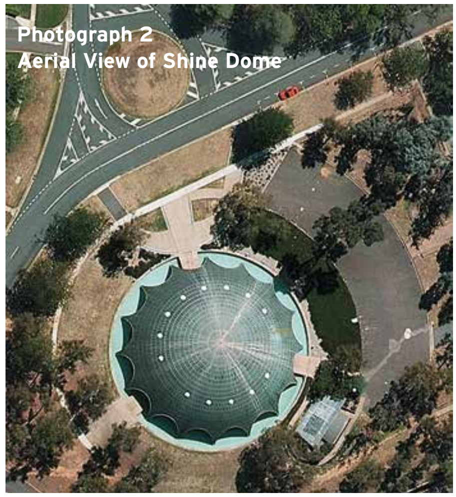
Douglas Wilcock

Shine On, Shine Dome

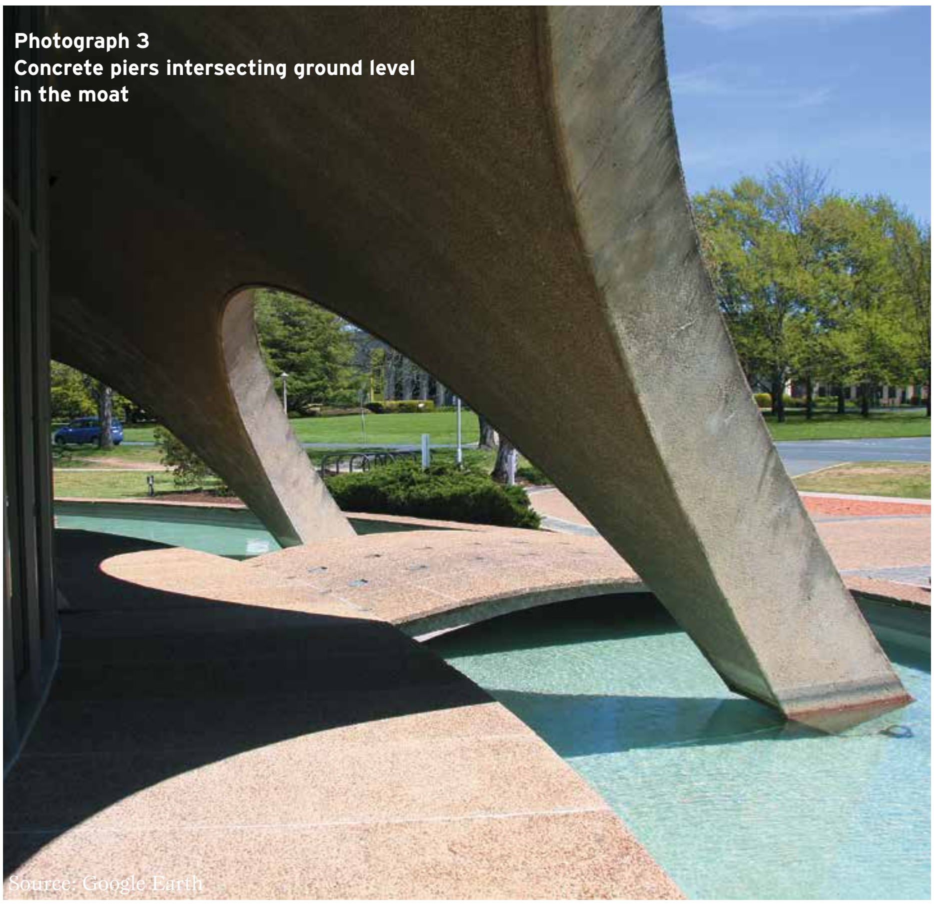
Shine Dome (**photographs 1** and **2**), named for the Australian biologist John Shine, a major contributor to the building's renovation, is a striking landmark in the Australian capital city of Canberra. Constructed in 1959 as the home of the Australian Academy of Science, the dome is a mathematically interesting structure featuring arches and a surrounding moat. The dome, a spherical cap, is supported by concrete piers grounded in the moat.

The most striking architectural features of Shine Dome are the sixteen arches at its base. They permit light, some of which is reflected from the water in the moat, to enter the building, and the net effect is a light, airy feel. The building's load is placed on sixteen piers located between the arches, thus giving a massive concrete structure an architectural grace. Further, the shadows cast by the arches add an additional pleasing aesthetic (see **photograph 3**).

