$, $2 \cdot 9$, and $3 \cdot 6$. However, 18/6 and 9/6 are not probabilities, so the only possibility uses 6 and 3.

27. $n = 222222$. Notice that the multipliers 11, 101, and 1001 are of interest because $2 \cdot 11 = 22$, $22 \cdot 101 = 2222$, and $222 \cdot 1001 = 222222$ maintain the condition of no digits other than 2. Consider the prime factorizations of $777 = 3 \cdot 7 \cdot 37$ and 222 is $2 \cdot 3 \cdot 37$. The multiplier $1001 = 7 \cdot 11 \cdot 13$ supplies the prime factor 7 and maintains all 2s, so $n = 222 \cdot 1001 = 222222$.

28. 2/3. Point D as shown in the diagram is an example of a point with the property that quadrilaterals of equal area can be drawn. Points 1/3 of the way along each side of the original triangle form a hexagon ($HKLMNP$), and any point inside the hexagon ($HKLMNP$) has this property. Points outside the hexagon but inside $\triangle ABC$—that is, points in the three triangles AHK, BLM, and CNP—do not share the property. The probability of interest equals the ratio of the areas of the hexagon and entire triangle. The area of each corner triangle is 1/9 the area of $\triangle ABC$, so the probability of a point landing inside the hexagon is $6/9 = 2/3$. (Points on one of the hexagon lines that are parallel to a side of $\triangle ABC$ allow for construction of one quadrilateral and two triangles all having the same area. Points inside the corner triangles cannot have lines drawn to the sides of the original triangle to form quadrilaterals of equal area.)



29. 7. Find pairs of primes whose differences are positive integers: $1 = 3 - 2$; $2 = 5 - 3$; $3 = 5 - 2$; $4 = 7 - 3$; $5 = 7 - 2$; and $6 = 11 - 5$. Because most of the primes are odd, finding differences that are even is relatively easy. Odd-number differences are found using the prime 2 with one of the odd integers. So 1, 3, and 5 could be found as differences, but 7 cannot occur because 9 is not a prime. Therefore, 7 is not in S.

30. 456, 3456, and 123456. Since 24 factors into 8 and 3, we can apply the divisibility tests for each of the numbers. A number is divisible by 8 when the number formed from the last three digits is divisible by 8. Consider one-digit numbers. The only one is 8, and it is not divisible by 24. Two-digit numbers with consecutive digits are 12, 23, 34, 45, 56, 67, 78, and 89. The only number divisible by 8 is 56, but it is not divisible by 24. Use the same technique to find possible three-digit numbers. The only one divisible by 8 is 456, which also happens to be divisible by 3, so it is divisible by 24. Add integers to the left of the number to find other possibilities: 3456 works; 23456 is not divisible by 3; but 123456 is divisible by 3 and 24.

31. b. Since a is the measure of the smallest interior angle, $180° - a$ is the measure of the corresponding exterior angle. Subtracting the largest interior angle gives $(180° - a) - c = 180° - (a + c) = b$.

DECEMBER

NCTM NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

Special date: 12/13/14

A movie theater ran an early-morning special for children. The cost of attending the show was the same for every child. All the children paid the fee using exactly six coins consisting of pennies, nickels, dimes, quarters, or half dollars. The total money brought in by the theater was \$32.77. How much did it cost each child to attend the show?

1

A grocer wishes to sell 1 lb. mixtures of walnuts and pecans. He buys 30 lb. packages of walnuts for \$150 and 40 lb. packages of pecans for \$140. The weight of the pecans will be twice the weight of the walnuts in the mixture. What should his selling price be if he is to make a 70% profit?

2

Find the ten integer ordered pairs (a, b) such that

$$(a+1)^2 - (b-1)^2 = 64.$$

3

Simplify:

$$81^{7/4} - \sqrt{3^{13} + 3\cdot 9^6 + 3\cdot 27^4}$$

4

Arrange the following five numbers from smallest to largest without using a calculator:

$$5\sqrt{2},\ 3\sqrt{7},\ 3\pi - 2,\ 7,\ 1+\sqrt{42}$$

5

When $2x^2 - (p+1)x + 3p$ is divided by $2x - 3$, the quotient will be $x - q$, and the remainder will be 9. Find the sum of

$$p + q.$$

6

In $\triangle ABC$, point D is on side $\overline{AB}$ and point E is on side $\overline{BC}$ such that $\overline{DE} \parallel \overline{AC}$. If the ratio of the areas of triangles BDE and BAC is $1:2$, what is the ratio of their corresponding altitudes?

7

Find the only pair of two-digit numbers whose squares are four-digit numbers with their digits reversed.

8

In the following sequence—

$$25,\ a,\ b,\ c,\ 25/49$$

—the first, third, and fifth terms form a harmonic sequence; the first three terms form an arithmetic sequence; and the last three terms form a geometric sequence. If a, b, and c are all positive numbers, find

$$a + b + c.$$

9

Compute the area of the triangle formed by the vertex and x-intercepts of the parabola defined by the equation

$$y = x^2 - 2x - 8.$$

10

Given

$$f(x) = x + 3$$
$$g(x) = ax^2 + bx + c$$
$$g(f(x)) = 2x^2 + 7x + 6,$$

find $a + b + c$.

11

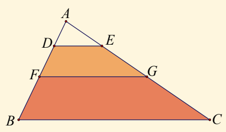
The base of isosceles $\triangle ABC$ is the hypotenuse of isosceles right $\triangle DBC$. If $m\angle A = 30°$, find the ratio of the area of $\triangle ABC$ to the area of $\triangle DBC$.

12

Today's date, 12/13/14, consists of three consecutive numbers in the order of month, day, and year (the last two digits only). How many times will three consecutive numbers occur in the dates of the century 2000–2099?

