Bogdan Vajiac

Can a Sixth Grader Do Trig?

Can sixth graders learn trigonometry? Although this material is taught at the high school and college levels, most sixth graders are smart enough to learn it. In fact, South Korean, Romanian, and American schools abroad introduce basic trigonometry in middle school. U.S. schools can do it, too. With the right preparation, any student can learn a bit of trigonometry; a simple curiosity is enough to get started.

Why is trigonometry important? Trigonometry uses algebra, which would give students good practice employing the algebraic skills of working with exponents and order of operations. Most important, this study benefits students' thinking processes. Students learn to make connections between a diagram and the original question. This process and method of thinking will be useful in subsequent math classes.

I had the opportunity to present a few trigonometry lessons in two sixth-grade classes a few years ago. The teacher was looking for some simple sine, cosine, and tangent problems to provide some exposure to trigonometric ratios for two honors classes. It is not easy to bring new and more advanced subjects to a class, even if it is an honors course. More preparation is required, practice problems should be created, and time needs to be allocated for the lessons. However, the teacher noted, "I don't normally teach things out of context, but these higher-ability kids take a hit on their scores due to them being so smart. Some exposure to more geometry would let them have a chance of getting more questions correct [on the state exam]."

Despite my limited experience with middle school students, I succeeded in making the subject interesting and keeping the class lively. In so doing, I experienced cooperative students. Any teacher can use some of the ideas to present this subject to most classes whenever the situation permits and the benefits are measurable.

Edited by **Joanne Snow**, jsnow@saintmarys.edu, Saint Mary's College, Notre Dame, Indiana. This department explores rich mathematical ideas by revisiting their origins and early investigations found in the history of mathematics.

The following list of topics, in order, can be covered in a sixth-grade class:

- Introduction to trigonometry (some history, use, and applications)
- Right triangles, the Pythagorean theorem, and the classification of triangles (some algebraic computations with radicals should be done, as well)
- Trigonometric functions of known angles (30°, 45°, 60°, and 90°)
- Definitions and some elementary uses of trigonometric ratios in a right triangle (sin, cos, tan, cot)

The following topics are potential extensions:

- Relations between trigonometric functions ($\sin^2 x + \cos^2 x = 1$, and others)
- Triangle solutions (finding all sides and angles, given just some of them)
- Radian measure and trigonometric values of angles greater than 90
- Laws of sine and cosine

This list constitutes a foundation for trigonometry. This article provides information on how to introduce some basics of the material with students in mind. Although you might have some trigonometry background, you may have had little to no experience teaching it at the middle school level and probably do not know much about its history. With a careful historical approach, you will discover that trigonometry is not that difficult for sixth graders' inquisitive minds.

An **activity sheet** appears at the end of this article. In addition, some sample projects that can be useful in teaching trigonometry are appended to the online version of this article at **www.nctm.org/mtms**. Most of the problems are self-explanatory, and students should be encouraged to persevere.

HISTORY

Although a story is always attractive for sixth graders, a little history can provide some information that is usually not covered in a traditional curriculum. Students learn about the origins of mathematical ideas and can start to make connections between history and the development of science.

Among teachers and professors, the history of trigonometry is not well known, although it follows a common pattern of mathematical discoveries. Specifically, it started in ancient Greece and was developed by Chinese, Indian, and Arabic mathematicians. Trigonometry was brought to Europe at a later date. This field of mathematics has interesting particularities in the way that names and concepts were created and used, for several practical reasons.

Trigonometry was originally created by the Greeks to aid in the study of astronomy. Historians can also trace trigonometry to ancient Egyptians and Babylonians, but these peoples were lacking in the concept of angle measure. Their study was limited to ratios of sides of triangles and the concept of similarity.

Ancient Greeks did not use angles; they used the notion of a *chord* instead (a line segment whose endpoints are on the circle). There is no trigonometry in the work of Euclid, but many trigonometric formulas are presented geometrically.

The first trigonometric table was created by "the father of trigonometry," Hipparchus of Nicaea, in present-day Turkey, who lived from 190 BC–120 BC. Hipparchus computed trigonometric values for corresponding values of arcs and chords for many angles. Other Greek mathematicians, such as Menelaus (70–140 CE) and Ptolemy (83–161 CE), expanded the use of trigonometry (for triangles on the sphere, for example) and created famous theorems that bear their names.