Photograph 1
Shine Dome, Canberra, Australia



Photograph 2
Aerial View of Shine Dome



Photograph 3
Concrete piers intersecting ground level in the moat

Source: Google Earth

Mathematical Lens uses photographs as a springboard for mathematical inquiry and appears in every issue of *Mathematics Teacher*. All submissions should be sent to the department editors.

Department editors
Ron Lancaster, *ron2718@nas.net, University of Toronto, ON, Canada, and* ***Brigitte Bentele,*** *brigitte.bentele@trinityschoolnyc.org, Trinity School, New York, NY*

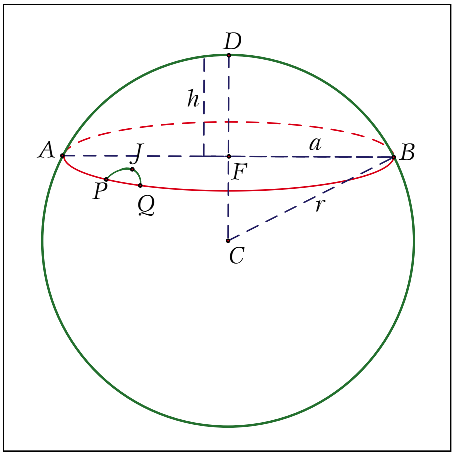


Fig. 1 A spherical cap represents the dome with its circular base through points *A* and *B*.

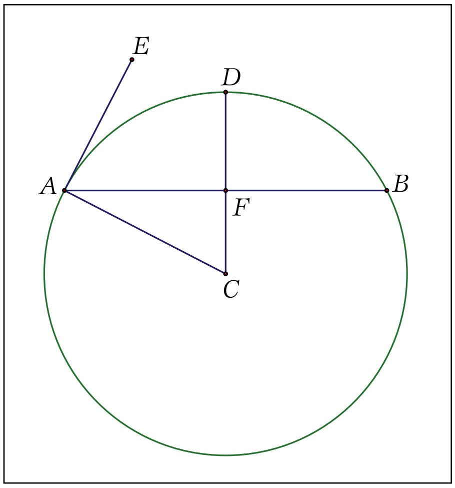


Fig. 2 A section shows *AE* as a tangent to the dome at ground level.

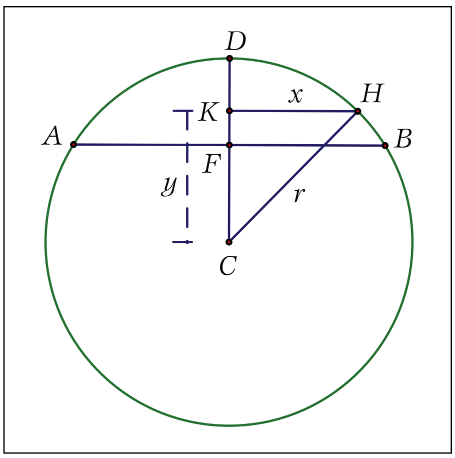


Fig. 3 Related lengths can be used to find the surface area of the dome.

1. Shine Dome is contained in a part of a sphere called a spherical cap (see **fig. 1**). If we cut through the dome perpendicularly to the circular floor, we get a section showing the height of the dome, DF, and the diameter of the circular floor, represented by segment AB. In **figure 2**, a tangent line to the dome at the ground level is labeled AE, and two radii of the sphere are AC and CD. The diameter of the dome, AB, is 45.75 meters.

   (*a*) Determine a formula for the height of the dome as a function of the radius of the sphere.

   (*b*) Restrict the height of the dome such that $DF \leq DC$ and modify your answer to part (*a*).

   (*c*) What is the height of the dome when the radius of the sphere is 22.875 m? How would you describe the shape of the dome?

2. The height of the dome represented in **figure 2** can be calculated as a function of the angle formed by the tangent line AE and the floor of the dome AB.

   (*a*) Determine the height of the dome if the measure of $\angle EAB$ is 30°, 45°, 60°, and 90°. List the results in **table 1**. (The radius of the sphere [see **table 1,** col. 2] will be an intermediate calculation in this process.)

