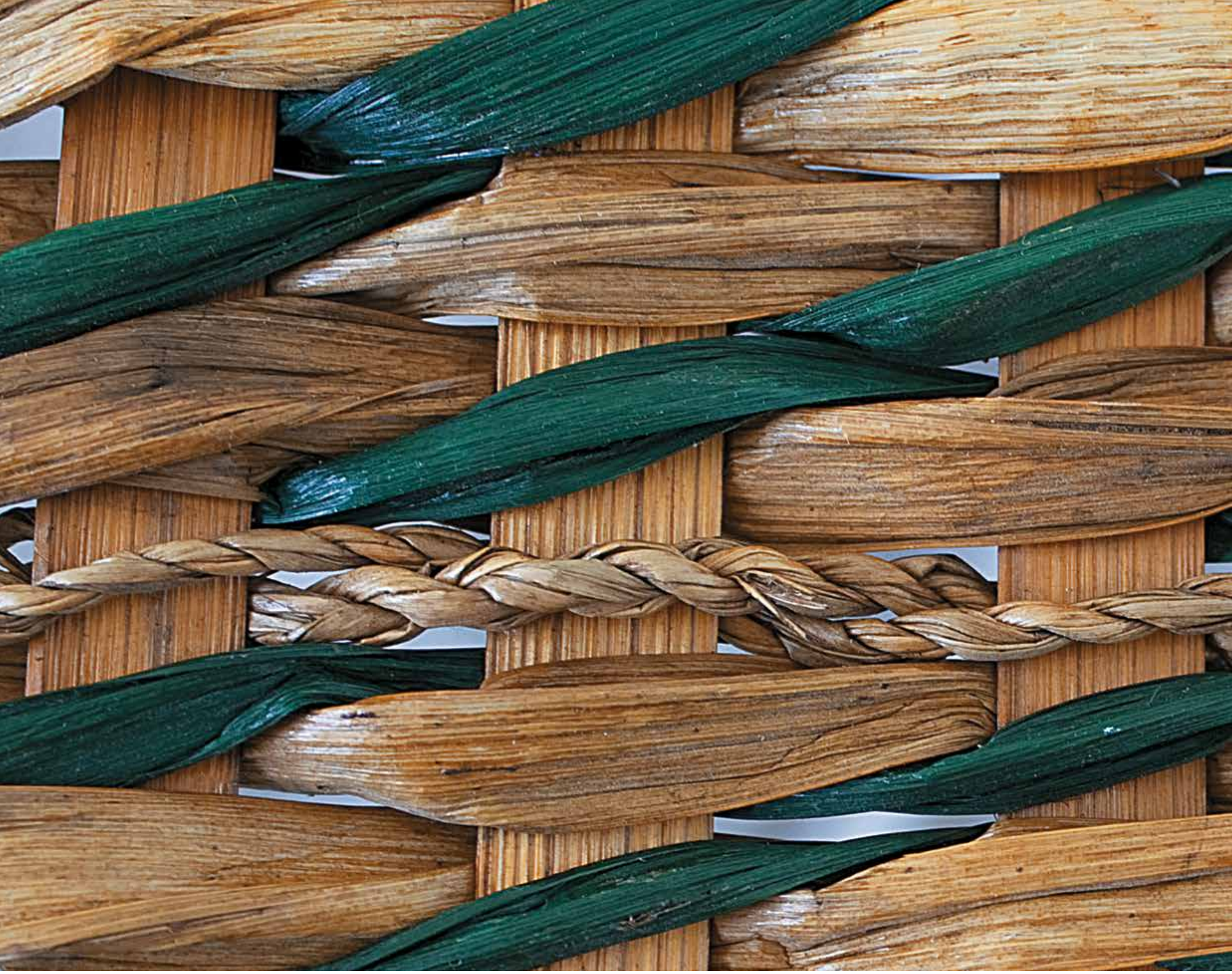