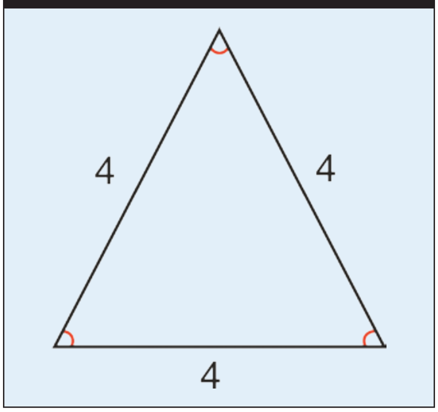
The next development of trigonometry occurred in medieval India, where the modern definitions of trigonometric ratios were established and from where the names of trigonometric functions originated. Both Indian and Arabic mathematicians are responsible for developing various trigonometric formulas and tables of trigonometric ratios (see Boyer 1991). They were also responsible for sending these trigonometric ideas to Europe. Trigonometry is an important subject because it assists in understanding many concepts in astronomy, physics, and other sciences.

The word *trigonometry* comes from the Greek, meaning "measurement of triangles." Here is a first question to ask about triangles:

> How can we determine the angles of a triangle when we know its sides? For example, if a triangle has sides 4, 4, and 4 inches, as shown in **figure 1**, then its angles are 60°, 60°, and 60°. Why?

If the sides are 6, 7, and 8 inches in length, then finding the angle measures is not simple. Students can answer some questions by using sticks measured to those lengths, putting them together to form a triangle, and drawing a sketch.

Fig. 1 Students can recognize that when all three sides of a triangle are the same length, the angles will each be 60°.



- How many different triangles can you draw that have the same measures of sides?
- Can you determine their angle measures?
- How precise can you be?

We will examine similar problems after we discuss the Pythagorean theorem.

The question of the possible number of distinct triangles requires more trigonometry than a teacher can explore in a sixth-grade class. However, students may notice that, given the three sides of a triangle, just one distinct triangle can be constructed with these side lengths. Students may then be ready to appreciate formulas that give us the angles in terms of their sides.

The law of cosines states:

> In a triangle with sides of length a, b, and c, and an angle opposite the side having length c with measure x, $c^2 = a^2 + b^2 - 2\,ab\cos(x)$.

The idea of finding relations between angles and sides of triangles is a good motivator. In the case of right triangles, the definition of trigonometric ratios is exactly what is needed.

RIGHT TRIANGLES AND THE PYTHAGOREAN THEOREM

Recall the question of finding the angles of a triangle when its sides are known. When the triangle in question is a right triangle, this question is easy to answer. At this point, we need to remember the Pythagorean theorem:

> If a and b are the lengths of the legs of a right triangle, and if c is the length of the hypotenuse, then $a^2 + b^2 = c^2$. Conversely if a, b, and c are positive numbers such that $a^2 + b^2 = c^2$, then there is a right triangle with legs of lengths a and b, and hypotenuse c.

Fig. 2 Students need to be familiar with the "labels" of each side of a right triangle, in relation to a specific angle α.

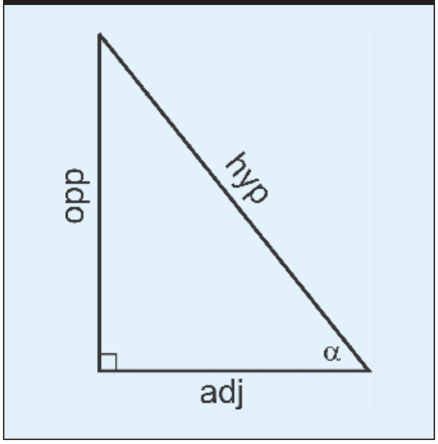
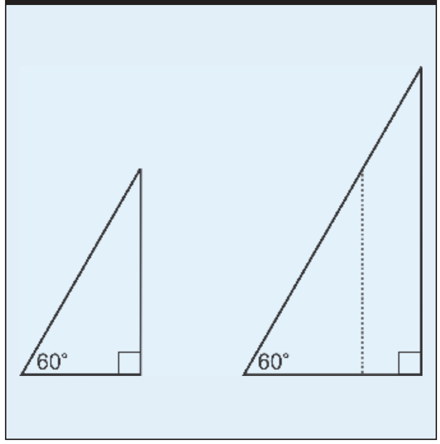


Fig. 3 Using trigonometry, students can see that the angle measures on similar triangles are the same.



Thinking about the Pythagorean theorem and the relation between the sides of the triangles and their angles can give us information about the type of triangle (right, acute, obtuse) that we have. To be more precise, if a triangle with sides of length a, b, and c, with c being the greatest one, has the property that $a^2 + b^2 \geq c^2$, then the triangle is acute; that is, each angle is less than 90 degrees.

A few activities can clarify this concept. Construct a triangle with sides of length 3, 4, and 5, respectively. This is a right triangle. Convince yourself that with these side lengths it is, indeed, a right triangle. Use sticks measured to these lengths and form a triangle. Suppose you need to construct a triangle that has sides of length 3, 4, and 4.2. What size are the angles of this new triangle? Suppose you construct a triangle with sides of length 3, 4, and 6. What can you say about the angles of this triangle?