   (*b*) Determine a general formula for the height of the dome as a function of $\theta = m\angle EAB$ and enter your formula in the last line of **table 1**. (Column 4 of **table 1** will be completed in question 4.)

   (*c*) If we restrict the domain of the function in part (*b*) to be positive values less than or equal to 90°, what is the range of the function?

3. (*a*) Refer to **figure 3**, a section of the dome, where segment AB represents the floor of the dome, and FD is the height of the dome. Use the center of the dome, C, and the (variable) point K on segment FD to define a point H (at the same height as K but on the surface of the dome) and variables $x = KH$ and $y = CK$ in right triangle CHK. Solve for x as a function of y and r, the radius of the sphere. Notice that x is the radius of a circular disk parallel to the floor at the same height as point K.

   (*b*) The formula for the (outside) surface area of the spherical cap is $SA = 2\pi rh$, where r is the radius of the sphere and h is the height of the dome. Develop this formula (using calculus).

4. Complete the last column in **table 1**, using the formula in question 3. That is, find the surface area of the dome for each angle listed.

Table 1 Dome Measurements for Questions

Tangent Angle $m\angle EAB$	Radius of Sphere $r = AC$ or DC (in m)	Height of Dome $h = DF$ (in m)	Gross Surface Area of Dome (in m^2)
30°			
45°			
60°			
90°			
θ			

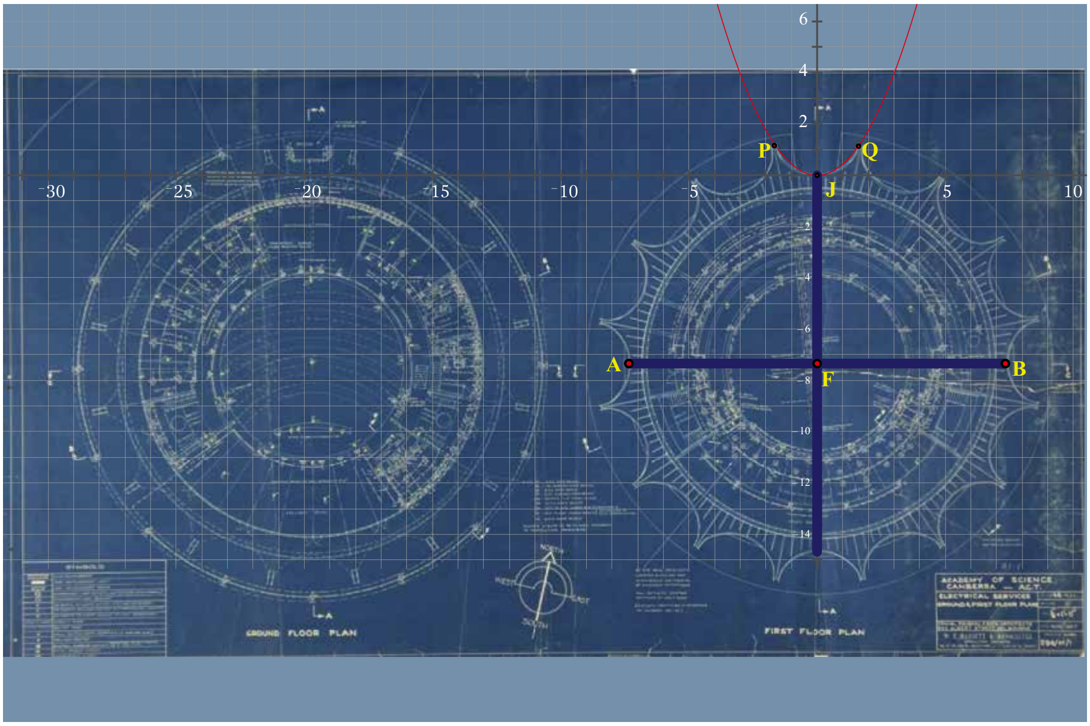


Fig. 4 A blueprint of dome shows a parabola representing one of the arches.

5. **Figure 4** shows a blueprint of the dome and a parabola superimposed over one of the arches. Point F is located at the center of the dome's floor; point J is a projection of the apex of the arch, or the vertex of the parabola, onto the floor; and point Q is located at the foot of the pier where the arch meets ground level. The diagram coordinates (in cm) of each point are F (0, –7.34), J (0, 0), and Q (1.68, 1.24). Point P is a reflection of point Q across the axis of symmetry of the parabola.

