SOLUTIONS to calendar

CALENDAR CONTRIBUTOR

Problems 2–12 were submitted by Richard Forringer of Durham Academy (retired) in Durham, North Carolina.

Looking for more Calendar problems?
Visit www.nctm.org/publications/calendar/default.aspx?journal_id=2 for a collection of previously published problems—sortable by topic—from *Mathematics Teacher*.

The Editorial Panel of *Mathematics Teacher* is considering sets of problems submitted by individuals, classes of prospective teachers, and mathematics clubs for publication in the monthly Calendar. Send problems to the Calendar editors. Remember to include a complete solution for each problem submitted.

Department editors
Margaret Coffey, *Margaret.Coffey@fcps.edu, Thomas Jefferson High School for Science and Technology, Alexandria, VA 22312; and* ***Art Kalish,*** *artkalish@verizon.net, Director of the Institute of MERIT at SUNY College at Old Westbury*

1. $(d - c)/2$. The median of the given numbers is $(c + d)/2$, the arithmetic mean of the third and fourth ordered values. Removing a or b or c increases the median. Since we now have an odd number of values, the new median is d. The difference is $d - (c + d)/2 = (d - c)/2$.

2. 5. If r_1 and r_2 are the two roots and a, b, and c are the coefficients of the quadratic, then $r_1 + r_2 = -b/a$ and $r_1 \cdot r_2 = c/a$. Therefore, the sum of the roots divided by the product of the roots is $(-b/a) \div (c/a) = -b/c$. In this case, $-b/c = 5$. Notice that the ratio of the sum of roots to the product of the roots is independent of the value of a.

Alternate solution: Without knowing the relationship between r_1 and r_2 and the coefficients a, b, and c, we can solve the quadratic to find the roots. Completing the square gives us $3x^2 + 5x - 1 = 0 \rightarrow x^2 + 5x/3 = 1/3 \rightarrow x^2 + 5x/3 + 25/36 = 1/3 + 25/36 \rightarrow (x + 5/6)^2 = 37/36 \rightarrow x = (-5 \pm \sqrt{37})/6$. The sum of the roots is $-10/6 = -5/3$. The product of the roots is

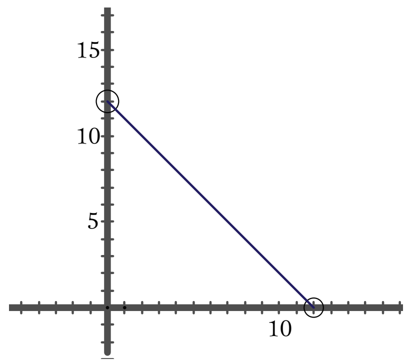
$$\left(\frac{-5+\sqrt{37}}{6}\right)\left(\frac{-5-\sqrt{37}}{6}\right) = \left(\frac{25-37}{36}\right) = -\frac{1}{3}.$$

The requested ratio is $(-5/3)/(-1/3) = 5$.

3. 18. Combine like terms to obtain $3(3^{17})$. Use the laws of exponents to write as $3^1 \cdot 3^{17} = 3^{1+17} = 3^{18}$.

4. 1. The sequence of digits in the units place of 3^n has a "cycle" of four digits: 3, 9, 7, 1, 3, 9, 7, 1, . . . Every fourth term in the sequence is 1, so 3^{444} must have 1 in the units place.

5. $x + y = 12$. We can write $(x + y)/2 = 6$ or, equivalently, $x + y = 12$. The graph, the set of points between (12, 0) and (0, 12), is shown in the figure.



6. (0, 7000), (8555 5/9, 0), and three answers that will vary—for example (–11, 7009), (11, 6991), and (11000, –2000). Any nonhorizontal, nonvertical line that does not contain the origin will pass through exactly three of the four quadrants. Since we are required to list the coordinates of five points, we must name a point in each of those quadrants plus the x- and y- intercepts. The y-intercept is 7000, so (0, 7000) is our first coordinate pair. The x-intercept is 77000/9 = 8555 5/9, so (8555 5/9, 0) is our second coordinate pair. Since the y-intercept is positive and the line has negative slope, the line passes through quadrants II, I, and IV. Let x be any negative number to obtain a point in quadrant II; choosing –11 for x results in $y = 7009$. Let x be any positive value less than 8555 5/9 to obtain a point in quadrant I; choosing 11 for x results in $y = 6991$. Let x be any value greater than 8555 5/9 to obtain a point in quadrant IV. If $x = 11000$, then $y = -2000$.

