Developing the Vertex Formula Meaningfully

A sequenced approach to quadratics helps students develop conceptual understanding of the vertex formula.

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Do your students think of mathematics as a collection of formulas to memorize? As teachers working with students in entry-level algebra classes, we realized that our instruction was a major factor in how our students viewed mathematics. We often presented students with abstract formulas that seemed to appear out of thin air.

One instance occurred while we were teaching students to graph quadratic equations. The algebra textbook in our school district introduced the concept by giving students a formula for the x-coordinate of the vertex as $x = -b/(2a)$. (Note: The vertex form $y = a(x - h)^2 + k$ is not introduced in the textbook and district standards until advanced algebra.)

For a number of years, we provided students with the vertex formula, and they successfully graphed by substituting values into the formula. Yet when asked where the formula came from or how it connected to the defining characteristics of quadratics, students did not know. They were performing procedures using this "magical" formula but did not understand how the formula developed.

Our own learning experiences and research on high-quality mathematics instruction illuminated the need for our students to understand mathematics at a deeper level (Papick 2011). We wanted to be teaching topics conceptually (NMAP 2008) and helping students reason and make connections among concepts (NCTM 2000) so that they could fully understand the mathematics (Kilpatrick, Swafford, and Findell 2001).

In an attempt to bring meaning to the vertex formula, we initially tried to teach our students to complete the square, a concept introduced and fully explored later in advanced algebra in our school district. When we used this approach, the students in our entry-level course quickly became confused as a result of the high level of abstractness

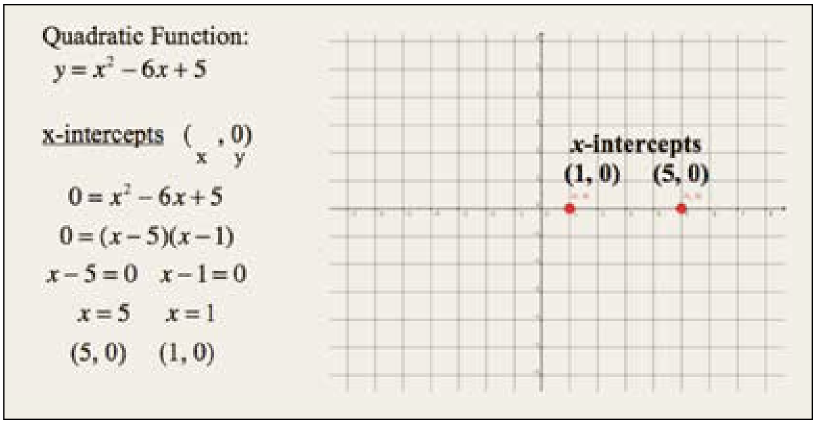

Fig. 1 Students begin to graph the quadratic function by finding the x-intercepts.

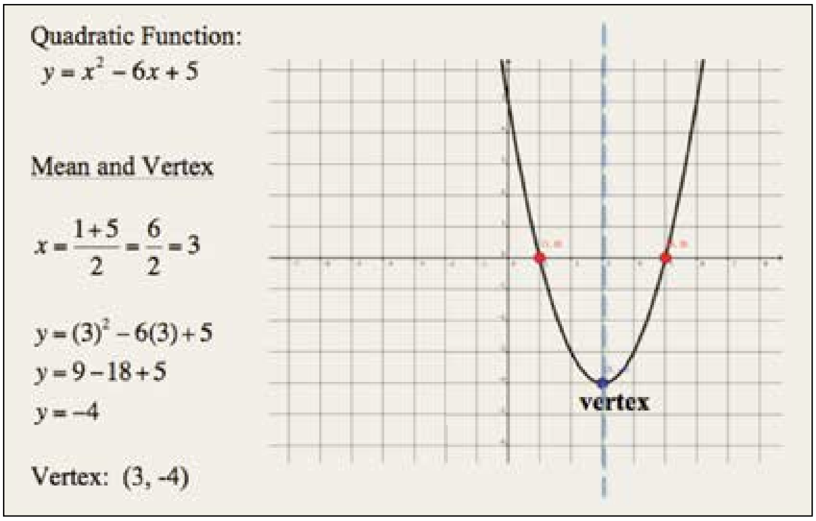

Fig. 2 The axis of symmetry is $x = 3$, which can be seen on the graph and in the calculation of the mean.

and symbolism involved. The confusion was compounded because most students were still in the novice stages of factoring.