13

In $\triangle ABC$, points D and F are on side $\overline{AB}$, and points E and G are on side $\overline{AC}$. If $\overline{DE}$ and $\overline{FG}$ are parallel to $\overline{BC}$ and if $AD = 4$, $DF = 5$, and $FB = 7$, find the ratio of the area of trapezoid $DEGF$ to the area of trapezoid $FGCB$.



14

Which one of the following transformations on the set $\{(-1, 1), (0, 2), (1, 5)\}$ does not result in a set of collinear points?

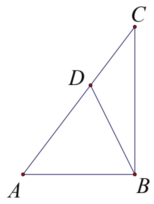
(a) $(x, y) \to (x, \sqrt{y-1})$
(b) $(x, y) \to (x^2, y)$
(c) $(x, y) \to ((x+1)^2, y-1)$
(d) $(x, y) \to (x^2 + 2x, y - 2)$

15

Two identical vats contain the same amount of liquid. The first vat takes 5 hours to empty, whereas the second vat takes only 3 hours to empty. If both vats start emptying at the same time, in how many hours will one vat have twice as much liquid as the other vat?

16

Right triangle ABC with right angle B has $m\angle C = 36°$ and point D on segment AC such that $AD = AB$. Find $m\angle CDB$.



17

Find the minimum distance that connects the points A $(0, y)$, B $(2, -1)$, C $(x, 7)$, and D $(5, 4)$ in the order A-B-C-D.

18

The volume of a rectangular prism is 864 cm^3. Two of its faces have areas of 72 cm^2 and 96 cm^2. Find the length of a diagonal that connects a pair of opposite vertices of the prism.

19

Find all integer values for n such that

$$(2n+3)/(2n-3)$$

is also an integer.

20

There exist many points inside a triangle such that segments can be drawn from the point to the sides of the triangle (not at the vertices) so that the three resulting quadrilaterals have equal areas. There is only one point inside a triangle such that the segments drawn to the vertices create three quadrilaterals of equal area. How is the point determined?

21

One of the less-frequently taught trigonometric identities is

$$(\cos x)(\cos y) = (1/2)(\cos(x+y) + \cos(x-y)).$$

Apply this identity to evaluate

$$(\cos 20°)(\cos 40°)(\cos 80°).$$

22

Solve for x:

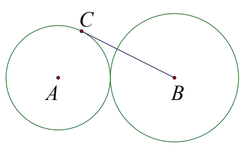
$$\log(\log 64) - \log(\log 4) = \log x$$

23

Find the reciprocal of the positive difference of the reciprocals of the two greatest one-digit primes.

24

Circles A and B are externally tangent to each other. From center B, a tangent is drawn to circle A, intersecting the circle at C. If BC is twice the radius of circle A, compute the ratio of the area of circle A to the area of circle B.



25

A game is played with two standard dice. Player A wins when the product of the two rolled dice is even; player B wins when the product is odd. Player A should win 27 out of 36 times, or 3/4 of the time. Create a pair of fair six-sided dice with numbers different from the usual ones to make the game fair (each player should win 1/2 the time).

26

The integer n consists of some number of 2s and no other digits. Find the least value of n such that the number 777 is a factor.

27

If a point inside a triangle is selected randomly, what is the probability that quadrilaterals of equal area can be formed as described in the problem for December 21?

28

Let S equal the set of all possible differences of primes. That is, S is the set of values of n where $n = p - q$ and both p and q are primes. What is the least positive integer not contained in set S?

29

Only three whole numbers exist such that the digits from left to right are consecutive integers in increasing order and the number is divisible by 24. Find the three numbers.

30

The three angles of a triangle have measures a, b, and c, with $a < b < c$. Express the difference in terms of a, b, and c when the largest interior angle is subtracted from the largest exterior angle.

31

JANUARY

NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

1

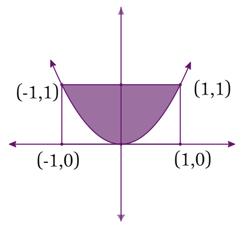
Let p_1, p_2, and p_3 be the prime factors of 2015. The two-digit primes contain the same digits but in reverse order. Find p_1, p_2, and p_3 and then find the next year that two of the (distinct) prime factors of the year contain the same digits.

2

An *equable* shape is one whose area and perimeter are numerically equal. What are the lengths of the legs of an equable isosceles right triangle?

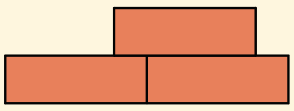
3

Consider the parabola with equation $y = x^2$ and the rectangle with vertices (1, 0), (1, 1), (–1, 1), and (–1, 0). Find the area of the parabolic segment (shaded) and compare its area with that of the rectangle.



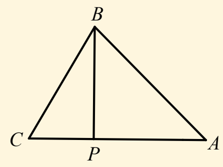
4

The shape shown consists of three congruent rectangles touching one another along certain sides, as shown. The perimeter of each rectangle is 21 cm. What is the perimeter of the shape?



5

In triangle ABC with $\overline{BP} \perp \overline{AC}$, $AC = 2$, the measure of $\angle BAC$ is 45°, and the measure of $\angle BCA$ is 60°. Find BP.



6

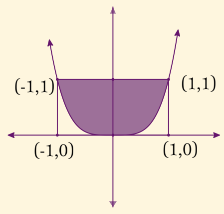
For two positive numbers a and b, the arithmetic mean is given by $(a + b)/2$, and the root mean square is given by

$$\sqrt{\frac{a^2+b^2}{2}}.$$

If two positive numbers have arithmetic mean of 15 and a root mean square of 20, then what is their product?