7. x. Factor the numerator to obtain

$$\frac{x\left(x^3 + x^4 + x^5 + x^6 + \cdots + x^{199}\right)}{x^3 + x^4 + x^5 + x^6 + \cdots + x^{199}},$$

which simplifies to x.

8. 32. The middle term of $(a + b)^2$ is $2ab$. Therefore, $2ab$ must equal $60x^3$, so $ab = 30x^3$. There are four factor pairs for 30, so the numerical coefficients of a and

b are 1 and 30; 2 and 15; 3 and 10; or 5 and 6. Both coefficients in a pair could be negative integers, too. For the exponents on x, there are four possibilities: 0 and 3; 1 and 2; 2 and 1; or 3 and 0. In this case, order *does* matter. The list that follows shows the possibilities when the coefficients are 5 and 6: $(5 + 6x^3)$, $(5x + 6x^2)$, $(5x^2 + 6x)$, and $(5x^3 + 6)$. Count eight possibilities, since the coefficients can both be negative. Count a total of 32 possibilities, since we found four pairs of coefficients that work.

9. 6 cows. Suppose that all the animals were horses and cows. Then the 24 animals would have 96 feet. Removing a horse or a cow and replacing it with a chicken decreases the number of feet by 2. We need to decrease the feet count by 20, so add 10 chickens and decrease the total of horses and cows to 14. Since Farmer Frank has 2 more horses than cows, he must have 8 horses and 6 cows.

Alternate solution: Let c be the number of chickens, let h be the number of horses, and let $h - 2$ be the number of cows. Then $c + h + h - 2 = 24$ and $2c + 4(2h - 2) = 76$. Simplify these to obtain $c + 2h = 26$ and $c + 4h = 42$. Subtracting the first equation from the second results in $2h = 16 \rightarrow h = 8$. Therefore, Farmer Frank has $h - 2 = 6$ cows.

10. We can look for a pair of Pythagorean triples such that the hypotenuse is the same. An example is (7, 24, 25) and (15, 20, 25); another example is (16, 63, 65) and (33, 56, 65). Thus,

$$\frac{7^2+24^2}{15^2+20^2}=\frac{49+576}{225+400}=\frac{625}{625}=1,$$

and

$$\frac{16^2+63^2}{33^2+56^2}=\frac{256+3969}{1089+3136}=\frac{4225}{4225}=1.$$

We can make a simple argument to prove that such pairs of triples are infinite. If a given set of values $\{A, B, C, D\}$ satisfies the equation, then $\{kA, kB, kC, kD\}$ satisfies it as well. We have

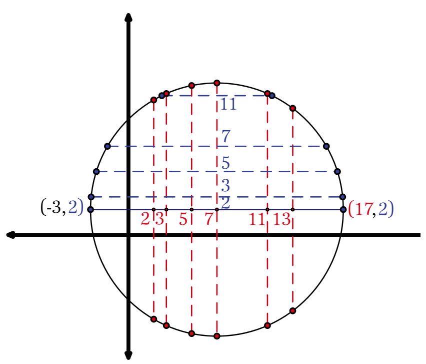
$$\frac{(kA)^2+(kB)^2}{(kC)^2+(kD)^2}=\frac{k^2(A^2+B^2)}{k^2(C^2+D^2)}=\frac{(A^2+B^2)}{(C^2+D^2)}=1.$$

There are an infinite number of positive integers k, so there are an infinite number of solutions.