The method for graphing quadratic equations in this article connects students' prior knowledge of graphing linear equations with intercepts and calculating mean. This atypical approach is effective only for a quadratic that has real roots. However, we use this method because the goal is to help students reason how to graph a quadratic with understanding rather than memorize the vertex formula. We believe that this initial approach to graphing is beneficial, despite the limitations, because students are making sense of the concept by connecting it to prior knowledge (NCTM 2009).

CHARACTERISTICS OF QUADRATIC EQUATIONS

For this method to be effective, students should understand various methods of solving quadratic equations. Once students have these skills, teachers can begin transitioning into graphing by developing students' understanding of the general features of quadratic equations. The specific characteristics that students should notice are the shape of quadratic graphs and the symmetry in both the table and the graph. These attributes are central to students' conceptual understanding of quadratics and are also critical to their future comprehension of the vertex formula.

Students begin by making their own observations about the graphical representation of quadratics. Each student is given several quadratic equations and the accompanying graph. For each quadratic, students locate five solutions (ordered pairs) and create a table of values. To explore quadratic equations further, students are given different quadratics, with the table of values completed, and are asked to graph the ordered pairs. If the technology is available, a similar activity could be completed using a graphing calculator such as a TI-Nspire.

After finishing these tasks, students make observations using the following prompts: "What do you notice about the graphs of the quadratic equations?" and "What do you notice about the tables of the quadratic equations?" Students may use phrases such as "pairs except for where it curves," "U-shaped," and "all have matches except the middle one" to describe the symmetry in both the graph and table. After a thorough discussion of their observations, students should be introduced to the terms *vertex* and *axis of symmetry* as they practice locating and labeling these on numerous quadratic graphs. This entire task is meant to help students understand that quadratic equations produce "U-shaped" parabolas that are symmetrical along the axis of symmetry and contain a unique point called the vertex. The development of this knowledge, along with previously learned skills of solving quadratic equations, lays the foundation for students to understand the vertex formula.

ESTABLISHING THE *X*-COORDINATE OF THE VERTEX AS THE MEAN

The next phase is to help students recognize the x-coordinate of the vertex as the mean of the x-coordinates of the two x-intercepts. Students can be encouraged to begin graphing $y = x^2 - 6x + 5$ by finding the x-intercepts, as they did with the linear equations. Students are often eager to apply what they know about finding the x-intercepts of linear equations and substitute zero for y for the quadratic equation. They should produce the x-intercepts (5, 0) and (1, 0) when solving the quadratic algebraically (see **fig. 1**).

After finding the two x-intercepts, students plot these solutions on the coordinate axis and brainstorm about ways to finish graphing the quadratic equation. Students may discuss their previous work with the characteristics of quadratic graphs, including the fact that the graph would be U-shaped and

symmetric. Some students may determine that the graph will not be a straight line because the equation is not linear and may conclude that more than two points are necessary. A student or group of students may sketch in the axis of symmetry and determine that if they find the vertex, they can complete their graph.

Students are now ready to reason how to find the axis of symmetry and the vertex. Through questioning, help students see that the middle of the parabola, or the axis of symmetry, can be found by calculating the mean of the x-coordinates of the two x-intercepts. For this example, students see that the axis of symmetry, and thus the x-coordinate of the vertex, is 3 (see **fig. 2**). Having them calculate the mean of the x-coordinates to find that $x = (1 + 5)/2 = 3$ supports this observation. Students then substitute 3 into the original equation to find the y-coordinate of the vertex (see **fig. 2**). Once the x-intercepts and the vertex are graphed, students sketch the remainder of the parabola.

