FEBRUARY

NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

Theme: Don't get mad.
Learn mod.

When 12 is divided by 5, the remainder is 2. Another way to say this is

$$12 \equiv 2 \pmod 5,$$

read as 12 is congruent to 2 modulo 5. Find a if $37 \equiv a \pmod 8$.

1

How many different remainders are possible when any positive integer is divided by 6?

2

How many positive integers less than 100 are congruent to 3 modulo 7?

3

Solve for all integer values of n in which

$$3n \equiv 2 \pmod 5 \text{ and } 0 \le n \le 20.$$

4

Find the positive remainder when -17 is divided by 3.

5

Find the remainder when 13^{20} is divided by 11.

6

For four given numbers, the sum of the first two numbers is p, the sum of the second and third numbers is q, and the sum of the last two numbers is r. Find an expression in terms of p, q, and r for the sum of the first and last numbers.

7

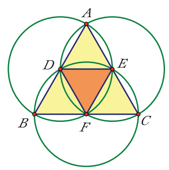
What is the greatest negative number that is congruent to 9 mod 23?

8

The *multiplicative inverse* (reciprocal) of a number p is $1/p$ because the product $p \cdot (1/p)$ equals 1, the identity element. Modular arithmetic has no fractions, but in some cases we can still have inverses. For example, $3 \cdot 5 = 15 \equiv 1 \pmod 7$, so 3 and 5 are multiplicative inverses modulo 7. Find an inverse of 23 modulo 17.

9

Three congruent circles each contain the centers of the other two circles. The circles, two at a time, intersect at three additional points. Find the ratio of the area of the triangle formed by these three additional points compared with the area of the triangle formed by the three centers of the circles.



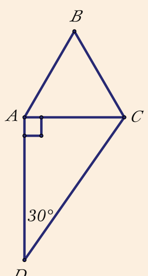
10

Solve for x given that a and b are two distinct positive numbers and that

$$\frac{x^{\log a}}{x^{\log b}} = \frac{b}{a}.$$

11

Suppose that $\triangle ABC$ is equilateral, $\angle CAD$ is a right angle, $m\angle D = 30°$, and $CD = 10$. Find the perimeter of quadrilateral $ABCD$.



12

Explain why the number 3 does not have a multiplicative inverse in arithmetic modulo 6.

13

For given numbers p and q, three of the following four expressions have the same value:

$$p + q,\ p - q,\ p/q, \text{ and } p \cdot q$$

Find the absolute value of the fourth expression.

14

Find the least positive integer congruent to 2014^{2014} modulo 7.

15

Solve for x:

$$\frac{x^3-3}{x^2-2}=\frac{3}{2}$$

16

How many positive integers less than 10 satisfy the equation $2x + 3 \equiv -1 \pmod 4$?

17

Point P is the center of the circle inscribed in $\triangle ABC$ with sides 6, 8, and 10. Find the product of the three lengths:

$$PA \cdot PB \cdot PC$$

18

Factor $x^4 + 4y^4$ into two polynomial expressions.

19

Find all positive integers m such that $11 \equiv 5 \pmod m$.

20

Prove that in every right triangle with integral side lengths, the product of the side lengths is a multiple of 60.

21

The International Standard Book Number (ISBN) is a unique numeric commercial book identifier. The tenth digit of ISBN-10 is a check digit. To find it, multiply the first digit by 10, the second by 9, the third by 8, and so on until the ninth digit is multiplied by 2. The tenth digit added to the sum of these nine products must result in 0 modulo 11. Find the check digit, c, for 0-9773045-6-c.

22

The ISBN-13 check digit is found by multiplying each of the first twelve digits by 1 and 3 alternately. The sum of these twelve products plus the check digit must be 0 modulo 10. Find the check digit for the ISBN-13 number 978-097730458-c.

23

Four circles are drawn, one centered at each vertex of a square. The circles all contain the center of the square. Find the ratio of the sum of the areas of the overlapping portions of the circles to the sum of the areas of the four circles.

24

Pencils cost 16¢ each. Sam bought n pencils and paid with exactly d dollar bills, receiving 4¢ change. What is the maximum number of pencils he could have purchased if $n \le 100$?

25

A square and an equilateral triangle are drawn so that the center of the square lies on the triangle and the incenter of the triangle lies on the square. If the side of the square measures 12, what is the maximum area of the triangle?

26

In problem 26, what is the minimum area of the triangle?