7

Consider the graph of $y = x^4$. The area bounded by the curve and the line connecting (–1, 1) and (1, 1) has been shaded in the figure. Compare this area with the area found in the problem for January 3. Give a mathematical explanation for your answer.



8

This figure consists of four congruent rectangles with a common vertex and coincident sides as shown. The perimeter of each rectangle is 16 cm. The perimeter of the shape is 48 cm. Find the area of the shape.

9

In how many ways can change be made for \$210 using any combination of \$5, \$10, and \$20 bills? You do not have to use each of the three types of bills simultaneously.

10

Four students took a standardized mathematics test and earned scores of a, b, c, and d, with $a \le b \le c \le d$. The mean of their scores was 645, the median was 640, and the range was 70. What is the largest possible value of c?

11

Find all solutions to the following equation:

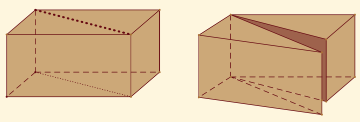
$$4-\frac{1}{x}=\sqrt{4-\frac{1}{x}}$$

12

A palindrome is a word or number that is the same when spelled backward. For example, 61488416, 131, and RADAR are all palindromes. Find the smallest six-digit palindrome that is divisible by both 4 and 9.

13

A block of wood in the shape of a right rectangular prism has dimensions 1 ft. × 1 ft. × 2 ft. If Tom saws it in half to create two triangular prisms with height 1 ft., what is the percentage increase in total surface area?



14

Consider the decimal expansion of the fraction 1/14. Find the sum of the first 40 digits in that decimal representation.

15

Consider a parabola with equation

$$x = -8y^2 + 8y - 1.$$

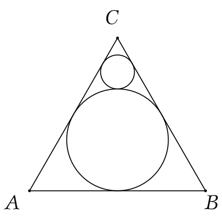
Find the area of the parabolic segment bounded by $x = -1$ and compare its area with that of the rectangle with vertices (1, 0), (1, 1), (–1, 1), and (–1, 0).

16

How many palindromes are there between 10 and 100,000?

17

Given equilateral triangle ABC with an inscribed circle and another smaller circle tangent to the first and to two sides of the triangle as shown, determine the ratio of the areas of the large circle to the small circle.

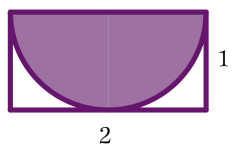


18

Loreena spent her entire life in four cities—Atlanta, Boston, Chicago, and Denver—in the order given. She spent the first 2/7 of her life in Atlanta. Of the remainder of her life, the first 3/5 was spent in Boston and Chicago. She spent the last 4/7 of her life in Chicago and Denver. What part of Loreena's entire life was spent in Chicago?

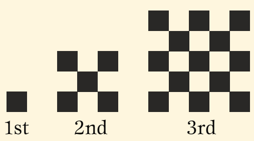
19

The endpoints of a semicircle are vertices of a 2 × 1 rectangle, as shown. What fraction of the rectangle's area lies in the interior of the semicircle?



20

The first figure was constructed with 1 tile. The second figure was constructed with 5 tiles. The third figure was constructed with 13 tiles. If the pattern continues, how many tiles will be needed to construct the 50th figure in this sequence?

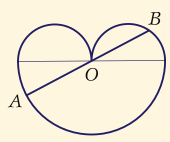


21

When the midpoints of a pair of opposite sides of any parallelogram are connected, two congruent parallelograms are created. But the newly formed parallelograms are not necessarily similar to the original parallelogram. Find a parallelogram such that joining the midpoints of a pair of opposite sides creates two parallelograms similar to the original.

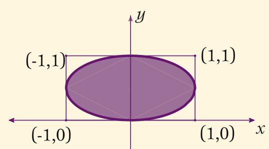
22

Points A and B lie on the perimeter of the figure, which consists of three semicircles. The smaller semicircles are congruent. Prove that chord AB, which passes through O, divides the perimeter in half.



23

Consider an ellipse inscribed in the rectangle with vertices (1, 0), (1, 1), (–1, 1), and (–1, 0) such that its major and minor axes are parallel to the x- and y-axes. What fraction of the rectangle's area lies in the interior of the ellipse?



24

Find the sum of the terms of the followng arithmetic sequence:

$$-403, -372, -341, \ldots, 2015$$

25

Examine the number triangle to determine how it has been constructed. Then append a row to the triangle below the last row shown.

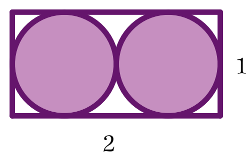
1
1 1 1
1 2 3 2 1
1 3 6 7 6 3 1
1 4 10 16 19 16 10 4 1

26

The coefficients of the trinomial $1 + x + x^2$ are 1, 1, and 1. What coefficients are obtained when $(1 + x + x^2)^2$ is expanded? What coefficients are obtained when $(1 + x + x^2)^3$ is expanded? Use your results along with the results of the problem for January 25 to expand $(1 + x + x^2)^5$ without paper and pencil.

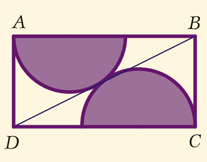
27

Two circles "fit" perfectly in a 2 × 1 rectangle because each can be inscribed in a 1 × 1 square. What fraction of the rectangle's area lies within the two circles?



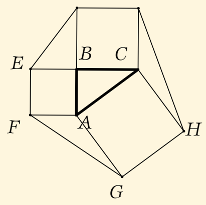
28

The diagonal of a 1 × 2 rectangle is drawn, and semicircular sectors are "inscribed," as shown. The arcs are tangent to the diagonal and to the sides of the rectangle at A and C. The diameters lie on sides AB and CD. What fraction of the rectangle's area lies within the two semicircles?