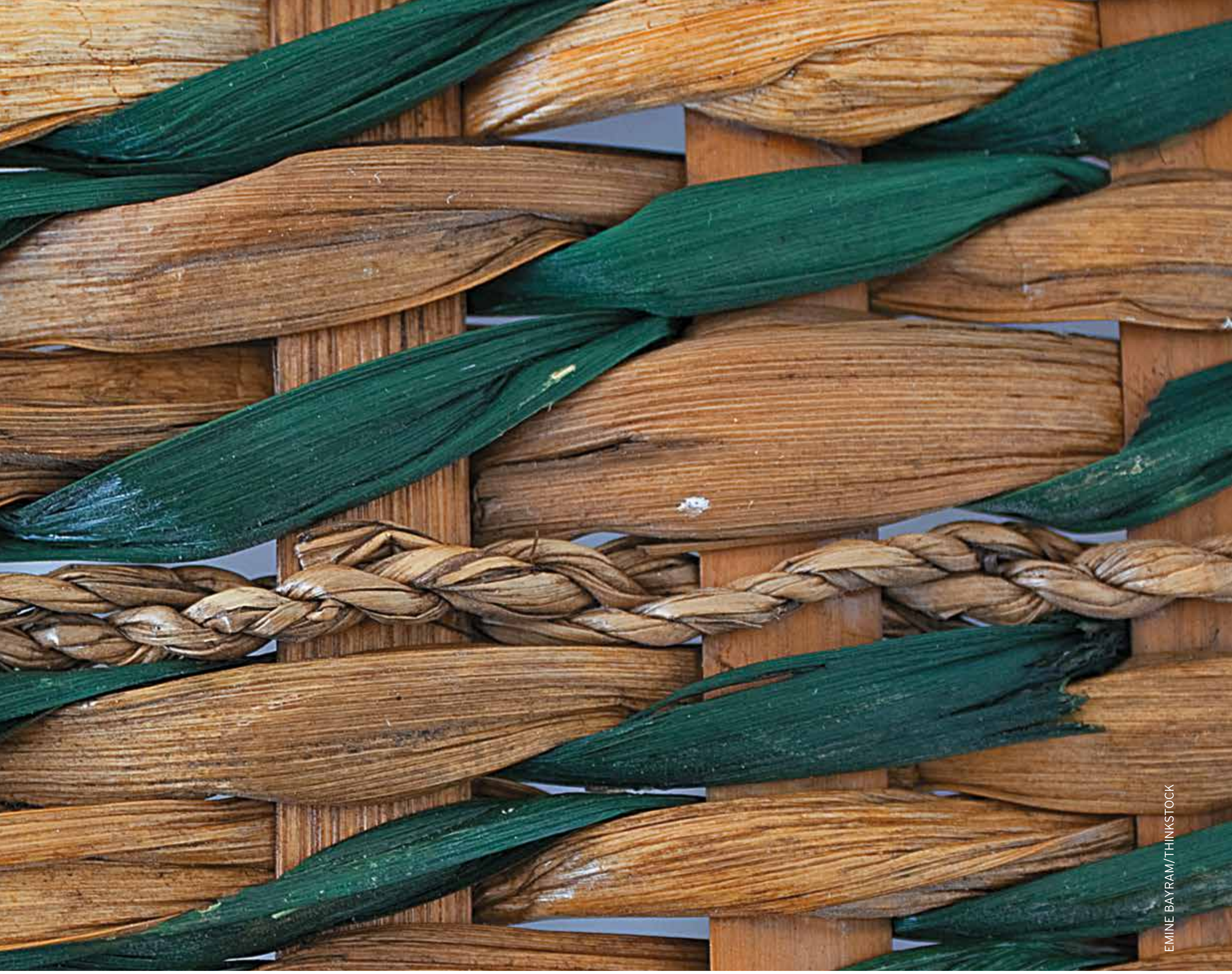