27

Solve

$$\frac{2^{n-4}\sqrt{2\sqrt{2^{n+1}}}}{2^{-3n/2}\sqrt{2^{n-2}}}=512.$$

28

SOLUTIONS to calendar

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1. 5. The largest multiple of 8 that is less than 37 is 32, and $37 - 32 = 5$.

2. 6. The remainder must be an integer in the set $\{0, 1, 2, 3, 4, 5\}$.

3. 14. The smallest number to satisfy the condition is 3, and the complete set may be written $\{3, 10, 17, \ldots, 94\}$. Each number is of the form $3 + 7n$ and must satisfy the inequality $3 \le 3 + 7n < 100$. Thus, $0 \le 7n < 97$, implying that $0 \le n < 13.857$.

4. 4, 9, 14, 19. Test values to exclude 0, 1, 2, and 3 (they do not make the sentence true). Substitute 4 for n in the expression $3n$ and then divide the result by 5 to find a remainder of 2. All numbers that satisfy the congruence relation must therefore be of the form $4 + 5k$. Since $0 \le n \le 20$, we have $0 \le 4 + 5k \le 20$, resulting in $0 \le k \le 3$. Thus, $n = 4, 9, 14$, and 19.

5. 1. Because the divisor, d, is 3, the only possible remainders, r, are 0, 1, or 2. We use the equation $n = qd + r$, with $n = -17$ and $d = 3$, to see that $q = -6$ and $r = 1$. (That is, $-17 = -6 \cdot 3 + 1$.)

6. 1. The question asks for a solution to the statement $13^{20} \equiv n \pmod{11}$, where $0 \le n < 11$. Because 13 is congruent to 2 modulo 11, consider powers of 2. (Convince yourself, perhaps using induction, that $13^k \equiv 2^k \pmod{11}$, so that we can work with a smaller base.) Consider $2^{20} = (2^4)^5 = 16^5$, written with a smaller exponent. But $16^5 \equiv 5^5 \pmod{11}$. Then, $5^5 = 25 \cdot 25 \cdot 5$ and $25 \equiv 3 \pmod{11}$, so we have $3 \cdot 3 \cdot 5 = 45$ and, finally, $45 \equiv 1 \pmod{11}$.

Alternate solution 1: Use $2^{20} = (2^{10})^2 = 1024^2$. By applying the divisibility rule for 11, we see that 1023 is a multiple of 11. Therefore, $1024^2 \equiv 1^2 \equiv 1 \pmod{11}$.

Alternate solution 2: $2^{20} = (2^5)^4 = 32^4$. Then $32^4 \equiv (-1)^4 \equiv 1 \pmod{11}$.

Alternate solution 3:
$13^{20} = 19004963774880799438801$.
Since the absolute value of the difference of the sum of the odd-positioned digits minus the sum of the even-positioned digits equals $58 - 57 = 1$, the remainder when dividing by 11 is 1.

7. $p - q + r$. Let the four given numbers be a, b, c, and d. Then $a + b = p$, $b + c = q$, and $c + d = r$. Subtract the second equation from the first to get $a - c = p - q$. Then add to this result the third equation to find that $a + d = p - q + r$.

Alternate solution: The given information can be written in matrix form as

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} p \\ q \\ r \end{bmatrix}.$$

We need to write the additional row vector $(1, 0, 0, 1)$ as a linear combination of the rows of the coefficient matrix. The first row plus the last row minus the middle row gives the desired vector, so $p + r - q$ will equal $a + d$. That is,

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} p \\ q \\ r \end{bmatrix}.$$

8. -14. The numbers congruent to 9 modulo 23 can be written as $9 + 23n$. Using $n = -1$, we find the greatest negative number to be -14.

9. 3. Because $23 \equiv 6 \pmod{17}$, we are

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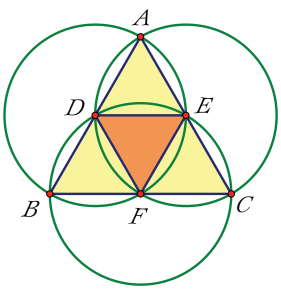
looking for a number n such that $n \cdot 6 \equiv 1 \pmod{17}$. Use $n = 3$, which results in $3 \cdot 6 = 18 \equiv 1 \pmod{17}$.

Alternate solution 1: Solve $n \cdot 23 \equiv 1 \pmod{17}$ by trial and error.

Alternate solution 2: Consider the congruence class $\{1, 18, 35, 52, 69, 86, \ldots\}$. Which number is divisible by 23? The answer is 69, the result of multiplying 23 by 3.

