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Calculus in the

The path of a thrown ball is one of many examples that can connect middle school math to calculus. Showing this calculus connection early may motivate students to undertake increasingly advanced high school and college math.

Middle School?

Calculus. Few words inspire more awe in the hearts and minds of students at all levels. In a room of twenty people, at least one will have a story to tell about his or her experience in this class. There will also be many who are positive that they could never make it through such a class. The common image of a calculus class is one in which the most mathematically talented high school or university students are working feverishly on long problems. The atmosphere of this room is solemn, silent, studious, and serious.

Consider this problem:

> Niran has 72 centimeters of molding to make a frame for a print. This is not enough molding to frame the entire print. How should he cut the molding to give the largest possible area for the print using the inside edge of the molding as the perimeter? (Lappan et al. 2006).

U.S. students have been presented problems similar to this one, which is from a popular middle school textbook. As they work to solve it, most students are unaware that their work is directly related to work done by calculus students at the high school and college level. In fact, many middle school students working on similar problems will never consider enrolling in such a high-level mathematics course as calculus. We contend that if the right foundation is laid in middle school, more students will be willing and eager to go further in math.

Algebra and calculus have been described as *gatekeeper* courses. Algebra has served as the gatekeeper course for high school mathematics and one in which only the successful were encouraged to continue into upper-level mathematics courses (Lott 2000; Moses 2001). Similarly, calculus has typically been the collegiate gatekeeper; students who were unable to successfully complete calculus often were forced to abandon their chosen major and look for a new career choice (Nelson 2004).

However, instead of algebra and calculus being gates that keep people out of careers and fields, we have been encouraged to make them pipelines through which students can go into advanced mathematics courses (NRC 1989).

To encourage more students to take advanced mathematics courses in high school, views of algebra needed to change. It is no longer just a course that focuses on manipulating variables, polynomials, and expressions. Instead, it has become a way of thinking. In so doing, the development of algebraic thinking at the elementary level has received increased attention (NCTM 1989, 2000; U.S. Department of Education 2008). Elementary teachers are being encouraged to think about how the content they teach provides a foundation for the study of algebra that occurs in middle school.

If such an approach, explicitly connecting the mathematics at the elementary school level with that taught in middle school, has been successful with algebra, wouldn't a similar approach work with calculus? Might not making these connections explicit to middle school students encourage more of them to enter the calculus pipeline? These were our thoughts as we began to look for areas where such connections could be made.

This article presents an example of how middle school teachers can lay this foundation for calculus. Although many middle school activities connect directly to calculus concepts, we have decided to look in depth at only one: the concept of change. We will show how teachers can lead their students to see and appreciate the calculus connection. Using this concept as a guide, teachers can develop other foundations. We include some suggestions for other activities at the end of the article.

REPRESENTING CHANGE

One big idea of calculus is *differentiation,* which refers to the process of computing the derivative of a function. A *derivative* is defined as the rate of change of a function at any instant with respect to one of its variables (Harcharras and Mitrea 2007). Typically, the derivative is found in terms of the change in the y-value as the x-value of the function increases. According to *Principles and Standards for School Mathematics*, "The study of mathematical change is formalized in calculus, when students study the concept of the derivative" (NCTM 2000, p. 40). However,