11. 22. We are interested in points that have at least one prime coordinate, so it is not necessary to find both coordinates for each point. Complete the squares to rewrite the equation of the circle:

$$x^2+y^2-4y-14x=47$$
$$x^2-14x+49+y^2-4y+4=47+49+4$$
$$(x-7)^2+(y-2)^2=100$$

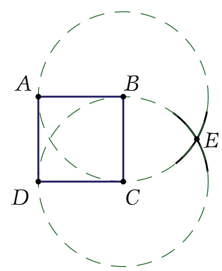
The circle has center (7, 2) and radius 10. The endpoints of a horizontal diameter are (17, 2) and (–3, 2). The endpoints of the vertical diameter of the circle are (7, 12) and (7, –8). Each prime number between 17 and –3—namely, 2, 3, 5, 7, 11, and 13—will correspond to two points on the circle whose abscissa is prime (the red points in the figure), and each of the prime numbers between –8 and 12—namely 2, 3, 5, 7, and 11—will correspond to two points whose ordinate is prime (the blue points in the figure). Note that the point (17, 2) is the only point that has two prime coordinates. There are 22 points.



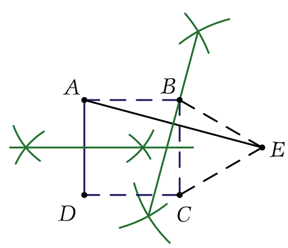
12. C, F. The integer 445,382,688, listed as C, cannot be a perfect square because a perfect square cannot have the digit 8 in the units place. Perfect squares have only the digits 1, 4, 9, 6, 5, and 0 in the units place. However, if a perfect square ends in zero, then the tens digits must also be zero. Therefore, the integer 484,664,755,610, listed as F, cannot be a perfect square.

13. 1. Students' ability to construct a square using a straightedge and compass or software construction program is assumed. Construct the equilateral triangle given side $\overline{BC}$ by constructing two circles of radius BC, one with center B and one with center C (see **fig. a**). Their intersection outside the square marks the location of vertex E. The center of circle O must be equidistant from points A, D, and E; hence, O is the circumcenter of $\triangle ADE$. Construct any two sides of $\triangle ADE$; then construct their perpendicular bisectors. (**Fig. b** shows the bisectors of $\overline{AE}$ and $\overline{AD}$.) The bisectors intersect at O. Finally, construct the circle using $OA = OD = OE$ as the radius.

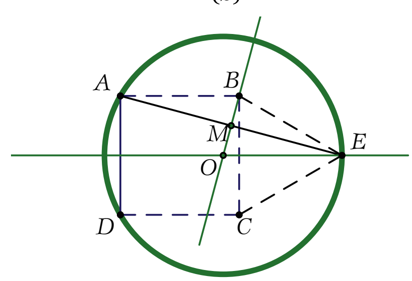
To find the radius of the circle that passes through points A, D, and E, note that $\triangle ABE$ is isosceles with leg lengths $AB = BE = 1$ and base AE. The perpendicular bisector of $\overline{AE}$ (see **fig. c**) necessarily contains point B. Let M be the midpoint of $\overline{AE}$; then $AM = ME$. The perpendicular bisector of $\overline{AD}$ necessarily contains point E, and it is parallel to $\overline{AB}$. The alternate interior angles $\angle ABM$ and $\angle EOM$ are congruent. We know that $\triangle ABM \cong \triangle EOM$ by AAS congruence, implying that $OE = AB = 1$. Thus, the radius is 1.



(a)



(b)

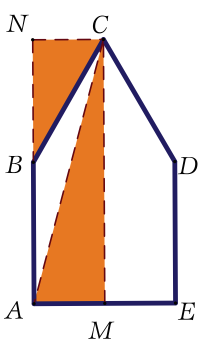


(c)

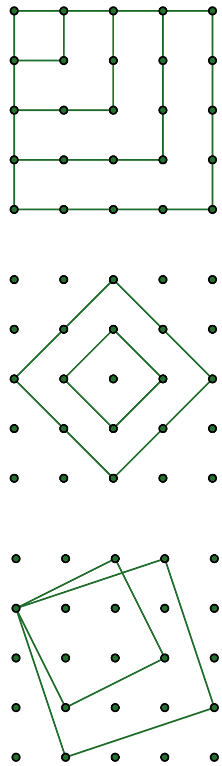
14. 1. The method of construction is the same as the method of construction given for March 13, except that vertex E lies in the interior of the square. The reasoning used for March 13 can be repeated here to find the radius of circle O. Surprisingly, the radius is again 1, although the inscribed triangle here, $\triangle ADE$, is not congruent to $\triangle ADE$ in the March 13 problem.