   (*a*) Segment FQ represents a radius of the dome's circular floor with an actual length of 22.875 m. Find the length (using coordinates in cm) of FQ on the blueprint and determine a scaling factor for the other coordinates and actual distances (in m).

   (*b*) Determine the area enclosed by one parabolic arch—that is, the region bounded by segment PQ and the parabola. The area of a parabolic region can be calculated with a formula as 2/3 of the area of the smallest rectangle containing the parabola or using calculus.

6. The answer that was determined in question 5 represents the area of a parabolic arch projected on the ground. However, the plane containing the dome's arch forms an angle with the ground. Assume that the plane of the parabolic arch forms a 45° angle with the ground and that the apex of the arch, G, is directly above the vertex of the parabola on the ground, J.

   (*a*) Determine the area of one arch in the tilted plane.

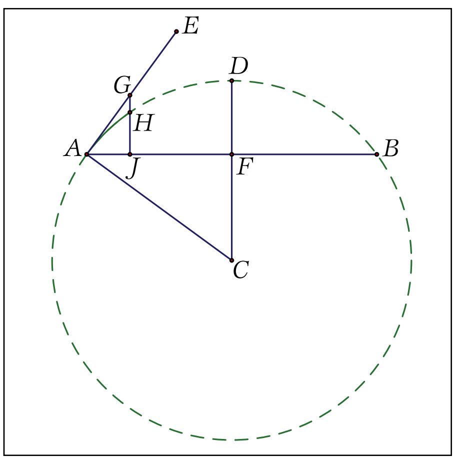


Fig. 5 The dome's arches could be represented by parabolas with vertices at G or H.

   (*b*) Determine the net surface area of the dome that results when 16 arches are removed.

   (*c*) Determine the dome's weight if its average thickness is 20 cm and the reinforced concrete weighs approximately 2500 kg/m^3. Convert the answer to metric tons (1 metric ton = 1000 kg).

   (*d*) At ground level, the piers are squares with side lengths of 0.6 m. Determine the load on each of the 16 piers in kg per pier and as metric tons per m^2.

7. The calculations in question 6 determine the area of a parabolic arch with an axis of symmetry represented in **figure 5** as AG. Because of the curvature of the dome, other approximations for the area would use the axis of symmetry of the parabola along segment or arc AH (see **fig. 5**).

   (*a*) Assume that, in **figure 5**, m$\angle EAB$ = 45° and find the position of H, relative to F, the point in the center of the dome's circular floor.

   (*b*) Use the results of your calculations in part (*a*) to determine the arc length of arc AH and the area of a parabolic arch with arc AH as its axis of symmetry.

   (*c*) Compare the parabolic-arch areas calculated with axis of symmetry AJ, AG, and AH. What are the safety implications for the engineers who designed the building?

   Readers might also consider another approximation that uses slices perpendicular to the dome's altitude so that the arch width at any height could be approximated as a circular arc length rather than the width of a parabola.

1. (a) In **figure 2**, we solve for the dome's height, h, by using the sphere's radius, r, and the relationship among the sides in right triangle AFC:

$$r^2 = \left(r-h\right)^2 + 22.875^2$$
$$r^2 = r^2 - 2rh + h^2 + 22.875^2$$
$$0 = h^2 - 2rh + 22.875^2$$
$$h = r \pm \sqrt{r^2 - 22.875^2}$$

(b) The dome's height would be the difference (using subtraction):

$$h = r - \sqrt{r^2 - 22.875^2}$$

Addition would make the height greater than the radius and put the dome floor in the bottom half of the surrounding sphere.

(c) If the radius is 22.875 m, the dome's height simplifies to r, or 22.875 m, and the resulting shape is a hemisphere.

2. (a) See **table 2**.

(b) Because segments AE and CF are perpendicular to each other, the measure of $\angle ACF$ is θ, so $\sin\theta = 22.875/r$. We use an expression for $\cos\theta$ to solve for h:

$$\cos\theta = \frac{r-h}{r}$$
$$r\cos\theta = r - h$$
$$h = r - r\cos\theta$$
$$h = r\left(1-\cos\theta\right).$$

(c) The range of the function h is from 0 to 22.875 m, the latter representing a tangent angle of 90°.