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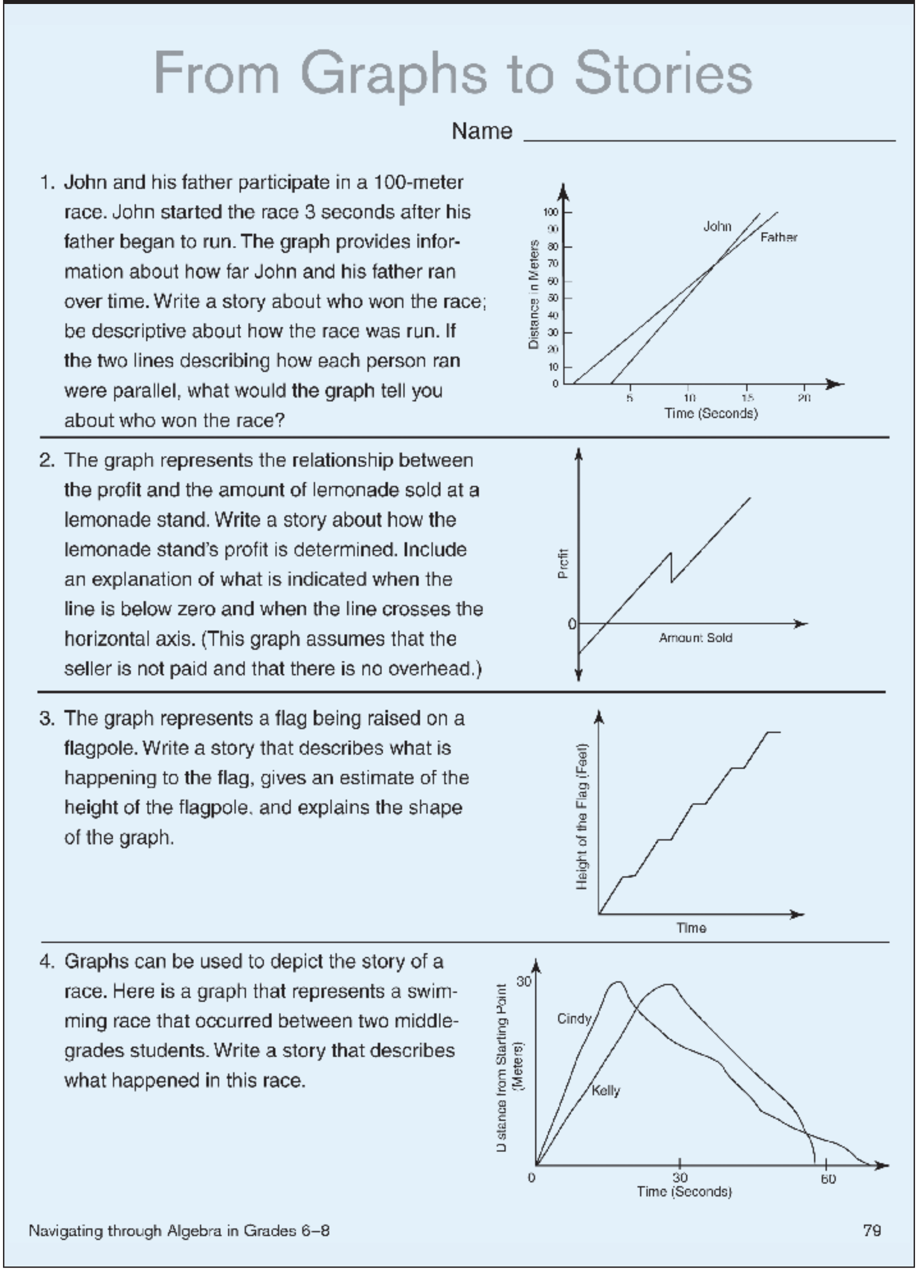
Fig. 1 These questions allow students to think about change, which is a crucial topic for understanding calculus (NCTM 2001).


From Graphs to Stories

Name ____________________

1. John and his father participate in a 100-meter race. John started the race 3 seconds after his father began to run. The graph provides information about how far John and his father ran over time. Write a story about who won the race; be descriptive about how the race was run. If the two lines describing how each person ran were parallel, what would the graph tell you about who won the race?

2. The graph represents the relationship between the profit and the amount of lemonade sold at a lemonade stand. Write a story about how the lemonade stand's profit is determined. Include an explanation of what is indicated when the line is below zero and when the line crosses the horizontal axis. (This graph assumes that the seller is not paid and that there is no overhead.)

3. The graph represents a flag being raised on a flagpole. Write a story that describes what is happening to the flag, gives an estimate of the height of the flagpole, and explains the shape of the graph.

4. Graphs can be used to depict the story of a race. Here is a graph that represents a swimming race that occurred between two middle-grades students. Write a story that describes what happened in this race.

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even after students complete calculus, research has indicated that the concept of change is not understood with much depth (NCTM 2000). "If ideas of change receive a more explicit focus from the early grades on, perhaps students will eventually enter calculus with a stronger basis for understanding the ideas at that level" (NCTM 2000, p. 40). The foundation for studying differentiation is built through several different activities at the middle school level.

One way that middle school students study change is by looking at various representations. One typical activity involves having students examine and compare graphs and write stories to fit a given graph. They might also be asked to match verbal, graphical, and tabular representations of a relationship involving change. Consider an activity found in NCTM's *Navigating through Algebra in Grades 6-8* (NCTM 2001). (See **fig. 1**.)

Such activities allow students to think about change and how it may be shown through a variety of representations. It is a small additional step to share the connection between this activity and calculus, and it can be accomplished in a variety of ways.