15. 1/4. There are many approaches to this problem. Consider one that uses trigonometry and another that requires only geometry. In a triangle, if we know the lengths of two sides and the measure of the included angle, we can use the area formula $A = (ab \cdot \sin\theta)/2$. In this case, $a = b = 1$, and $\theta = 90° + 60° = 150°$. The area is $A = (1 \cdot 1 \cdot \sin 150°)/2 = (1/2)(1/2) = 1/4$.

The second approach refers to the figure. Construct a line parallel to $\overline{AE}$ through point C and extend $\overline{AB}$ to intersect that line at N. Let M be the midpoint of $\overline{AE}$. The figure shows rectangle $ANCM$ divided into three triangles: $\triangle ACM$, which has half the area of the rectangle; $\triangle BCN$, which has area equal to half that of $\triangle BCD$; and $\triangle ABC$, the area of interest. Find the height of $\triangle BCD$ using special right-triangle relationships or the Pythagorean theorem. Its height is $\sqrt{3}/2$, so rectangle $ANCM$ has area $(\sqrt{3}/2 + 1) \cdot (1/2)$. Therefore, the area of $\triangle ACM$ (or, equivalently, $\triangle ANC$) is $(\sqrt{3} + 2)/8$. Subtract the area of $\triangle BCN$ from it. The area of equilateral triangle BCD is $1^2\sqrt{3}/4$, so the area of $\triangle BCN$ is $\sqrt{3}/8$. Subtracting gives us area 1/4, as before.



16. 8. Four different sizes of squares can be constructed with vertical and horizontal sides. Additional squares can be constructed with sides of lengths $\sqrt{2}$, $\sqrt{5}$, $\sqrt{8} = 2\sqrt{2}$, and $\sqrt{10}$, as shown in the figures.



17. 28. If the average for the group of 6 is 18 text messages per hour, then the group sends a total of 18(6) = 108 messages per hour. If Finley's messages are not counted, then the remaining 5 friends send 16(5) = 80 messages per hour. The difference is Finley's rate: 28 messages per hour.

18. 2/3. Once John observes that he has pulled a blue marble from his pocket, only three scenarios can describe this chance experiment. The first possibility is that the original marble is blue and John selected the added marble. The second possibility is that the original marble is yellow and John selected the added marble. The third is that the original marble is blue and John selected the original marble. In two of these three equally likely scenarios, the marble still in John's pocket is blue.

19. ± 40. If the equation has a double root, then the left side can be rewritten as a perfect square: $16x^2 + bx + 25 = 0 \rightarrow x^2 + bx/16 + 25/16 = 0 \rightarrow (x \pm 5/4)^2 = 0$. Therefore, $b/16 = \pm 10/4$ and $b = \pm 40$.

Alternate solution: If the equation has a double root, then the discriminant, $b^2 - 4ac$, is zero. Set $b^2 - 4 \cdot 16 \cdot 25 = 0 \rightarrow b = \pm 40$. Confirm this answer by solving the equation. We find that when $b = 40$, the double root is –5/4. When $b = -40$, the double root is 5/4.

20. 2077. The first sequence has a common difference 6, and the second has a common difference 13. The least common multiple of 6 and 13 is 78. If k is an element of both sequences, then $k + 78$ will also belong to both sequences. We need to find any term that is common to both sequences. If we write a few additional terms in sequence A = {7, 13, 19, 25, 31, 37, 43, 49, … }, we find that 49 is common to both sequences. Therefore, a multiple, x, of 78 added to 49 gives us a common term. We have $49 + 78 \cdot x > 2015 \rightarrow x > 25.2$. Use $x = 26$ to find the common term:

$$49 + 78(26) = 2077$$

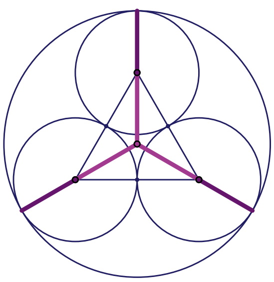
Solution 22

Base	Sum	Representation
2	$2^6 + 2^5 + 2^4 + 2^3 + 1$	1,111,001
3	$3^4 + 3^3 + 3^2 + 3^1 + 1$	11,111
4	$4^3 + 3(4^2) + 2(4) + 1$	1321
5	$4(5^2) + 4(5) + 1$	441
6	$3(6^2) + 2(6) + 1$	321
7	$2(7^2) + 3(7) + 2$	232
8	$8^2 + 7(8) + 1$	171
9	$9^2 + 4(9) + 4$	144

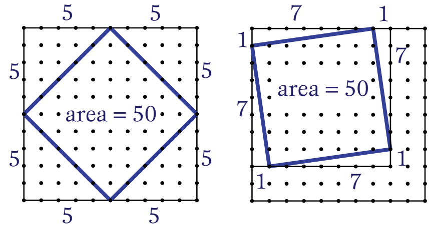
21. $11k$. If the mean of nine numbers is k, their sum is $9k$. The new mean, $(9k + c)/10$, equals $2k$. Solving for c results in $c = 11k$.

22. Base 3, base 7, and base 8. The results of converting 121 to bases from 2 to 9, inclusive, are shown in the table for solution 22. Highlighted rows indicate the palindromes.

23. $(3 + 2\sqrt{3})/3$. By symmetry, the centers of the three congruent circles are the vertices of an equilateral triangle, and the center of that triangle and the center of the circumscribed circle must be the same point. Since the small circles have radius 1, the triangle has side length 2 and height $\sqrt{3}$. The radius of the large circle can be thought of as having two parts: (1) the distance from the center to a vertex of the triangle and (2) the distance from a vertex to the nearest point of tangency of the larger and smaller circles. The former distance is the radius of the equilateral triangle, which is 2/3 the height, or $2\sqrt{3}/3$; the latter distance is the radius of the smaller circle, 1. Thus, the radius of the larger circle is $(3 + 2\sqrt{3})/3$.



24. Since 50 is not a perfect square, the square we need cannot have vertical or horizontal sides. Each side of our square, with length $\sqrt{50}$, will be the hypotenuse of a right triangle with vertical and horizontal (and integral) legs. There are two such right triangles, because there are two ways to write 50 as the sum of two squares: $1^2 + 7^2 = 5^2 + 5^2 = 50$. Note that only one copy of the square framed by the isosceles right triangles will fit in the 11×11 lattice, whereas multiple translations and reflections of the square framed by 1-7-$5\sqrt{2}$ right triangles will fit.



25. 5/12. The table below shows the scores. Highlighted cells are prime. The probability is $15/36 = 5/12$.

	1	4	9	16	25	36
1	2	5	10	17	26	37
4	5	8	13	20	29	40
9	10	13	18	25	34	45
16	17	20	25	32	41	52
25	26	29	34	41	50	61
36	37	40	45	52	61	72

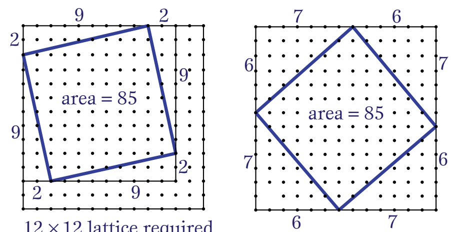
26. $(-1/5, 18/5)$. If P were the midpoint of $\overline{AB}$, then we would find the average of the x-coordinates and the average of the y-coordinates. In this case, we will find a *weighted average* of the x- and y-coordinates. Since $AP:PB = 3:2$, P is closer to B than it is to A. We can, therefore, think of point B as "pulling harder." Point B will have weight 3, and point A will have weight 2; the denominator is the sum of the weights. The coordinates of P are

$$\left(\frac{2(-5)+3(3)}{5}, \frac{2(0)+3(6)}{5}\right) = \left(-\frac{1}{5}, \frac{18}{5}\right).$$

Alternate solution: Point P is 3/5 of the way from A to B. Because vector AB can be written as (8, 6), we find P at $(-5, 0) + (3/5) \cdot (8, 6) = (-1/5, 18/5)$.