3. (a) The relationship is given by the Pythagorean theorem. Solving for x, we get

$$x = \sqrt{r^2 - y^2}.$$

(b) The surface area can be found by integrating the circumference of stacked circular disks with varying radius x (where x is as a function of y found in part (a)) along sections of arc ds—that is, by applying the formula

$$SA = \int 2\pi \bullet x \bullet ds$$

with appropriate limits of integration. You may find s as a function of y from $s = r\theta = r\sin^{-1}(y/r)$, so

$$\frac{ds}{dy} = \frac{r}{\sqrt{r^2 - y^2}};$$

thus, the integral becomes

$$\int 2\pi x\, ds = \int 2\pi\sqrt{r^2 - y^2}\,\frac{r}{\sqrt{r^2 - y^2}}\,dy\,.$$

Alternatively, you may use the formula

$$ds = \sqrt{1 + \left(x'\right)^2}\,dy.$$

Then, taking a derivative of x with respect to y, we have

$$x' = \frac{-y}{\sqrt{r^2 - y^2}}.$$

Thus,

$$\begin{aligned} ds &= \sqrt{1 + \left(x'\right)^2}\,dy \\ &= \sqrt{1 + \frac{y^2}{r^2 - y^2}}\,dy \\ &= \sqrt{\frac{r^2}{r^2 - y^2}}\,dy \\ &= \frac{r}{\sqrt{r^2 - y^2}}\,dy\,, \end{aligned}$$

which will lead to the same integral.

The limits of the integral will correspond to points F and D; thus, the surface area becomes

$$\begin{aligned} SA &= \int_{s(F)}^{s(D)} 2\pi x\, ds \\ &= \int_{y(F)}^{y(D)} 2\pi\sqrt{r^2 - y^2}\,\frac{r}{\sqrt{r^2 - y^2}}\,dy \\ &= \int_{r-h}^{r} 2\pi r\, dy \\ &= 2\pi r\left(r - \left(r-h\right)\right) \\ &= 2\pi rh. \end{aligned}$$

4. See the results in **table 2**.

5. (a) In the blueprint (see **fig. 4**), FP and FQ are the radii of the circular floor of the dome measuring 22.875 m in actual length. To find the scaling factor, we first calculate the blueprint length of FQ:

Table 2 Dome Measurements Depending on Tangent Angle at the Base

Tangent Angle m$\angle EAB$	Radius of Sphere r = AC or DC (in m)	Height of Dome h = DF (in m)	Gross Surface Area of Dome (in m^2)
30°	45.75	6.13	1761.91
45°	32.35	9.48	1925.93
60°	26.41	13.21	2191.85
90°	22.875	22.875	3287.77
θ	$r = 22.875/\sin\theta$	$h = r(1 - \cos\theta)$	

$$FQ = \sqrt{1.68^2 + (1.24 - (-7.34))^2} \approx 8.743$$

Then we use

$$\frac{22.875 \text{ actual length in m}}{8.74 \text{ blueprint length in cm}} \approx 2.617.$$

(Values are rounded to answer the questions, but nonrounded values are used at the intermediate calculations.)

(*b*) The distance from the vertex of the parabola, point J, to line segment PQ measures 1.24 cm in the blueprint (see **fig. 4**) and represents an actual distance of 1.24 • 2.617 ≈ 3.244 m. Segment PQ represents an actual distance of (1.68 – (–1.68)) • 2.617 ≈ 8.791 m. The product of these lengths multiplied by 2/3 gives a parabolic-arch area of 19.014 m^2.

Using calculus, we can evaluate an integral to determine the area. Because Q lies on a parabola with vertex (0, 0), its coordinates are scaled from (1.68,1.24) in cm on the blueprint to a corresponding position (4.396, 3.244) in m on the parabola with an equation

$$y = 0.1679x^2 \text{ or } x = \sqrt{5.955y}.$$

Integrating with respect to y, we determine the area of the parabolic section to be

$$Area = 2\int_0^{3.244} \sqrt{5.955y}\, dy \approx 19.014.$$

6. (*a*) If the parabola is tilted at an angle of 45° to the ground and stretched so that the apex of the arch, G, is directly above point J on the ground, then the effective height of the parabolic section is multiplied by $\sqrt{2}$. That is, $AG \approx 4.588$ m. Because the base, PQ, of the section is unchanged, the area of slanted arch is $\sqrt{2}$ times 19.014, or approximately 26.89 m^2.