10. 4 : 1. Let 1 be the lengths of the sides of the triangle formed by the centers of the circles: D, E, and F. The other three points—A, B, and C—form the endpoints of the diameters of the circles and are length 2. Since the triangles are similar and the ratio of the sides is 2 : 1, the ratio of their areas is 4 : 1.



11. 1/10. Take the log of both sides of the equation and apply the rules for logs:

$$\log\left(\frac{x^{\log a}}{x^{\log b}}\right) = \log\left(\frac{b}{a}\right)$$
$$\rightarrow \log\left(x^{\log a}\right) - \log\left(x^{\log b}\right) = \log b - \log a$$
$$\rightarrow \log a \log x - \log b \log x = \log b - \log a$$
$$\rightarrow \log x = \frac{\log b - \log a}{\log a - \log b} = -1$$
$$\rightarrow x = 10^{-1} = 1/10$$

12. $20 + 5\sqrt{3}$. In a 30-60-90° triangle, the side opposite the 30° angle is half the hypotenuse. Thus, AC equals 5, as do AB and BC. The missing side length, AD, equals $5\sqrt{3}$, so the perimeter is $10 + 5 + 5 + 5\sqrt{3} = 20\sqrt{3}$.

13. The options to consider in arithmetic modulo 6 are 0, 1, 2, 3, 4, and 5. The products of each of these and the number 3 are 0, 3, 6, 9, 12, and 15, respectively, equivalent to 0, 3, 0, 3, 0, and 3 modulo 6, respectively. The number 1 is never a result. A theorem concerning modular inverses states that the only time p has an inverse modulo m is when p and m are relatively prime. This means that GCF$(m, p) = 1$. When m and p are relatively prime, integers a and b exist such that $am + bp = 1$.

14. 3/2. If $p + q = p - q$, then $q = 0$, which is not a valid option because q is the denominator of one of the expressions. Therefore, we must omit either $p + q$ or $p - q$ as one of the three equivalent expressions. It must be that $p \cdot q = p/q$, so $q^2 = 1$, giving $q = \pm 1$. If $q = 1$, then either $p + 1 = p$ or $p - 1 = p$, neither of which is possible. Therefore, $q = -1$. If we omit $p + q$, so that $p - q = p \cdot q$, we see that $p - (-1) = p(-1) \rightarrow p = -1/2$. Under these conditions—namely, that $q = -1$ and $p = -1/2$—we have $p - q = p/q = p \cdot q = -1/2$ and $|p + q| = |(-1) + (-1/2)| = 3/2$. A similar argument follows if we omit $p - q$ from the three other expressions. Under this condition, in which $q = -1$ and $p = 1/2$, we found that $p + q = p/q = p \cdot q = -1/2$. This condition leads to the solution $p - q = 1/2 - (-1) = 3/2$, the same answer.

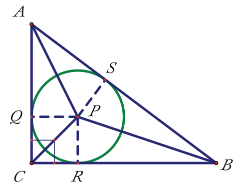
15. 3. When 2014 is divided by 7 the remainder is 5. Thus, $2014 \equiv 5 \equiv -2 \pmod{7}$. Evaluate $(-2)^{2014} \bmod 7$ by taking advantage of the simplification $(-2)^3 \equiv -8 \equiv -1 \pmod{7}$. Thus,

$$\begin{aligned}(-2)^{2014} &\equiv (-2)^{3 \cdot 671 + 1} \\ &\equiv (-1)^{671} \cdot (-2) \equiv 2 \pmod{7}.\end{aligned}$$

16. 0, 3/2. The solution $x = 0$ may be apparent by inspection. Rearranging $2(x^3 - 3) = 3(x^2 - 2)$, we find that $0 = x^2(2x - 3)$, so the second solution is at $x = 3/2$.

17. Four. Subtract 3 from both sides of the given statement to get $2x \equiv -4 \equiv 0 \pmod{4}$. Thus, $x = 0 + 4n$ is a solution. But, $x = 2 + 4n$ is also a solution. Notice that subtracting 3 from members of the congruence statement is a valid step but that dividing by 2 is not. The four positive solutions less than 10 are 2, 4, 6, and 8.