# Weaving Geometry and

Michelle Cetner

When thinking about student reasoning and sense making, we teachers must consider the nature of tasks given to students along with how we plan to use the tasks in the classroom. Students should be presented with tasks in a way that encourages them to draw connections between algebraic and geometric concepts. To this end, a dynamic geometry program can be used with high school mathematics students to convey standard algebra and geometry concepts in a connected way through the use of carefully constructed questions and directed discussions. The platform used in the following examples is The Geometer's Sketchpad®, but the tasks could be adapted to any dynamic geometry software platform.

The use of tasks that encourage both visual and analytic reasoning has been shown to enable students to be better problem solvers and lead to better long-term conceptual memory (Banchoff 2008; Chinnappan and Lawson 2005; NCTM 2003; Noss, Healy, and Hoyles 1997). One reason may be that the concurrent and connected use of both visual and analytic thinking about a problem encourages more connections to be made between the left and right halves of the brain (Battista 2007; Thornton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# Algebra Together

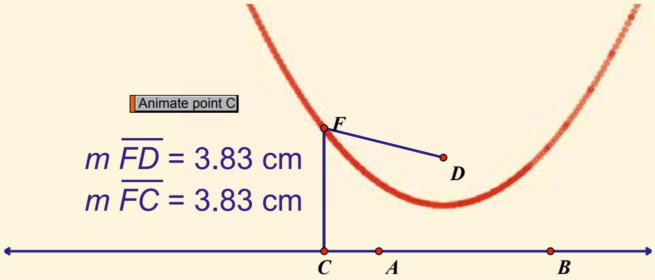
*Two activities help create a tapestry of connections.*

2001). Engaging students with connected activities in which both visual and analytic reasoning skills are used instead of giving students tasks that are entirely of one nature or the other greatly benefits all students, particularly English language learners and those with multiple learning styles (Thornton 2001).

In *Principles and Standards for School Mathematics* (NCTM 2000), the Connections Standard includes recognizing and understanding connections among mathematical ideas and how those ideas build on one another to produce a coherent whole (pp. 64–66). The Common Core State Standards for Mathematics (CCSSI 2010) do not make any broad statements about connecting different mathematical concepts, but making meaningful connections between mathematical strands is implied through its discussion of the mathematical practice standards of modeling, structure, and repeated reasoning (e.g., pp. 8–72), along with numerous specific examples within the content standards portion of the document (e.g., pp. 66–67, 69). Together, these documents demonstrate the mathematics education community's commitment to helping students form deep mathematical connections across all topics in school mathematics.

Two ideas prevail about the teaching of geometry

## THE LOCUS OF POINTS EQUIDISTANT BETWEEN A POINT AND A LINE




1. Select the arrow tool and then **Edit**, **Preferences**, and **Text**. Check the box labeled as **For all new points** and **OK**.
2. Select the point tool and plot two points near the bottom of your sketch.
3. Select the arrow tool, both points, and **Construct**, **Line**; then **Construct**, **Point On Line**.
4. Select the point tool and plot a point above line $AB$.
5. Select the arrow tool, points $D$ and $C$, **Construct**, **Segment**; then **Construct**, **Midpoint**, segment $DC$, **Construct**, **Perpendicular Line**, and deselect.
6. Select point $C$, line $AB$, **Construct**, **Perpendicular Line**, the perpendicular line through $E$, **Construct**, **Intersection**, point $D$, **Construct**, **Segment**; **Measure**, **Length**, and deselect.
7. Select points $F$ and $C$, **Construct**, **Segment**; **Measure**, **Length**.
8. Select lines $EF$ and $CF$, segment $DC$, point $E$, **Display** and **Hide Objects**.
9. Select point $F$, **Display**, **Trace Intersection**, and deselect.
10. Select point $C$, **Edit**, **Action Buttons**, and **Animation**. Then select the tab named **Label**; type **Animate point C** and **OK**.
11. Click on the **Animate Point C** button to observe the shape that is being generated.
12. Select the **Text** tool; double-click to open a dialog box. Then answer the following questions:
    (*a*) Describe the relationship between the segments $FC$ and $FD$.
    (*b*) Describe the relationship between point $F$ and the objects point $D$ and line $AB$.
    (*c*) If the curved shape that you made is called a parabola, write a geometric definition for the term *parabola*.