27. (1209, 1612, 2015), (775, 1860, 2015), (496, 1953, 2015), and (1023, 1736, 2015). The factors of 2015 and our knowledge of primitive Pythagorean triples will be used to find the four triples. The factors of 2015 (excluding 1 and 2015) are 5, 13, 31, 65, 155, and 403. We know that (3, 4, 5) is a primitive triple, so we multiply each value by $2015/5 = 403$ to obtain (1209, 1612, 2015). Another primitive triple is (5, 12, 13); we multiply each value by $2015/13 = 155$ to obtain (775, 1860, 2015). Since 65 can be written as the sum of two squares in two different ways—namely, $1^2 + 8^2 = 4^2 + 7^2 = 65$—we can use the integer pairs (1, 8) and (4, 7) to generate two Pythagorean triples that have 65 as the largest number. In the first case, $2 \cdot 1 \cdot 8 = 16$, and $8^2 - 1^2 = 63$ for the triple (16, 63, 65). In the second case, $2 \cdot 4 \cdot 7 = 56$, and $7^2 - 4^2 = 33$ for the triple (33, 56, 65). Multiply both triples by $2015/65 = 31$ to obtain (496, 1953, 2015) and (1023, 1736, 2015).

28. 12×12 lattice. Note that 85 can be written as the sum of two squares in two different ways: $2^2 + 9^2 = 6^2 + 7^2 = 85$. Therefore, the right triangles that frame the lattice square have legs 2 and 9 or 6 and 7. Since the perimeter of a right triangle with legs 2 and 9 is less than that of a right triangle with legs 6 and 7, the smallest lattice required will have $2 + 9 + 1 = 12$ dots on a side.



12×12 lattice required

14×14 lattice required

29. $n(n + 1)(2n + 13)/6$ or, equivalently $S_n = n^3/3 + 5n^2/2 + 13n/6$. Each term in the series has the form $k(k + 4)$. Using summation notation, we are looking for

$$\sum_{k=1}^{n}\left(k^2 + 4k\right) = \sum_{k=1}^{n} k^2 + 4\sum_{k=1}^{n} k.$$

These two sums have formulas that should be familiar to students in precalculus. Eliminating the summation notation, we have

$$\frac{n(n+1)(2n+1)}{6} + \frac{4n(n+1)}{2}.$$

Find a common denominator and combine like terms to obtain $n(2n^2 + 15n + 13)/6$ or $n(n + 1)(2n + 13)/6$ in factored form.

Solution 29 (alternate solution)

Term Number	1	2	3	4	5	6
Sum	5	17	38	70	115	175
First differences		12	21	32	45	60
Second differences			9	11	13	15
Third differences				2	2	2

Alternate solution: Another approach uses the method of finite differences. Write the sums for $n = 1$ to, say, $n = 6$: 5, 17, 38, 70, 115, and 175. Then find the differences between successive sums. Next, find the differences between successive differences. Repeat until the differences are all the same. (If the differences never reach a constant, the series cannot be expressed as a polynomial function of n.) Differences are shown in the table for solution 29.

Because the third differences are (a nonzero) constant, the series can be described by a cubic polynomial. The general model is $an^3 + bn^2 + cn + d = S_n$. Create a system of four equations to solve for the four coefficients above:

$$
\begin{aligned}
a(1)+b(1)+c(1)+d&=5\\
a(2^3)+b(2^2)+c(2)+d&=17\\
a(3^3)+b(3^2)+c(3)+d&=38\\
a(4^3)+b(4^2)+c(4)+d&=70
\end{aligned}
$$

$$
\rightarrow\quad
\begin{aligned}
7a+3b+c&=12\\
19a+5b+c&=21\\
37a+7b+c&=32
\end{aligned}
$$

$$
\rightarrow\quad
\begin{aligned}
12a+2b&=9\\
18a+2b&=11
\end{aligned}
$$

$$\rightarrow a=\frac{1}{3}$$

To complete the solution, we determine that $b = 5/2$, $c = 13/6$, and $d = 0$. The cubic model is $S_n = n^3/3 + 5n^2/2 + 13n/6$, which is equivalent to the previous result.