29

Right triangle ABC has squares constructed on the legs and the hypotenuse. The vertices of the squares are connected to create hexagon $DEFGHI$. If $AB = 3$ and $BC = 4$, find the area of the hexagon.



30

The points A (–2, 1) and B (4, 4) lie on the parabola $y = x^2/4$. Let l be the line parallel to $\overline{AB}$ and tangent to the parabola. If C is the point of tangency, then the area of $\triangle ABC$ gives a rough approximation of the area bounded by $\overline{AB}$ and the parabola. Find the coordinates of C and also the area of $\triangle ABC$.

31

Continuing the problem of January 30, let j and k be lines tangent to the parabola and parallel to $\overline{AC}$ and $\overline{BC}$, respectively. Let D and E be the respective points of tangency. Find the coordinates of D and E and the areas of $\triangle ADC$ and $\triangle BEC$. Add the areas of the three triangles to improve the area approximation found for the problem of January 30.

SOLUTIONS to calendar

JANUARY

CALENDAR CONTRIBUTORS

Problem 2 was submitted by Jordan Brooks, who teaches mathematics at Nexus Academy of Royal Oak in Beverly Hills, Michigan.

Problems 4, 5, 6, 8, 9, 10, 11, 12, 16, 17, 18, and 20 were submitted by Charles Kicey. He and coauthors John Seppala and Arsalan Wares wrote the problems for the 2013 High School Mathematics Tournament hosted by Valdosta State University (Georgia), where the three teach mathematics.

Looking for more Calendar problems?

Visit www.nctm.org/publications/calendar/default.aspx?journal_id=2 for a collection of previously published problems—sortable by topic—from *Mathematics Teacher.*

The Editorial Panel of *Mathematics Teacher* is considering sets of problems submitted by individuals, classes of prospective teachers, and mathematics clubs for publication in the monthly Calendar. Send problems to the Calendar editors. Remember to include a complete solution for each problem submitted.

Department editors

Margaret Coffey, *Margaret.Coffey@fcps.edu, Thomas Jefferson High School for Science and Technology, Alexandria, VA 22312; and* ***Art Kalish,*** *artkalish@verizon.net, Director of the Institute of MERIT at SUNY College at Old Westbury*

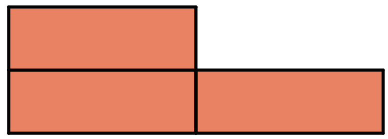
1. 5, 13, 31; 2414. The prime factorization of 2015 is 5 • 13 • 31. Since 13 is the smallest two-digit prime with two distinct digits, we examine the product 7 • 13 • 31 = 2821. We can obtain a smaller product by using the next-larger two-digit prime and its reversal: 2 • 17 • 71 = 2414. We cannot do better than this. If we try a pair of two-digit primes without a third factor, the smallest candidate is 37; but 37 • 73 = 2701, which is larger than 2414.

2. $4 + 2\sqrt{2}$. Let the length of each leg equal x. Using knowledge of special right triangles, we find that the length of the hypotenuse is $x\sqrt{2}$. The area of the triangle is $x^2/2$, and the perimeter is $2x + x\sqrt{2}$. Since the triangle is equable, $x^2/2 = 2x + x\sqrt{2}$, implying that $x^2/2 - 2x - x\sqrt{2} = 0$. Solving for x, we have $x^2 - 4x - 2x\sqrt{2} = 0 \rightarrow x(x - 4 - 2\sqrt{2}) = 0$. Since $x \neq 0$, $x = 4 + 2\sqrt{2}$. (Interested readers will find additional problems about equable polygons in the *MT* May 2014 Calendar.)

3. 4/3; 2/3 the area of the rectangle. The formula for the area of a parabolic segment is $A = (2/3)hw$, where w is the length of the segment that connects a point on the curve with its reflection in the axis of symmetry and h is the distance from the midpoint of that segment to the vertex. In this case, $w = 2$ and $h = 1$, so the area $A = 4/3$, which is 2/3 the area of the rectangle.

***Alternate solution*:** The formula that Archimedes developed gives the area of a parabolic segment as 4/3 times the area of the triangle formed by the endpoints and the vertex. In this case, we have $A = (4/3)(2 \cdot 1/2) = 4/3$.

4. 42 cm. Let x and y be the horizontal and vertical dimensions of the three congruent rectangles; so $2x + 2y = 21$. Notice that we can slide the upper rectangle left to the position shown, so the perimeter of the figure itself is $4x + 4y = 2(2x + 2y) = 2 \cdot 21 = 42$.



5. $3 - \sqrt{3}$. If $AP = x$, then $CP = 2 - x$. The sides of any 45-45-90° triangle are in the ratio $1:1:\sqrt{2}$, so $BP = AP = x$. The sides of any 30-60-90° triangle are in the ratio $1:\sqrt{3}:2$, so $BP = \sqrt{3} \cdot CP = \sqrt{3}(2 - x)$. Thus, $x = BP$ satisfies $x = \sqrt{3}(2 - x)$, implying that $(1 + \sqrt{3})x = 2\sqrt{3}$. Solving for x, we obtain

$$x = \frac{2\sqrt{3}}{1+\sqrt{3}} = \frac{2\sqrt{3}(1-\sqrt{3})}{-2} = 3-\sqrt{3}.$$

6. 50. From $(a + b)/2 = 15$ and $\sqrt{(a^2 + b^2)/2} = 20$, it follows that $a + b = 30$ and $a^2 + b^2 = 800$. Squaring the first equation and subtracting the second gives $2ab = 100 \rightarrow ab = 50$.