Source: Adapted from Cinco and Eyshinsky (2006)

**Fig. 1** Steps to create a parabola on The Geometer's Sketchpad are listed.

and algebra in a connected way. The first is that geometry acts as a motivator in teaching algebra because visualization and demonstration of patterns can be instrumental in teaching students how to generalize and think algebraically (Presmeg 2006; Saha, Ayub, and Tarmizi 2010). Although this perspective is useful in many situations in an algebra class, this article will focus on the second idea—that it is possible to teach both geometry and algebra concurrently and in an integrated way (NCTM 2003). Although the definition of *integrated* varies widely among teachers and schools, a survey of teachers (NCTM 2003) revealed that teachers often emphasize connections between subjects and topics within a subject through problem solving and by using mathematical reasoning and technology.

With the availability and versatility of today's technology, we can help students make powerful connections between algebraic and geometric concepts. Through the use of dynamic software, we can construct classroom activities that will help students draw such connections, encouraging the shift to higher levels of student reasoning. For example, in the exterior angle activity discussed (task 2), the use of the click-and-drag feature of the program makes it possible to manipulate an otherwise static image, creating a dynamic number experience for students. Students actually see the number change as the figure changes and thus have an enduring image that they would otherwise not have (Mackrell 2011; Noss, Healy, and Hoyles 1997). Seeing the figure and number change concurrently can help all students draw deeper connections with higher levels of reasoning about variable, incorporating the visual into every stage of the activity (Saha, Ayub, and Tarmizi 2010).

The availability of technology, however, will have little bearing on student learning without the influence and encouragement of the teacher (Lu 2008). Teachers should help students move back and forth, relating the analytic with the perceptual investigations and the rigorous with the intuitive approaches (Noss, Healy, and Hoyles 1997); in addition, they should help students differentiate between *convincing evidence* and *proof* and understand when each is appropriate (Olive 2000). The tasks in and of themselves are relatively standard lessons in ninth- or tenth-grade mathematics curricula, but the teacher's implementation and guidance through careful questioning enhances the number and nature of connections that students will make.

## TASK 1: THE CONSTRUCTION AND EQUATION OF A PARABOLA

This task uses a geometric lab approach to show that the set of points equidistant to a point and a line is a parabola. This step-by-step type of activity (see **fig. 1**), adapted from Cinco and Eyshinsky (2006), guides those who are just learning to use The Geometer's Sketchpad (GSP). This task addresses the following Common Core standards: "Derive the equation of a parabola given a focus and directrix" (CCSSI 2010, p. 78) and "Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function" (CCSSI 2010, p. 69).

This activity asks students to construct and recognize a parabola (steps 1–11) and to create a definition for the parabola that they constructed. Students will generally recognize the shape of the parabola quickly but will need more guided discussion to construct the geometric definition (in step 12), even after answering the first question on the worksheet: Describe the relationship between segments $\overline{FC}$ and $\overline{FD}$. The teacher may encourage them by rewording the second question: "If $\overline{FC}$ is congruent to $\overline{FD}$, what does that tell you about the distance from $F$ to $D$ and from $F$ to $\overleftrightarrow{AB}$?" From this, students should determine that the distances are equal.

Before moving on to the third question, the teacher may ask, "Is this always true? What if you move point $F$?" The exchange that follows may begin with the student conjecturing that the relationship holds because you can see the measurements remain the same when you click and drag. The discussion may evolve into students trying to deductively convince either the teacher or one another about features of the diagram, such as that $\overline{FC}$ and $\overline{FD}$ have to always have the same length because $E$ is the midpoint of $\overline{CD}$. Depending on the students' understanding as well as time constraints and the general purpose of the lesson, the teacher could decide to develop this discussion further or include other observations about parts of the parabola, such as the vertex and the axis of symmetry, and why they will always be present.

Students are generally eager to write the answers to all the questions posed on their worksheet. As a result, they will invariably bring the conversation back to the third question—how to write the geometric definition for the trace that they created. By now students understand that a parabola is the set of all points equidistant from a given point and line and are ready to move on.