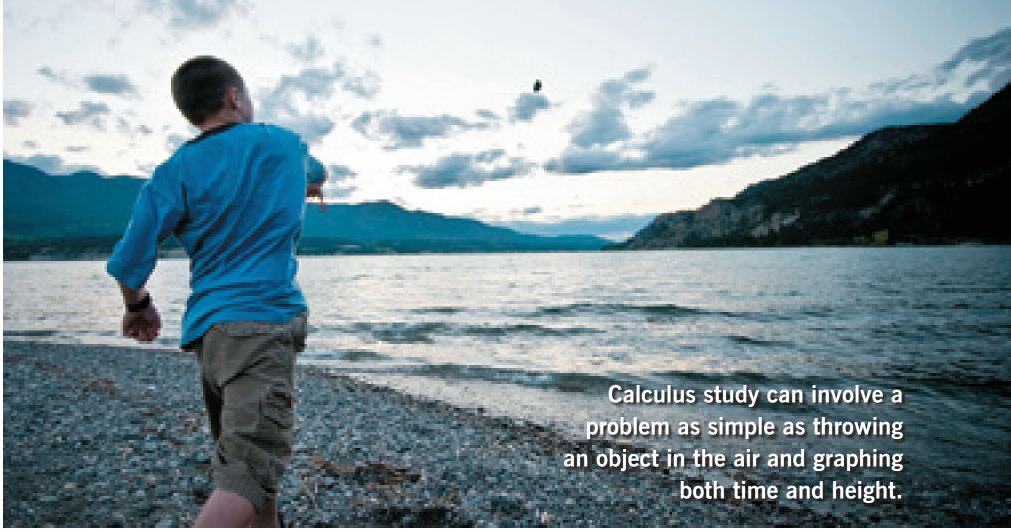
One method would be to show the students a calculus problem and tell them you found it in a calculus textbook. Follow this with a discussion about the common use of different representations embedded in the two activities. Here is our hypothetical example of how the middle school discussion might have unfolded:

Ms. Erikson's eighth-grade students have being working with tabular, graphical, and verbal representations of change. They have just completed the "From Graphs to Stories" activity shown in **figure 1**. The students have shared the stories they wrote with their classmates, and Ms. Erikson is leading a discussion about their work by asking them to identify what is changing in the situation. The students responded with the following:

- "How much money they made is changing."
- "The amount they've sold is also changing. The longer they sell, the more they sell."

Ms. Erikson pointed out the increases and drops in the graph and asked what the drops might represent. Some responses included these:

- "Our group said that they probably had to buy some more supplies."
- "Maybe some of their money got stolen."
- "I think that it could mean that a batch of lemonade tasted bad and they had to refund some money."


M. ERIC HONEYCUTT/ISTOCKPHOTO.COM

Calculus study can involve a problem as simple as throwing an object in the air and graphing both time and height.

The teacher praised these responses and followed up with this question: "Can you tell the amount of money they've made?" Some students said:

- "I can't tell. I think it would be easier if there were some numbers, and the graph doesn't have any numbers."
- "We can't tell how much money exactly, but we can tell that they are making money because the line is going up."

Next, the teacher shared a problem from a calculus textbook. She told them that calculus is a special type of math and that part of calculus is thinking about change, similar to what they had just been doing. She also mentioned that this particular problem is similar to ones they will encounter in high school. The students read the problem silently while she read it aloud to the class.

> If an object is thrown vertically upward from the ground with an initial speed of 160 feet per second, its height (in feet) t seconds later is given by the function $h(t) = -16t^2 + 160t$. Sketch the graph of $h(t)$. Use the graph to estimate how high the object will rise. (Hoffmann and Bradley 2004, p. 24)

Ms. Erikson asked the students if they noticed anything that makes this problem similar to those they had been working on. One student recognized that the problem required a graph, as did their previous tasks. Another student commented that the problem was posed "as a story," although she also found it difficult to understand.

Ms. Erikson then asked her students to think more about this problem in groups and to write a story similar to those they had been writing. Before they began writing, she pointed out that the problem discusses an object being thrown in the air. She asked the class to consider what the object could be. Students suggested that baseballs and sometimes rocks are thrown, which she confirmed as good ideas before reminding the students to describe the path of the object in their writings.

One student asked if the stories needed to have all the numbers in them. Ms. Erikson pointed out that the stories and graphs used thus far gave sufficient information without using lots of numbers. She suggested that the class continue this practice of not using specific numbers. The teacher then asked students to share their stories:

Brian: Tim was practicing catching baseballs. He threw one up in the air and then tried to catch it. He watched the ball go up for a while and then start to come back down.