Discover, or recall, some of the properties of right triangles by completing **activity sheet 1** (see p. 180).

INTRODUCING BASIC TRIGONOMETRIC RATIOS

Look at the keys on a scientific calculator that are marked sin and cos. What do they mean? Where do we use them?

Sin, short for "sine," comes from the Sanskrit word *jya-ardha*, which means "chord-half." Sanskrit is an extinct Indo-European language that was once spoken in India. *Cosine* comes from the same source. Indian mathematicians called *kojya* the "perpendicular sine," a companion for sine. Tan, or the tangent ratio, was introduced and named by the Persian mathematician Abul Wafa (940–997 CE).

See **figure 2** to visualize sine at work. Draw a right triangle with an angle labeled α, or alpha. Call the side opposite to α *opp* (which stands for opposite) and the side adjacent to α *adj* (which stands for adjacent). The term *hyp* refers to the hypotenuse. See the following definitions:

$$\sin\alpha = \frac{opp}{hyp}$$

$$\cos\alpha = \frac{adj}{hyp}$$

$$\tan\alpha = \frac{opp}{adj}$$

We will also compute some basic function values for familiar angles. Students were very happy to compute, for example, sin 60°, sin 45°, and others in a triangle or a similar triangle, as illustrated in **figure 3**. They were convinced that those functions are independent of the triangle, since they got the same results.

Some interesting facts are easy to discover:

- Consider an acute angle α in a right triangle. Incorporate our angle labels from **figure 2** into the Pythagorean theorem, as follows:

$$\begin{aligned} a^2 + b^2 &= c^2 \\ (opp)^2 + (adj)^2 &= (hyp)^2 \end{aligned}$$

If we divide by $(hyp)^2$, we obtain

$$\left(\frac{opp}{hyp}\right)^2 + \left(\frac{adj}{hyp}\right)^2 = 1.$$

This means that $\sin^2\alpha + \cos^2\alpha = 1$. Therefore, we can find the cosine of an angle if we know its sine.

- If the angles α and β are complementary ($\alpha + \beta = 90°$), then $\sin(90° - \alpha) = \cos\beta$ and $\sin\beta = \cos(90° - \alpha)$. Think about this from the point of view of α. Which leg is *opp*, and which is *adj*? How about from the point of view of β?

CONCLUSIONS

Basic trigonometry concepts are within reach for sixth graders. With testing programs including trigonometry, school districts may start to introduce elementary trigonometry in sixth-grade honors classes. It is not difficult, and it can be done with minimum effort. “High school trigonometry” will not be words that students fear. Learning concepts of trig at an early age can change their attitude toward learning mathematics. With a little work, a sixth grader can and will be able to do trig.

BIBLIOGRAPHY

Gelfand, I. M., and Mark Saul. *Trigonometry.* New York: Birkhäuser Boston, 2001.

Baker, C. C. T. *Introduction to Trigonometry.* New York: Philosophical Library, 1967.

Boyer, Carl B. *A History of Mathematics.* 2nd ed. New York: John Wiley & Sons, 1991.

International Union of Geodesy and Geophysics. www.iugg.org.

Bogdan Vajiac, bvajiac@saintmarys.edu, teaches at Saint Mary’s College in Notre Dame, Indiana. He is particularly interested in preparing future school teachers, organizing math clubs, and assisting math competitions for grades K–12 students.

Note: For solutions to the activity sheet and additional projects, view the online version of this article at **www.nctm.org/mtms**.

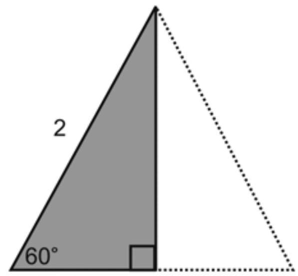
(*Activity sheet 1 is on the next page.*)

activity sheet 1

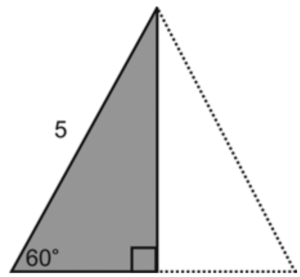
Name ______________________________

Right Triangles and the Pythagorean Theorem

1. a. In a right triangle with a 60° angle, the hypotenuse is 2 inches long. What is the length of each of the legs? Hint: Use the triangle and its reflection, as shown in the figure below.

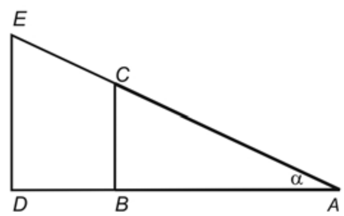


b. What is the length of each of the legs in the shaded triangle below? Hint: Use the triangle and its reflection, as shown in the figure below.



2. Create a table with the values of sin, cos, and tan for the angles of 30°, 45°, and 60°. Draw appropriate triangles to help you.