7. Area > 4/3. For $0 < |x| < 1$, $x^4 < x^2$. Using words instead of symbols, we note that multiplying numbers between –1 and +1 (excluding 0) makes their absolute values smaller. For example, (0.3)(0.3)(0.3)(0.3) must be less than (0.3)(0.3). So every point on the curve of $y = x^4$ between $x = -1$ and $x = 1$ (except 0) will be closer to the x-axis than the corresponding point on the curve of $y = x^2$. Hence, the shaded region must be larger.

***Extension*:** Use technology to find the exact area. In this case, the area bounded by $y_1 = x^4$ and $y_2 = 1$ is 8/5. Students may enjoy observing the emerging pattern

when the exact area is bounded by $f(x) = x^{2n}$ for a positive integer n and $g(x) = 1$. In general, the area is $4n/(2n + 1)$.

8. 48 cm^2. Let x and y be the dimensions of the rectangles with $y > x$. So $2x + 2y = 16$ or $x + y = 8$. Moving from bottom to top on the left side of the figure, the total vertical measure is $y + x + (y - x) = 2y$, so by symmetry the perimeter of the figure is $4(2y) = 8y$. Since $8y = 48$, $y = 6$. So $x = 2$, and the area of the shape is $4(6 \cdot 2) = 48$.

9. 132. We organize the combinations in a table. Note that one \$10 may be changed to two \$5s and one \$20 to two \$10s. There are two ways (one \$10 or zero \$10s) using ten \$20s. Develop a pattern by first using the greatest number of \$20s and then the greatest number of \$10s. The table shows four ways (three, two, one, or zero \$10s) using nine \$20s. When these nine \$20s are reduced to eight, the number of combinations increases from four to six, and the pattern continues. So the total number of combinations is $2 + 4 + 6 + \cdots + 22 = 11 \cdot 12 = 132$.

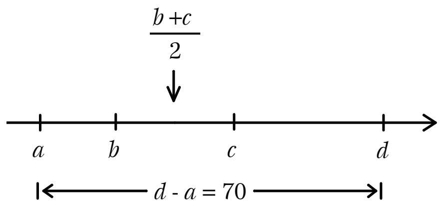
10. 665. A simple diagram like the one below may be helpful. The information can be expressed as

$$\left\{\begin{array}{c} d - a = 70 \\ (b + c)/2 = 640 \\ (a + b + c + d)/4 = 645 \end{array}\right\}.$$

Equivalently,

$$\left\{\begin{array}{c} d - a = 70 \\ b + c = 1280 \\ a + (b + c) + d = 2580 \end{array}\right\}.$$

Inserting the second equation into the third gives $a + d = 2580 - 1280 = 1300$. Combine with the first equation to reach $a = 615$ and $d = 685$. Now b and c must be equidistant from their mean. Let $b = 640 - x$ and $c = 640 + x$. The largest deviation x from 640 that maintains $a \le b$ and $c \le d$ is $x = 25$. So $640 + 25 = 665$.



11. 1/4 or 1/3. If we let $u = 4 - 1/x$, then we obtain the equation $u = \sqrt{u}$. Since 0 and 1 are the only numbers unchanged by the square root, we have $u = 0 = 4 - 1/x \rightarrow 1/x = 4$ or $u = 1 = 4 - 1/x \rightarrow 1/x = 3$. Therefore, $x = 1/4$ or $1/3$.

***Alternate solution*:** Square both sides of the equation to obtain $16 - 8/x + 1/x^2 = 4 - 1/x$. Combine like terms to obtain a quadratic in $1/x$: $(1/x)^2 - 7(1/x) + 12 = 0 \rightarrow (1/x - 3)(1/x - 4) = 0$. Therefore, $x = 1/4$ or $1/3$, as before.

12. 216612. Recall the divisibility test for 4—the number formed by the last two digits must be divisible by 4—and the divisibility test for 9—the sum of the digits must be divisible by 9. Call the unknown palindrome *abccba*, where a, b, and c are digits and $a \neq 0$. So the smallest possible first digit (in the largest place value) is $a = 1$, but this would also be the last digit, and divisibility by 4 would fail. Try instead $a = 2$. The smallest second digit to satisfy divisibility by 4 is $b = 1$, resulting in 21*cc*12. Choose $c = 6$ to also have divisibility by 9.

13. 44.7%. The surface area of any (right) prism can be found with the formula that uses the perimeter times the height plus two times the area of the base. The surface area of the prism before sawing is $6 \cdot 1 + 2(2) = 10$ ft.^2. Sawing exposes two rectangular surfaces. Each has length equal to the length of the diagonal cut $d = \sqrt{1^2 + 2^2} = \sqrt{5}$ and height 1 ft. The sum of these two rectangles is $2(\sqrt{5} \cdot 1) = 2\sqrt{5}$. The percentage increase in surface area is $2\sqrt{5}/10 \approx 0.447$, or 44.7%.

14. 174. Although a calculator may make solving this question easier, it can certainly be solved without a calculator, as the following solution will make clear. Many students know the decimal expansion of 1/7 by heart: $0.\overline{142857}$. Since 1/14 is half of 1/7, we expect to see 6 digits in the repetend, which we can obtain by division:

$$2\overline{)0.142857142857\ldots} = 0.0\overline{714285}$$

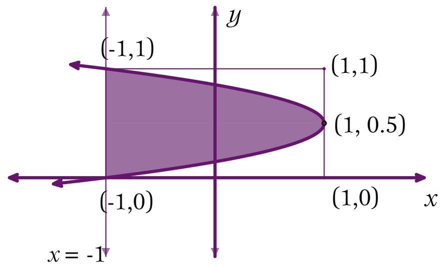
Every set of 6 consecutive digits after the first digit, 0, has a sum of 27. Therefore, the sum of the first $1 + 6(6) = 37$ digits is $0 + 6(27) = 162$. The next 3 digits are 7, 1, and 4, so the first 40 digits of 1/14 sum to $162 + 7 + 1 + 4 = 174$.