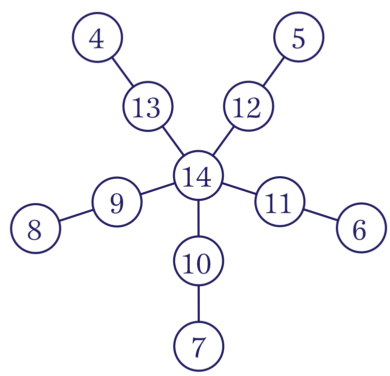
30. Row 45. One approach looks at the first integer in each row; we need two consecutive first integers such that 2015 lies between them. Observe the differences between consecutive first integers: $3 - 1 = 2$; $7 - 3 = 4$; $13 - 7 = 6$; $21 - 13 = 8$. These differences increase by 2; that is, the second differences are constant. Therefore, we can write a quadratic model to link the first term in a row to the row number. Compare the row number squared to the first integer: 1^2 and 1; 2^2 and 3; 3^2 and 7; 4^2 and 13; and 5^2 and 21. The square is a little bit too big in each case; subtracting 1 less than the row number gives the correct value. (The method of finite differences, shown in detail in the previous problem, can be used again here.) Because 45 is the integer closest to $\sqrt{2015}$, we will begin by finding the first integer in row 45. If n is the row number, then our model gives us $n^2 - (n - 1) = 45^2 - 45 + 1 = 1981$. The next row begins with $46^2 - 46 + 1 = 2071$. So 2015 appears in row 45.

31. The method of trial-and-success is probably the fastest road to solving this puzzle, and that method is not without benefit for students. What follows are some comments on a logical, systematic approach.

Focus on the integer in the center circle. Suppose that it is 1. Then one spoke must have the integer 2, and that spoke will require 28 to have a sum of 31. But we are required to use eleven consecutive integers. So the center circle cannot be 1. Use a similar argument to show that the middle number must be at least 7.

Can we find an upper bound for the number in the center circle? Suppose that it is 17. Then one spoke must have the integer 16. But those two integers have a sum larger than 31, so the center number must be less than 17. Will 16 work in the center? One spoke must have 15, and that spoke will require 1 to have a sum of 31. But this solution violates the requirement of eleven consecutive integers. For the same reason, we cannot have 15 in the center.

We have managed to reduce the number of possibilities for the center integer. It must be greater than 7 and less than 15. Trying 14 in the center gives us a solution, as shown here.



NATIONAL COUNCIL OF
TEACHERS OF MATHEMATICS

THE NATION'S PREMIER MATH EDUCATION EVENT

2015 NCTM ANNUAL MEETING & EXPOSITION

April 15–18 • Boston

EFFECTIVE TEACHING TO ENSURE MATHEMATICAL SUCCESS FOR ALL

PULLING TOGETHER

Join NCTM in Boston as we bring together thousands of education professionals for the nation's largest math education event.

Go beyond the classroom to examine the innovative ideas that can improve the quality of learning for every student.

- Gain insights into implementation and assessment of the **Common Core State Standards** for Mathematics.
- Learn about ***Principles to Actions: Ensuring Mathematical Success for All***, which describes what is essential for a high-quality mathematics education.
- Collaborate with peers on concepts that diversify learning and **support student learners**.
- Explore **more than 700 sessions** to help you grow and learn as a teacher.
- Discover fresh ways to **integrate mathematics into other disciplines**.

WHO SHOULD ATTEND?

- Pre-K–12 teachers
- Math teacher educators
- New and soon-to-be-teachers
- Math coaches and specialists
- Math researchers
- School and district administrators

Learn more at **www.nctm.org/boston** and follow us on

#NCTMBOSTON