### *Making Connections*

One valuable aspect of dynamic geometry software is that students can then take this geometric figure that they have constructed and overlay coordinate axes. The teacher may encourage students to describe the parabola analytically by asking them to conjecture about what the equation of this function should be. Students may ask where they should put the axes, leading to a discussion about the rigidity of the parabola $y = a(x - h)^2 + k$, in which the shape (wideness or narrowness) of the parabola completely depends on $a$. Placing the origin at the vertex of the parabola makes a much neater equation: $y = ax^2$. This realization would ideally be developed by having students experiment with and discuss the placement of the origin on their own constructed sketches.

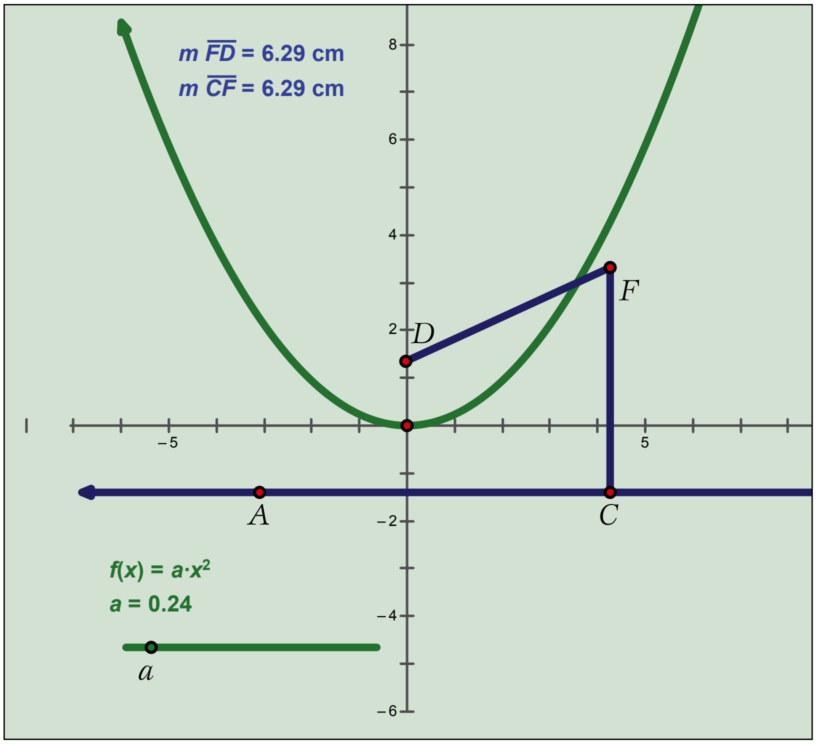



**Fig. 2** A slider for $a$ is used to graph $f(x) = ax^2$. Adjusting the value of $a$ opens the parabola to include point $F$.

Depending on their skill and comfort level with the software, students may plot the function $y = ax^2$ for different values of $a$ to guess and check, or else they may construct a parameter and slider for $a$ so that they can observe the changes in the graph more easily. Once students have the correct equation, their function plot overlays the trace exactly (see **fig. 2**). This "geometric approach enables students to manipulate functions as an entity," making new connections that would not be possible by using an analytic or numeric approach alone (Mousoulides and Gagatsis 2004, p. 391).

### *Extending the Task*

Depending on the focus of the class, the lesson may stop here or go on to explore an analytic proof of the parabola's equation using the Pythagorean theorem or distance formula. Because our parabola has its vertex at (0, 0) and students already defined it as the set of points equidistant from the focus and directrix, they can define the coordinates of the focus $D$ $(0, p)$ and the equation of the directrix $y = -p$. They

**The discussion may evolve into students trying to deductively convince either the teacher or one another about features of the diagram.**