15. 4/3; 2/3 the area of the rectangle. The equation indicates that the parabola opens to the left. The equation of the axis of symmetry is $y = -b/(2a) = -8/(2 \cdot (-8)) = 1/2$. Find the x-coordinate of the vertex by substitution: $x = -8(0.5^2) + 8(0.5) - 1 = -2 + 4 - 1 = 1$. The vertex has coordinates $(1, 1/2)$. Since $x = -1$ is a boundary line for the region, we find the y-values when $x = -1$: $-8y^2 + 8y - 1 = -1 \rightarrow y^2 - y = 0 \rightarrow y = 0$ or $y = 1$. We sketch the figure to

Solution 9

\$20s	\$10s	\$5s	Number of Combinations
10	1	0	Two, using 0 or 1 \$10s
10	0	2	
9	3	0	Four, using 0, 1, 2, or 3 \$10s
9	2	2	
9	1	4	
9	0	6	
8	5	0	Six, using 0, 1, 2, 3, 4, or 5 \$10s
⋮	⋮	⋮	
8	0	10	
⋮	⋮	⋮	

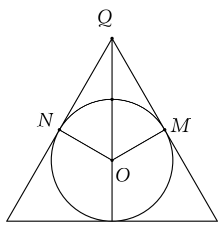
see the parabolic segment that we are working with. In this case, $h = 2$ and $w = 1$, so the area is $A = (2/3)(2)(1) = 4/3$. This represents 2/3 of the rectangle's area.



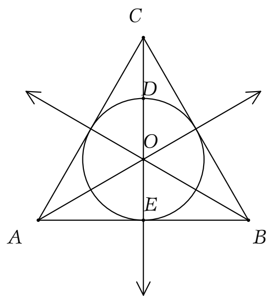
16. 1089. There are 9 two-digit palindromes *aa* for $a = 1, 2, \ldots, 9$. A three-digit palindrome has the form *aba* for $a = 1, 2, \ldots, 9$, but $b = 0, 1, 2, \ldots, 9$, so there are 9 choices for a and 10 choices for b, or $9 \cdot 10 = 90$ three-digit palindromes. Similarly, there are 90 four-digit palindromes *abba* and 900 five-digit palindromes *abcba*. Altogether there are $9 + 90 + 90 + 900 = 1089$ palindromes in the required range.

17. 9. Consider any circle of radius r and center O inscribed in an equilateral triangle as shown; let M and N be the points of tangency indicated. Since tangents to a circle from a point are congruent, $MQ = NQ$, and so, by SSS, triangles NOQ and MOQ are congruent. Thus, $m\angle MQO = 30°$. Therefore, in triangle MOQ we have $OQ = 2 \cdot OM$; it follows that the height of the original equilateral triangle is $3r$.

Returning to the original problem and its diagram, we denote the radii of the large and small circles by r_L and r_S, respectively. The small circle is inscribed in a small equilateral triangle with its base parallel to $\overline{AB}$; so the height of this triangle is $3r_S$, and the height of the original triangle is $3r_L$. It follows that $3r_S = r_L$. Thus, the desired ratio is $\pi(r_L)^2 : \pi(r_S)^2 = \pi(3r_S)^2 : \pi(r_S)^2 = 9:1$.



Alternate solution: Let O be the center of the inscribed circle. Since $\triangle ABC$ is equilateral, O is the intersection of the medians as well as of the angle bisectors. Let D and E be the intersections of the circle with the median from C to $\overline{AB}$. The centroid of a triangle divides each median into two segments with ratio $2:1$; thus, we have $CO = 2OE = 2OD \rightarrow OD = CO/2$. This result implies that $CD = CO/2$, which is equivalent to $CD = CE/3$. If we construct a line through D parallel to $\overline{AB}$, we have circumscribed the smaller circle in an equilateral triangle with 1/3 the height of the original triangle. Conclude that the ratio of areas is $9:1$, as above.



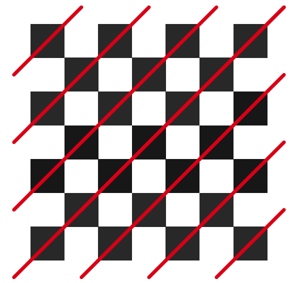
18. 2/7. Let $T = A + B + C + D$, where A, B, C, and D represent the times Loreena lived in the cities that correspond to the alphabet letters. Then $(2/7)T = A$; $(3/5) \cdot (T - (2/7)T) = B + C$; and $(4/7)T = C + D$. Rewrite the system as follows:

$$(5/7)T = B + C + D \qquad (1)$$
$$(3/7)T = B + C \qquad (2)$$
$$(4/7)T = C + D \qquad (3)$$

Substitute (2) into (1) to obtain $(5/7)T = (3/7)T + D \rightarrow (2/7)T = D$. Substitute this result into (3) to obtain $(2/7)T = C$. So Loreena lived two-sevenths of her entire life in Chicago.

19. 0.785, or roughly 11/14. The semicircle has radius 1, so its area is $A = \pi(1^2)/2 = \pi/2 \approx 1.57$. The rectangle has area 2, so the semicircle's area is $1.57/2 \approx 0785$, or roughly 11/14, of the rectangle's area.

20. 4901. There are many approaches to solving this problem; this one uses the fact that the sum of the first n consecutive odd numbers is n^2. In the fourth figure, counting diagonally, we need $1 + 3 + 5 + 7 + 5 + 3 + 1 = 4^2 + 3^2$ tiles. Starting with the first figure, the pattern is $1^2 + 0^2 = 1$, and then $2^2 + 1^2 = 5$ for the second figure, and so on. The number of tiles for the fiftieth figure is $50^2 + 49^2 = 50^2 + (50 - 1)^2 = 50^2 + 50^2 - 2(50) + 1 = 5000 - 100 + 1 = 4901$.