To help students establish the relationship between the x-coordinate of the vertex and the x-intercepts, be sure to choose quadratic equations with two real roots. Although quadratic equations with one real root can be graphed using this method, the relationship between the vertex and x-intercepts would not be apparent. Students should practice graphing quadratic equations on their own by finding the x-intercepts algebraically, calculating the mean of the two x-coordinates of the x-intercepts to find the x-coordinate of the vertex, and then substituting to find the y-value of the vertex (see **fig. 3**).

DEVELOPING THE VERTEX FORMULA

Graphing by calculating the mean of the x-coordinates of the x-intercepts applies only to quadratics with two real roots. This learning progression is purposeful in the sense that using the x-intercepts as a way to find the x-coordinate of the vertex establishes a foundation for giving meaning to the vertex formula. Yet students must also realize that this method of graphing is not effective for quadratic equations with repeated roots or no real roots.

Provide students with the equation $y = x^2 + 16$ and ask them to graph using the method that they have just learned. When trying to compute the x-intercepts algebraically, students may conclude that it is not possible to graph the equation on a coordinate plane. If so, use a graphing calculator to show that a parabola does indeed exist. Pose questions to prompt a discussion about a quadratic equation with no real solutions: "How does this graph compare with the quadratics that we previously graphed?" and "Why did our graphing method not work even though a graph does exist?" Students will likely notice that the graph still exhibits the same