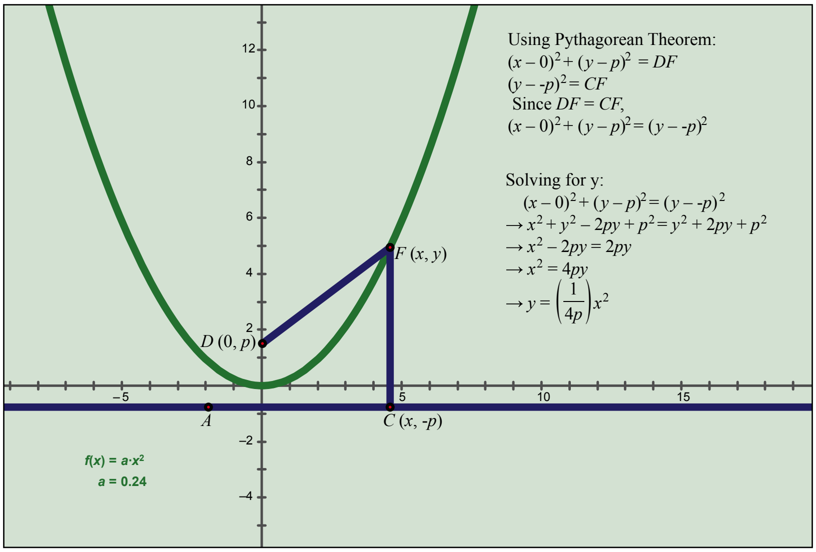



**Fig. 3** The geometric features of the parabola—its focus and directrix—are related to its equation.

could then explore further to use a combination of analytic proof and geometric conjecture about what effect the coordinates of the focus and the point $C$ on the directrix have on the equation (see **fig. 3**).

The equation that students develop, $y = 1/(4p)x^2$, shows that the $a$ in $y = ax^2$ is given by the expression $1/(4p)$. So the width of the parabola, controlled by $a$, is inversely related to the distance, $2p$, between the focus and directrix. Advanced students may be encouraged to explore the same methods when the vertex is not at the origin or even when the directrix is not horizontal.

This task could be extended to calculus by thinking of the relationship of the tangent line that the student constructed to the curve (hidden in the figure). Students might measure the slope of the line and compare that with what they already know about the derivative of the function at a point. They also might go on to argue mathematically why this line ends up being the tangent line.

## TASK 2: EXTERIOR ANGLE OF A TRIANGLE

This activity is designed as a teacher-led classroom demonstration but could also be given to students as a preconstructed sketch, written as a lab, or generated by students as in the previous activity. The purpose of this activity is to teach that the measure of the exterior angle of a triangle equals the sum of the measures of the alternate interior angles (see **fig. 4**). However, equally important in this lesson is the algebraic interpretation and representation of the situation with expressions and equations. This activity addresses the following Common Core standards: "Prove theorems about triangles" (CCSSI 2010, p. 77) and "Interpret expressions that represent a quantity in terms of its context" (CCSSI 2010, p. 64).

The teacher may initiate this lesson by asking students to conjecture and discuss what relationships exist within the diagram. One technique for fielding student responses is to have GSP calculate different combinations or measurements as students suggest and then ask students to classify and sort the calculations that they generated.

### *Making Connections*

Many students do not have a good understanding of variable, and activities such as this one uses the concept of dynamic number to reinforce the concept that variables change (Mackrell 2011; Olive 2000). For example, the teacher may encourage students to click and drag any point in the diagram and discuss what happens to the numbers. Students will notice that the measure of the exterior angle $DBC$ is equal to the sum of the measures of the two interior angles. Beyond that, they should notice that the angle measures change along with the diagram but that they keep the same relationship. This connection between the geometric and the numeric has a profound effect on student understanding of dynamic number.