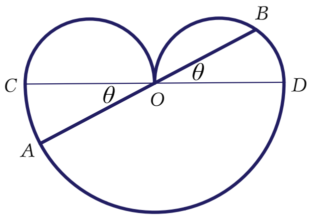


Alternate solution: A second approach compares the white tiles (the empty spaces), black tiles, and the "outside" square (which comprises both the black and white tiles). If each tile has unit area, then the outer squares have areas 1, 9, 25, . . . , with about half the area white. Since there are an odd number of rows, we can match up black and white tiles in pairs of rows until the last row, and we find that we will always have exactly one more black tile than white tile in the last row. We can count the total number of black and white tiles. Let n represent the figure number. The first figure, with $n = 1$, has 1^2 tiles; for $n = 2$, we have 3^2 black and white tiles; for $n = 3$, we have 5^2 tiles. In general, the nth figure will have $(2n - 1)^2 = 4n^2 - 4n + 1$ tiles, where the number of black tiles is one more than the number of white tiles. Thus, $2n^2 - 2n + 1$ tiles are black. The fiftieth figure requires $2(50)^2 - 2(50) + 1 = 5000 - 100 + 1 = 4901$ tiles.

21. The sides must be in the ratio $\sqrt{2}:1$. Consider parallelogram $ABCD$, with M and N the midpoints of $\overline{AB}$ and $\overline{DC}$, respectively. To prove that $ADNM$ and $MNCB$ are similar to $ABCD$, we must show that the angle measures of the three parallelograms are equal and that the corresponding sides are in proportion. $ADNM$ and $MNCB$ each share two angles with $ABCD$; because opposite angles of parallelograms are congruent, we have that the angle measures are equal regardless of the ratios of

corresponding sides. We desire $AM/AD = AD/AB \to AD^2 = AM \cdot AB$. Since M is a midpoint, $AM = AB/2$. Substituting, we obtain $AD^2 = AB^2/2 \to AD = AB/\sqrt{2}$ or $AB : AD = \sqrt{2} : 1$.

22. Let diameter CD have length d. Then the length of semicircle $\widehat{CAD}$ is $\pi d/2$. Since $OC = OD = d/2$, we find the lengths of semicircles $\widehat{OC}$ and $\widehat{OD}$ are each $\pi d/4$. Thus, chord CD divides the perimeter in half. If we can show that the length of $\widehat{CA}$ equals the length of $\widehat{DB}$, then we have shown that chord AB divides the perimeter in half as well. Angles BOD and COA are vertical angles; let their measure be θ. The length of $\widehat{CA}$ is $(\theta/360°)\pi d$, because $\angle COA$ is a central angle. Angle BOD is an inscribed angle; the measure of the central angle that intercepts $\widehat{DB}$ has measure 2θ. Therefore, the length of $\widehat{DB}$ is $(2\theta/360°)\pi(d/2) = (\theta/360°)\pi d$, which equals the length of $\widehat{CA}$. Thus, any chord through point O divides the perimeter of the figure in half.



23. 0.785, or approximately 11/14. The formula for the area of an ellipse is customarily expressed as $A = \pi ab$, where a and b represent half the lengths of the major and minor axes. The major axis for our ellipse has length 2, and the minor axis has length 1, so we have $A = \pi(1)(1/2) = \pi/2$, or approximately 1.57 units^2. The rectangle has area 2, so the ellipse's area is $1.57/2 \approx 0.785$, or roughly 11/14, of the rectangle's area.

24. 63674. Arithmetic sequences are characterized by a common difference between successive terms. In this sequence, the common difference is $-372 - (-403) = -341 - (-372) = 31$. If the sequence contains n terms, then there are $n - 1$ intervals of size 31 between -403 and 2015: $-403 + 31(n - 1) = 2015 \to 31n = 2449 \to n = 79$. To find the sum, we use the idea of pairing terms with the same sum: first and last terms, second and second-to-last terms, and so on. The number of pairs is half the number of terms, so we will have $79(-403 + 2015)/2 = 63674$.

25. (1, 5, 15, 30, 45, 51, 45, 30, 15, 5, 1). Each element in this triangle, except the single 1 in row 0, is the sum of the elements directly above and to the left and right of the number above. For example, 16 is the sum $3 + 6 + 7$.

26. (1, 2, 3, 2, 1), (1, 3, 6, 7, 6, 3, 1), and $1 + 5x + 15x^2 + 30x^3 + 45x^4 + 51x^5 + 45x^6 + 30x^7 + 15x^8 + 5x^9 + 1x^{10}$. The expansion of $(1 + x + x^2)^2$ is $1 + 2x + 3x^2 + 2x^3 + x^4$, and the expansion of $(1 + x + x^2)^3$ is $1 + 3x + 6x^2 + 7x^3 + 6x^4 + 3x^5 + x^6$. Observe that the coefficients constitute rows 2 and 3, respectively, of the number triangle. We guess that the expansion of $(1 + x + x^2)^5$ uses the coefficients from row 5 of the triangle—namely, (1, 5, 15, 30, 45, 51, 45, 30, 15, 5, 1)—which can be confirmed with pencil and paper.

27. 0.785, or approximately 11/14. The radius of each circle is 0.5, so the sum of the areas of the two circles is given by $A = 2\pi(0.5^2) = \pi/2 \approx 1.57$ units^2. The rectangle has area 2, so the circles' area is $1.57/2 \approx 0.785$, or roughly 11/14, of the rectangle's area.