3. In the triangle below, measure the sides of both the small triangle *ABC* and the large triangle *ADE* in inches. Compute sin (α), cos (α), and tan (α).



4. Use a ruler that measures in centimeters and millimeters. Determine the values of sin (α), cos (α), and tan (α). What happens if you use the larger triangle and repeat both measurements?

(Continued from page 180)

PROJECT 1

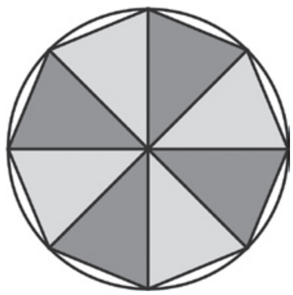
You are a consultant for a play about the Greek mathematician, Eratosthenes. The director of the play asks for your advice about how to incorporate in the play his computation of the circumference of Earth. You can read about Eratosthenes' measurements in *A History of Mathematics* (Boyer 1991). The play is for a middle school audience, and the director wants you to change the ancient Egyptian example and use a more modern one. For example, instead of using the ancient Egyptian cities of Syene and Alexandria, refer to two cities in the United States that are about 800 miles apart. Design an experiment that will measure the radius of Earth. What angles do you need to measure so that your estimate of Earth's radius, at about 3961 miles, is accurate? (This number comes from the International Union of Geodesy and Geophysics; www.iugg.org.)

Comments: This project is commonly used in K–12 education. For example, the Center for Innovation in Engineering and Science Education at Stevens University sponsors one of these projects. The Web site (www.k12science.org/noonday/) contains a lot of useful information on measuring Earth's circumference using Eratosthenes' method.

PROJECT 2: THE OCTAGON

The circle shown has a radius of 10 inches. You want to inscribe the regular octagon shown below inside the circle. Imagine that you can walk around that polygon.

- How long is it until you return to the starting point if you walk at a speed of 1 inch per second?
- Suppose now that you double the speed, but you also increase the number of the sides of your polygon to 16. Do you get the same time as in the answer to the first question?



Comments: Students discover that the side of a regular n-polygon does not depend linearly (in a proportional way) on n. In fact, a teacher can ask students to compute the side length in term of the sine function of the central angle. For example, a regular inscribed polygon with 8 sides (in a circle of radius 10) has a perimeter of $160 \cdot \sin(22.5) = 61.23$, and the polygon with 16 sides in the same circle has a perimeter of $320 \cdot \sin(11.5) = 62.43$. The general formula that a teacher may be aware of is the one below. It can be discovered by "cutting" one of the isosceles triangles in the diagram above into two congruent triangles and applying the sine function.

$$P = 2nR \cdot \sin\frac{360}{2n},$$

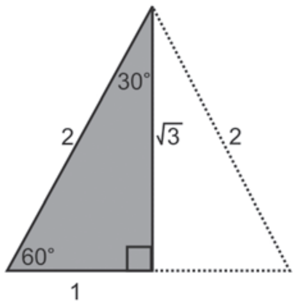
where R is the radius of the circle, and n is the number of sides of the polygon.

 (Continued from page 180)

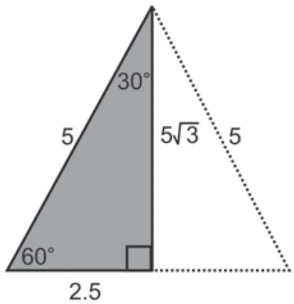
Right Triangles and the Pythagorean Theorem

1a. In the diagram, we see that the original triangle and its image from a reflection form an equilateral triangle. We see that the length of the leg opposite to the 30° angle is half the hypotenuse (which is 2 inches long), or 1 inch long. The length of the other leg can be found using the Pythagorean theorem.

So $x^2 + 1^2 = 2^2$, or $x = \sqrt{3}$.



b. With work similar to part a, see the side lengths, shown below.



2. Students will draw a 30°-60°-90° right triangle and a 45°-45°-90° triangle. The trigonometric ratios of the 30°, 45°, and 60° angles can be found from these triangles. A teacher can choose to incorporate a "rationalizing denominators" activity, for example, to discover that

$$\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} = \frac{\sqrt{8}}{4}.$$

Here are the values of sin, cos, and tan for the angles of 30°, 45°, and 60°:

	30°	**45°**	**60°**
Sin	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$
Cos	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$
Tan	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$

In this exercise, students discover the concept of similar triangles. If a teacher chooses, this concept can be discussed in detail. For students' understanding of trigonometry, it is sufficient that they measure the sides of both the small and the large triangle using different rulers and discover that the trigonometric ratios depend only on the angle, not on the triangle, in which we write the particular ratio. Students can compute sin, cos, and tan on their calculators. A teacher can emphasize that only an angle needs to be used as an input in the calculator.