To develop student understanding of the equivalent expressions that are present, it also helps to

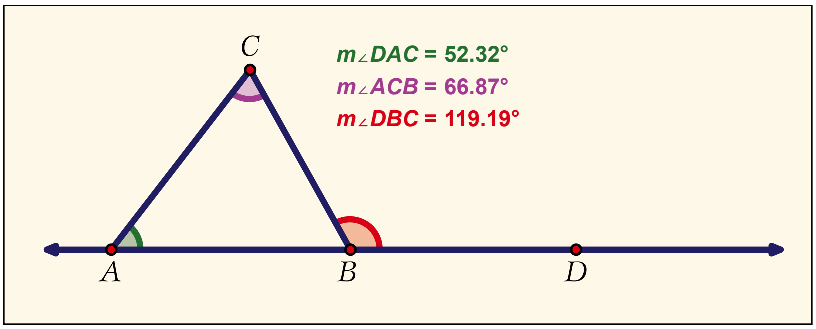



**Fig. 4** Angle measures exemplify the exterior angle theorem.

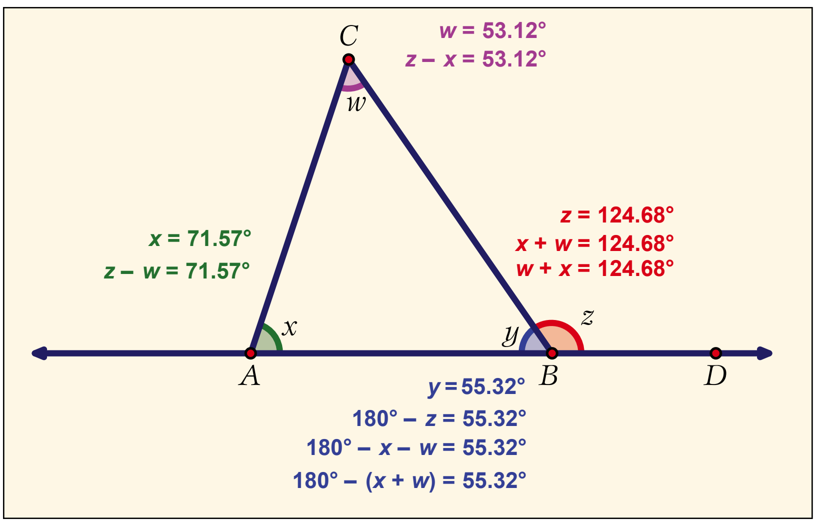



**Fig. 5** Equivalent expressions, grouped by color, are evaluated as the same measurement.

change the labels for the angles in front of students so that they can tell that naming the angles by one letter, three letters, or even words does not change their relationships. Through a class discussion and dynamic manipulation, students may also see equivalent expressions, such as $180 - z$ and $180 - (x + w)$ (see **fig. 5**).

Finally, we want students to think creatively about arguments that would constitute a formal proof of their conjectures. As in the parabola activity above, students should compare, contrast, argue, and evaluate their reasoning. For example, a student might intuitively know that the angle measure labeled $z$ is the same as the sum of angle measures $x$ and $w$; however, in finding ways to convince others that this conjecture is true, the student will learn to think logically through an argument.

One student may cite the theorem that the sum of the measures of the angles in a triangle is 180°, as is the sum of the measures of the angles in a linear pair. Geometrically, the angle $ABC$, common to both these relationships, leads to a proof of the conjecture about the exterior angle. Another student may use the blue set of equations in **figure 5** to show the same theorem algebraically.

### *Extending the Task*

The teacher may take the task one step further and ask other students which argument they prefer, whether and how the arguments are the same or different, and how they could use one argument to support the other. This discussion tends to lead students away from the get-the-answer-as-quickly-as-possible approach to problem solving and in the direction of reasoning and sense making about generalizations in mathematics.

**We want students to think creatively about arguments that would constitute a formal proof of their conjectures.**

## THE TEACHER'S ROLE

These two tasks demonstrate two of many ways in which teachers can use dynamic geometry software to help students learn to draw deep connections between geometry and algebra. The teacher's role as classroom facilitator is crucial in encouraging students to reason, generalize, question, and prove their conjectures.

One responsibility that we have as teachers is to constantly ask our students important questions—Why?" "How?" "Where?" "When?" and "How do you know?"—and encourage them to ask themselves the same questions as they mature mathematically. Students need to know that mathematics is more than just a dry list of formulas and procedures to memorize and imitate. Making creative mathematical connections enhances the ability to model and understand the world around us.