18. 80. Since the triangle side lengths are 6, 8, and 10, the triangle is a right triangle. Draw the three segments from P to each of the vertices. Drop perpendicular segments from P to the three sides of the triangle. Let the lengths of the segments be r. The sum of the areas $\triangle APB + \triangle BPC + \triangle CPA$ equals the area of $\triangle ABC$, so $AC \cdot r + BC \cdot r + BA \cdot r = 6 \cdot 8$. That is, $(6 + 8 + 10)r = 48 \rightarrow 24r = 48 \rightarrow r = 2$. PC is the diagonal of square $CQPR$, so its length is $2\sqrt{2}$. We can find the other two lengths by applying the Pythagorean theorem: $2^2 + 4^2 = PA^2$, so $PA = \sqrt{20} + 2\sqrt{5}$; and $2^2 + 6^2 = PB^2$ so $PB = \sqrt{40} = 2\sqrt{10}$. The product is $2\sqrt{2} \cdot 2\sqrt{5} \cdot 2\sqrt{10} = 80$.



19. $(x^2 + 2y^2 - 2xy)(x^2 + 2y^2 + 2xy)$. The sum of perfect squares does not factor over the real numbers, but the difference does. Thus, we make the given polynomial a difference of perfect squares as follows:

$$\begin{aligned}&x^4 + 4x^2y^2 + 4y^4 - 4x^2y^2 \\ &\quad = (x^2 + 2y^2)^2 - 4x^2y^2 \\ &\quad = (x^2 + 2y^2 - 2xy)(x^2 + 2y^2 + 2xy)\end{aligned}$$

20. 1, 2, 3, 6. Since $11 \equiv 5 \bmod m$, $11/m = q + 5/m$, where q is the integral quotient. Multiply by m to get $11 = mq + 5 \rightarrow mq = 6$. Therefore, the only positive values for m are the factors of 6: 1, 2, 3, and 6.

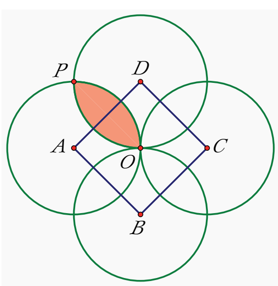
21. All primitive triples can be found using pairs of positive integers, m and n, where m and n are relatively prime and exactly one of them is even. For $m > n$, the formulas $m^2 - n^2$, $2mn$, and $m^2 + n^2$ give all primitive triples. To show that the product of these three is always a multiple of 60, show that at least one of them is divisible by 3, one by 4, and one by 5. Since one of m or n is even, then $2mn \equiv 0 \pmod{4}$. If $m \equiv 0 \pmod{3}$ or $n \equiv 0 \pmod{3}$, then $2mn \equiv 0 \pmod{3}$. If neither m nor n is a multiple of 3, then $m \equiv \pm 1 \pmod{3}$ and $n \equiv \pm 1 \pmod{3}$. Thus, $m^2 \equiv 1 \pmod{3}$ and $n^2 \equiv 1 \pmod{3}$,

so $m^2 - n^2 \equiv 0 \pmod 3$. In a similar fashion, if $m \equiv 0 \pmod 5$ or $n \equiv 0 \pmod 5$, then $2mn \equiv 0 \pmod 5$. If m is not a multiple of 5, then m is congruent to ± 1 or ± 2 modulo 5; thus, $m^2 \equiv 1 \pmod 5$ or $m^2 \equiv 4 \pmod 5$. Equivalently, $m^2 \equiv \pm 1 \pmod 5$. Similarly, $n^2 \equiv \pm 1 \pmod 5$. When m^2 and n^2 are congruent modulo 5 (both are either 1 or –1), their difference, $m^2 - n^2$, is congruent to 0 modulo 5. When one is 1 and the other –1, the sum, $m^2 + n^2$, is congruent to 0 modulo 5. Therefore, every Pythagorean triple includes numbers that are multiples of 3, 4, and 5, and the product is a multiple of 60.

22. 6. The tenth digit is the additive inverse of the sum mod 11. Multiply and add the following: $10(0) + 9(9) + 8(7) + 7(7) + 6(3) + 5(0) + 4(4) + 3(5) + 2(6) = 247$. Then we need to find the value of c that satisfies $247 + c \equiv 0 \pmod{11}$. Since $247 \equiv 5 \pmod{11}$, use $c = 6$.

23. 5. Multiply each of the twelve known digits by 1 and 3 alternately and then add the products: $1(9) + 3(7) + 1(8) + 3(0) + 1(9) + 3(7) + 1(7) + 3(3) + 1(0) + 3(4) + 1(5) + 3(8) = 125$. The check digit, c, must be added to this result to get a multiple of 10, so $c = 5$.