(*b*) In question 3 (*b*), we determined that the surface area of the spherical cap is 1925.93 m^2. We now remove 26.89 m^2 for each of the 16 arches, or 430.24 m^2 total. The net surface area for the dome is approximately 1495.69 m^2.

(*c*) The volume of the dome is approximately 1495.69 • 0.2 = 299.14 m^3. This volume gives a weight of approximately 282.857 • 2500 = 747845 kg = 747.845 metric tons.

(*d*) This weight distributes as 747845/16 ≈ 46740 kg per pier, or 46740/0.36 ≈ 129834 kg per m^2 ≈ 129.8 metric tons per m^2. This is a tremendous load, so it is not surprising to learn that a scale model (1/40 scale) was built before construction of Shine Dome to determine whether the piers could carry the load.

7. (*a*) A and H are points on the circle centered at C with radius 22.875 • $\sqrt{2}$ ≈ 32.35 m. We know that the distance CF = 22.875 m, and in the solution to question 5(*b*), we determined that the length AJ is 3.244 m. Thus, FJ = 22.875 – 3.244 = 19.631 m. It follows that vertical distance from C to H is

$$\sqrt{32.35^2 - 19.63^2} \approx 25.71 \text{ m},$$

so the point H is 25.71 – 22.875 = 2.84 m above the floor.

(*b*) To determine arc length AH, use the x-coordinates (relative to point C) for points H and A as –19.63 and –22.875, respectively. Then, points H and A correspond to rotations of $\cos^{-1}(-19.63/32.35)$ ≈ 127.36° and 135°, respectively. Thus, $m\angle ACH \approx 7.64°$, and

$$\text{arc } AH \approx \frac{7.64}{360} \cdot 2\pi \cdot 22.875\sqrt{2} \approx 4.314.$$

The area of the arch, then, is (2/3) • 8.791 • 4.314 ≈ 25.283 m^2.

(*c*) In calculating the parabolic arch areas as 2/3 the product of a base length PQ times a height, we found the heights $AJ \approx 3.244$, $AG \approx 4.588$, and arc $AH \approx 4.314$. Had the engineers used the area in the 45° plane (with parabola vertex at G), they would have underestimated the actual load on the piers, a dangerous simplifying assumption.

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DOUGLAS WILCOCK, dougw_96@comcast.net, retired as a mathematics, statistics, and economics teacher at Cape Cod Academy in Osterville, Massachusetts. He remains interested in finding mathematical applications in the world around us.

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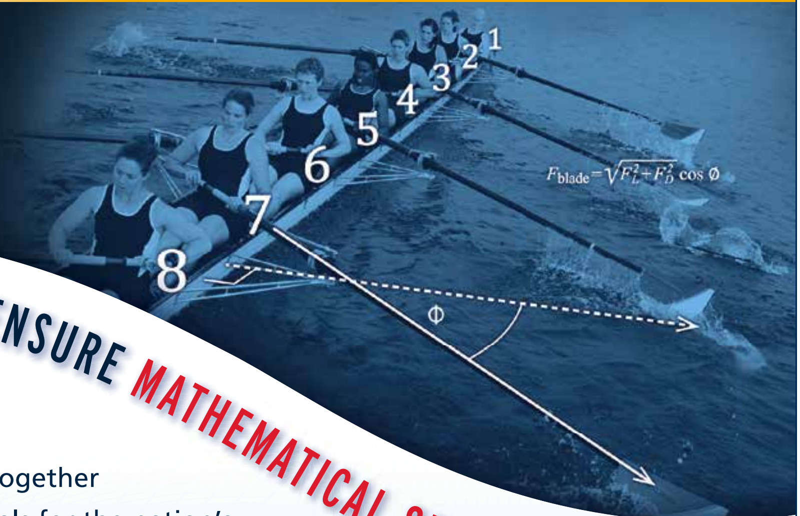

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