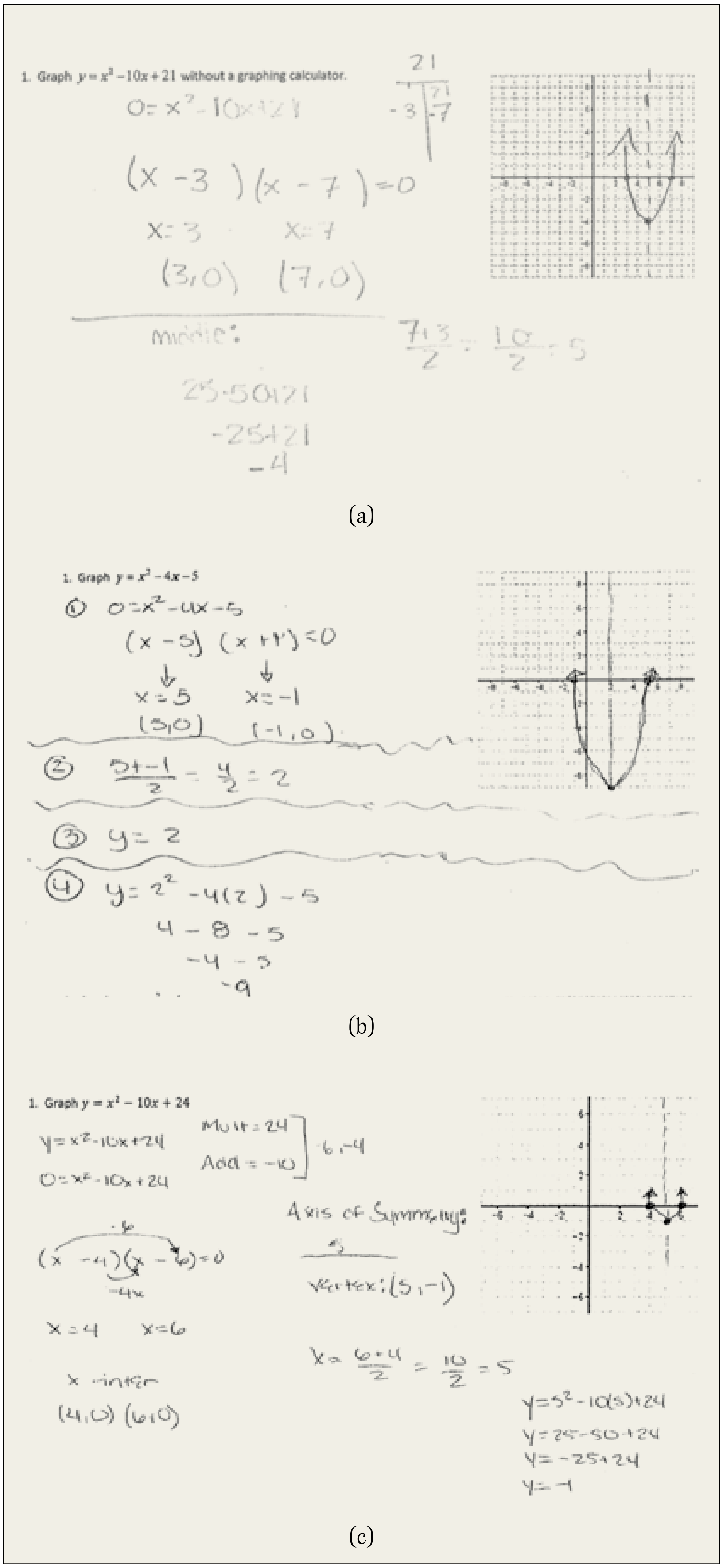

Fig. 3 Students reinforce the relationship between the x-intercepts and the x-coordinate of the vertex by graphing quadratics with two real roots.

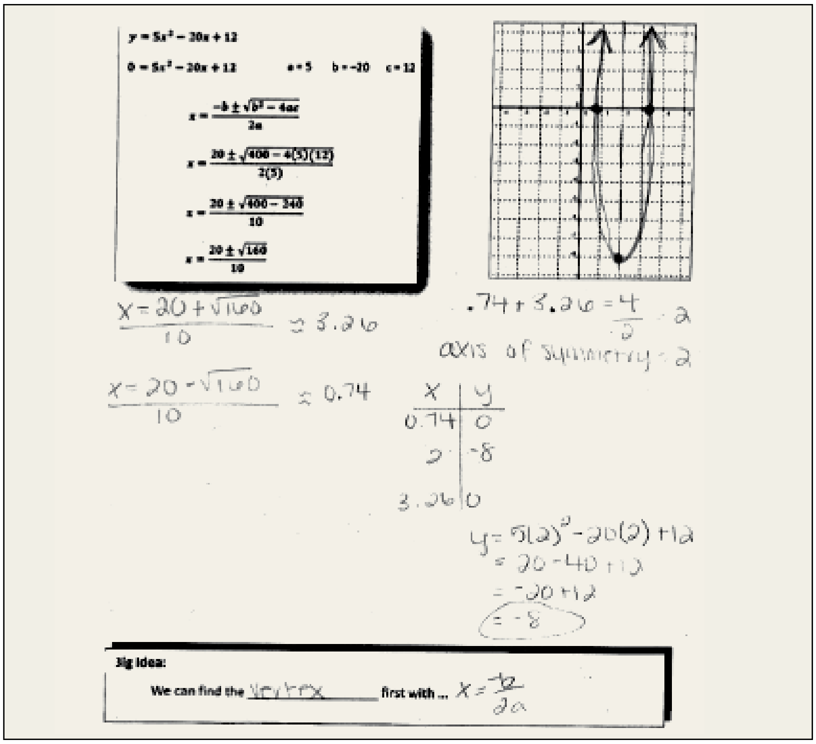

Fig. 4 With calculations provided, students can focus on work that supports conceptual development.

characteristics of shape and symmetry and that the vertex is still an essential component. In addition, students should also note that the graph does not have x-intercepts, making our previous graphing technique ineffective. By attempting to graph $y = x^2 + 16$, we help students realize that another approach to graphing is necessary. This need for another approach, along with the previous understanding of the role that the mean plays in finding the vertex, helps construct the vertex formula.

Students will return to the equation $y = x^2 + 16$, but first we want to develop the vertex formula. Ask students to finish graphing the equation $y = 5x^2 - 20x + 12$ (see **fig. 4**). Although our students should already know how to use the quadratic formula to solve a quadratic equation, the majority of the computational work is completed for them so that they can focus on how the quadratic formula is used to find the two x-intercepts.