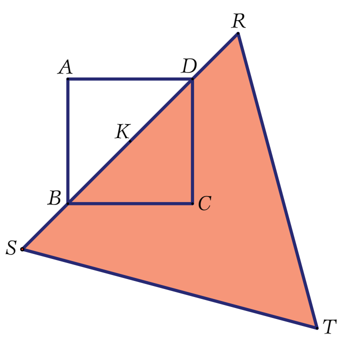
24. $(\pi - 2)/(2\pi)$. Since there are four congruent circles and four identical overlapping regions, the ratio of the sum of the areas is the same as the ratio of a pair of areas. Let the radius of each circle be 1; then its area is π. An overlapping region, seen shaded in the diagram, has an area determined by finding the area of sector ODP, subtracting the area of $\triangle ODP$, and then doubling that result. Since $\angle PDO$ is 90°, the area of the overlapping region is $2(\pi/4 - 1/2) = \pi/2 - 1$. Thus, the ratio we seek is $(\pi/2 - 1)/\pi = (\pi - 2)/(2\pi)$.



25. 81. Since each pencil costs 16¢, the total number of cents spent is $16n$. Sam used d dollar bills and got back 4¢, so he spent $100d - 4$ cents. Set these expressions equal and solve for integer solutions: $100d - 4 = 16n$. Possible solutions can be found using a spreadsheet, but modular arithmetic gives another approach. First, simplify the equation to $25d - 1 = 4n$. It follows that $25d - 1 \equiv 4n \pmod{25}$, which reduces to $24 \equiv 4n \pmod{25}$, giving $n = 6$ as a potential solution. (When $n = 6$, we have $d = 1$.) Since we want the maximum number of pencils less than or equal to 100, find numbers equivalent to 6 modulo 25. These are $6 + 25 = 31$, 56, and 81. The largest possible number is 81. Readers might consider a variation of this approach using the slope of the line whose equation is $(25x - 1)/4 = y$.

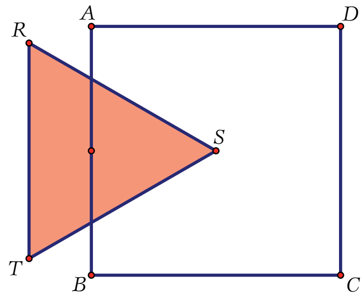
26. $216\sqrt{3}$. The side length, s, of the triangle determines the area of the triangle since $A = s^2\sqrt{3}/4$. The maximum altitude, a, of the triangle maximizes the side of the triangle since $s = 2a\sqrt{3}/3$. A side of the triangle must contain the center of the square, K, and the triangle's altitude to that side must contain the incenter of the triangle. Thus, to maximize, we want the incenter of the triangle to be as far from K as possible. This occurs when the incenter is at a corner of the square, C. The side length of the square is 12, so the distance KC is $6\sqrt{2}$. But KC is 1/3 of the altitude. Thus, the altitude is $18\sqrt{2}$, and a side of the triangle is $12\sqrt{6}$. The area of the triangle is

$$A = s^2\sqrt{3}/4 = \left(12\sqrt{6}\right)^2\sqrt{3}/4 = 216\sqrt{3}.$$



27. $27\sqrt{3}$. To minimize the area of the triangle, minimize the distance from the center of the square to the incenter of the triangle. We do so by placing a vertex of the triangle at the center of the square and placing the incenter of the triangle at a midpoint of the side of the square. This distance is 6, which is 2/3 of the altitude of the triangle. The altitude of the triangle is 9 so a side is $6\sqrt{3}$, and the area is

$$\left(6\sqrt{3}\right)^2\sqrt{3}/4 = 27\sqrt{3}.$$



28. $n = 5$. The right side equals 2^9. Use the rules of exponents to simplify the left side of the equation. The numerator is

$$2^{n-4}\sqrt{2\sqrt{2^{n+1}}} = 2^{n-4}\left(2^1\left(2^{n+1}\right)^{1/2}\right)^{1/2} = 2^{n-4}2^{1/2}2^{(n+1)/4} = 2^{(5n-13)/4},$$

and the denominator is

$$2^{-3n/2}2^{(n-2)/2} = 2^{-n-1}.$$

Then

$$\frac{2^{(5n-13)/4}}{2^{(-4n-4)/4}} = 2^9.$$

Thus, $(9n - 9)/4 = 9$, and $n = 5$.