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**MICHELLE CETNER,** mcetner@ncsu.edu, who previously taught high school mathematics, is pursuing a doctoral degree in mathematics education at North Carolina State University in Raleigh. She is interested in mathematics teaching practices and enjoys giving professional development workshops about making mathematical connections more accessible to students.

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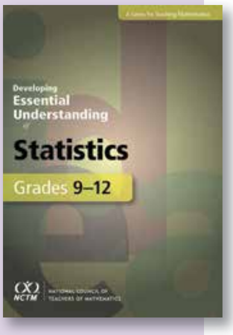


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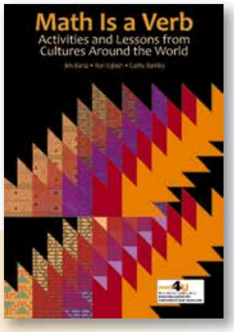


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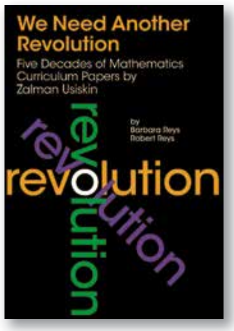


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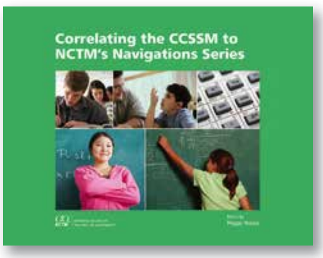


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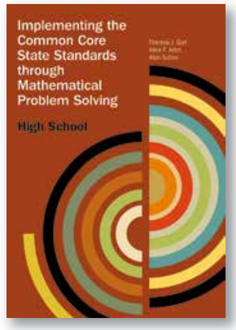


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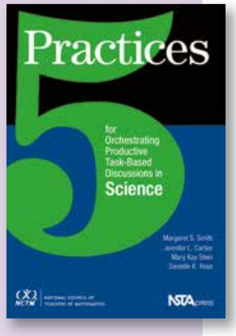


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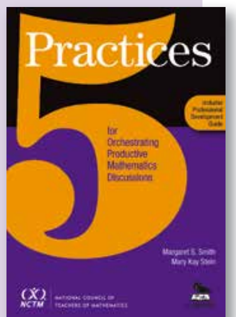


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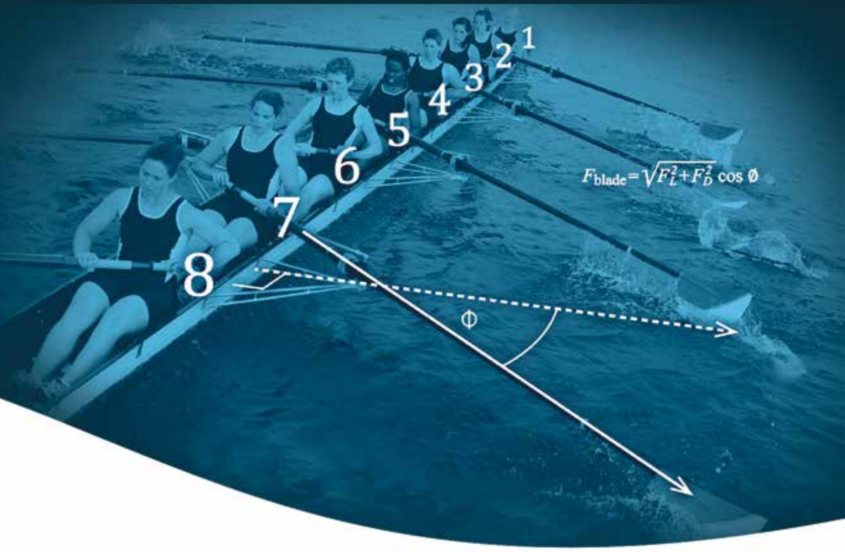


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