As students find the x-intercepts, the teacher should simultaneously plot these points on a coordinate plane that is visible to the whole class. Next to

$$0.74 \quad + \quad 3.26 \quad = \quad 4 \qquad\qquad 4 \div 2 = 2$$

$$\frac{-b-\sqrt{b^2-4ac}}{2a}+\frac{-b+\sqrt{b^2-4ac}}{2a} = \frac{-b+-b}{2a} = \frac{-2b}{2a} = \frac{-b}{a} \qquad \frac{-b}{a} \div 2 = \frac{-b}{2a}$$

Fig. 6 The mean of the two x-intercepts is calculated using values from the example and the symbolic solutions from the quadratic formula.

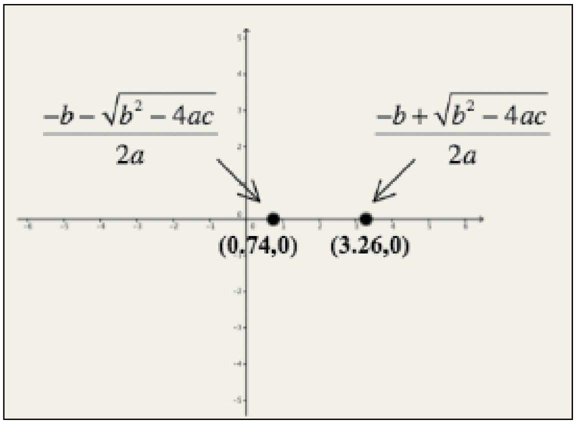

Fig. 5 Students should see the abstract and concrete solutions simultaneously.

the x-intercepts, write the two generic solutions from the quadratic formula (see **fig. 5**). We want students to see the abstract solutions, $(-b - \sqrt{b^2 - 4ac})/2a$ and $(-b + \sqrt{b^2 - 4ac})/2a$, next to the concrete solutions that they just computed, 0.74 and 3.26, as part of the vertex formula development.

Once students find the x-intercepts, encourage them to continue graphing by finding the mean of these two solutions, or the x-coordinate of the vertex. The work used to obtain the x-coordinate of the vertex should also be written so that the whole class can see it. Directly under this work, show students how to find the mean with the two generic quadratic formula solutions. Because this symbolic representation is coupled with the concrete examples, students can make the connection between finding the mean with numbers and finding the mean with variables (see **fig. 6**). This process helps students obtain the vertex formula, a formula that will always help them calculate the x-coordinate of the vertex, no matter how many x-intercepts are in the quadratic equation.

Verifying that the vertex formula is equivalent to finding the mean of the two x-intercepts is an important piece of the connection. Substitute the values of the original quadratic equation into the vertex formula and notice how doing so produces the same x-coordinate as previously computed using the mean. To complete the graph, students calculate a coordinate on one side of the vertex and reflect it over the axis of symmetry. Students should continue graphing quadratics using the vertex formula, beginning with the equation introduced earlier, $y = x^2 + 16$. All quadratic equations—those with graphs of one, two, or no x-intercepts—can now be used because students now know the vertex formula.

MEANING, NOT MAGIC

Using this approach to develop the vertex formula, students essentially learn two methods of graphing quadratics. They are then able to decide which

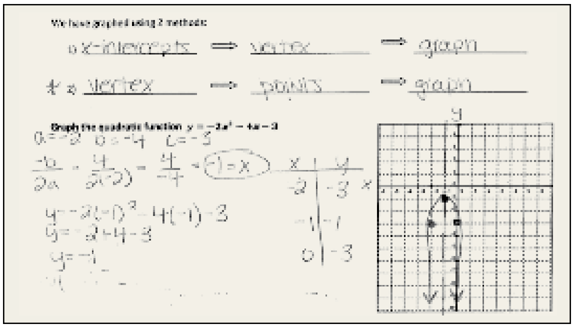

Fig. 7 Students determine which of two methods to use when graphing.

method to use while graphing (see **fig. 7**). The first method will work only for quadratics with real roots but is helpful in developing understanding of the vertex formula. The second method is the traditional technique to graphing quadratics, which uses the vertex formula. By developing the vertex formula using this overall developmental approach, students create meaning for the vertex formula rather than viewing it as a "magical" formula to memorize.

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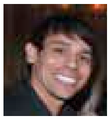

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