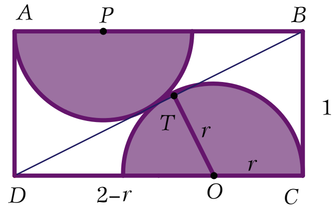
28. 3/5, or 60%. Let T be the point of tangency for semicircle O and diagonal $\overline{BD}$. Then $OT = OC = r$, and $\overline{OT} \perp \overline{BD}$. Since $\triangle OTD : \triangle BCD$, we have

$$\frac{OT}{BC} = \frac{OD}{BD} \to \frac{r}{1} = \frac{2-r}{\sqrt{5}},$$

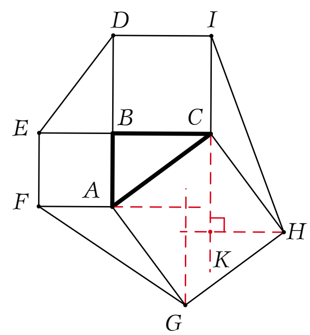
so that $r\sqrt{5} = 2 - r$ or $r = (\sqrt{5} - 1)/2$. The area of the two semicircles is

$$\pi\left(\frac{\sqrt{5}-1}{2}\right)^2 = \pi\left(3-\sqrt{5}\right)/2 \approx 1.200.$$

This result represents 1.2/2, or 60%, of the rectangle's area. Some readers may have observed that the value of $r = (\sqrt{5} - 1)/2 \approx 0.618034$ is the reciprocal of the golden ratio.



29. 74. Triangle $ABC \cong \triangle EBD$; each has area 6. Squares $EFAB$, $BCID$, and $CAGH$ have areas 9, 16, and 25, respectively. It remains to find the areas of triangles ICH and GAF. Through H construct a line l parallel to $\overline{BC}$. Extend $\overline{IC}$ to meet l at K. Since $\overline{HK} \perp \overline{AB}$ and $\overline{IK} \perp \overline{BC}$, $\angle K$ is a right angle. We have $\angle BAC \cong \angle ACK$, and these two angles are complementary to $\angle ACB$ and $\angle KCH$, respectively; thus, $\angle ACB \cong \angle KCH$. We now have $\triangle ABC \cong \triangle HKC$ by AAS, so we can find the areas of triangles HKC, HKI, and IHC. Triangle HKC has area $(3 \cdot 4)/2 = 6$, and $\triangle HKI$ has area $(3 \cdot 8)/2 = 12$, so $\triangle ICH$ has area 6 by subtraction. The area of $\triangle GAF$ is 6 also; it can be found in the same way as the area of $\triangle ICH$. The total area of the hexagon is the sum of the areas of four triangles and three squares: $4 \cdot 6 + 9 + 16 + 25 = 74$.

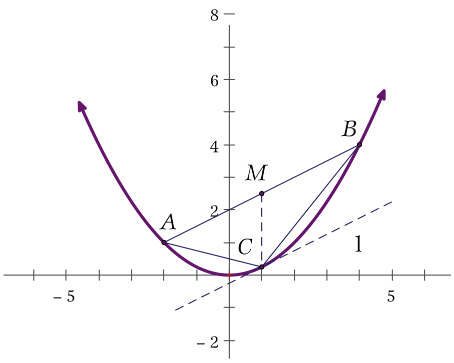


30. (1, 1/4); 6.75. The slope of $\overline{AB}$ is 1/2. The equation of l is $y = x/2 + b$. Since l and the parabola intersect at C, we have $x/2 + b = x^2/4$. The quadratic formula gives us

$$x = \frac{2 \pm \sqrt{4-4(1)(-4b)}}{2} = \frac{2 \pm \sqrt{4+16b}}{2}.$$

Since l is tangent to the parabola, there must be a unique solution to the quadratic, so the discriminant must equal zero. Thus, C has x-coordinate $2/2 = 1$ and y-coordinate 1/4. There are many ways to find the area of $\triangle ABC$. We can

find the midpoint of $\overline{AB}$ by inspection, and we let $\overline{CM}$, with length 2.25, serve as the base of $\triangle ACM$ with height 3 and of $\triangle BCM$ with height 3 as well. The area of $\triangle ABC$ is thus 2.25(3) = 6.75.



31. D (–1/2, 1/16), E (5/2, 25/16); area of $\triangle ADC$ equals the area of $\triangle BEC$, which equals 27/32. The slope of $\overline{AC}$ is –1/4, so the equation of line j is $y = -x/4 + b$. To find D, the point of tangency, we have $-x/4 + b = x^2/4$. Solve for x with the quadratic formula:

$$x = \frac{-1 \pm \sqrt{1-4(1)(-4b)}}{2} = \frac{-1 \pm \sqrt{1+16b}}{2}$$

As before, the discriminant must equal zero, so D has x-coordinate –1/2 and y-coordinate 1/16. It is convenient to find the midpoint of $\overline{AC}$, point N in the figure, with coordinates (–1/2, 5/8). Use $\overline{ND}$, with length 9/16, as the base of $\triangle AND$ and $\triangle BND$. Each of these has height 1.5, so the area of $\triangle ADC$ is (9/16)(3/2) = 27/32, or 0.84375. Repeat these steps for point E and the area of $\triangle BEC$. The slope of the tangent line is 5/4; the coordinates of E are (5/2, 25/16). The midpoint of $\overline{BC}$ is P with coordinates (5/2, 17/8); $\overline{PE}$ has length 9/16. The height of $\triangle BPE$ equals the height of $\triangle CPE$, which equals 1.5, so the area of $\triangle BEC$ is (9/16)(3/2) = 27/32. To complete the estimate, add the areas of the three triangles: 6.75 + 2(0.84375) = 8.4375. Observe that $\triangle ADC$ and $\triangle BEC$ have the same area; each is exactly 1/8 the area of $\triangle ABC$.