Jen: Ours is kind of like that. Some friends were throwing rocks into a pond. The rocks went up, seemed to kind of slow down and curve over, and then fell down into the pond.

Ms. Erikson: What is changing in your stories?

Maria: Well, how high the ball or rock is changes. It goes up and then comes back down.

Michael: It's kind of like the flagpole problem that was on the sheet with the lemonade stand. Time is going by and the height of the flag is different, depending on the amount of time that's passed.

Ms. Erikson then asked the class to graph this situation. She pointed out that the height of the object changes as time passes and suggested that students put *time* on the horizontal axis and *height* on the vertical axis. The students worked in their groups to sketch a graph showing the height of the object as time passed. Students described their graphs in these ways:

- "The line on our graph starts low and then goes up and then comes back down."
- "Ours looks like that except we made it curve a little at the top because we think the rock would kind of hang in the air for a little bit."

Ms. Erikson mentioned that although "From Graphs to Stories" did not involve tables, students had used tables as representations in the past. Since an equation is given, a table of values is possible. Some students expressed concern that it was a difficult task, but Ms. Erikson persisted and explained:

> Let's talk about it. The equation given is $h(t) = -16t^2 + 160t$, and the problem states that h represents the height of the object while t represents the number of seconds that have passed. You might be more familiar with the equation if we wrote it as $y = -16t^2 + 160t$. One of the things you'll learn about as you go on in math is the way that equations and functions are related. When the problem says $h(t)$, it's giving the equation in function notation.

One student described the process:

> Just square the number of seconds and multiply by –16. Then we would multiply 160 by the number of seconds and add those two together.

Groups completed the tables, one of which is shown in **table 1**. When the students finished making a table of values, Ms. Erikson asked her students what they noticed:

- "The height values go up for a while and then come back down."
- "I think it shows that our group was right! We said we thought the rock would hang in the air and if you look at the numbers, you can see that it kind of seems to slow down before it starts back down!"

Ms. Erikson then pointed out that the class had seen three different representations for this problem: a graph, a table, and an equation. She mentioned that each representation can be used to convey information and asked the class if one representation helps more than the others in understanding this problem. The students seem pleased with their work and responded:

- "I like the graph because you can see just by looking at the graph what is happening."
- "I'd rather look at the table of heights. The numbers tell exactly how high the object is."

Ms. Erikson remarked, "Give yourselves a pat on the back because you've just worked a problem that appears in a calculus book! The same representations we've been working with—graphs, stories, tables—will be used as you work with mathematics in the future."

Table 1 One group's table explored the function $h(t) = -16t^2 + 160t$.

SECONDS	0	1	2	3	4	5	6	7	8	9	10
HEIGHT	0	144	256	336	384	400	384	336	256	144	0

EXTENSION

Middle school materials contain many other data-collection activities that allow students to investigate change. Another example from Connected Mathematics, *Thinking with Mathematical Models* (Lappan et al. 2004b) asks students to create bridges composed of layers of paper and then to test each bridge to determine the number of pennies each will hold. The students use half sheets of paper and fold one inch on each long edge to form the bridge. They first create and test a bridge made of one layer of paper and then test increasing numbers of layers. Each bridge is suspended between two books, and a small paper cup is placed near the center of the bridge. Students add pennies until the bridge crumples. The number of pennies the bridge held is then recorded.

After completing the exploration, students examine the data to discover the change that is occurring. They graph the collected data and notice that it approximates a linear relationship. They can find a line of fit, write an equation for the line, use the line to predict the number of pennies that additional layers will support, or change the design of the experiment depending on the objectives of the teacher. Such middle school activities allow students to physically investigate change, which becomes more difficult to accomplish as students study calculus formally and the applications become more difficult to replicate in the classroom.

The activities and classroom discussion regarding change show only one of the many connections that can be made between middle school math and calculus. **Figure 2** lists some areas discussed in calculus and some middle school math concepts that can provide a foundation for these topics.

CONCLUSION

In the same way that algebraic thinking at the elementary level is better preparing students for their formal introduction to algebra in middle school, middle school mathematics can do much to help prepare students for their formal introduction to calculus in high school or college. By showing middle school students the connections between what they are studying and actual problems from calculus textbooks, we can help to demystify calculus. Teachers can obtain a copy of the calculus textbook used in high school or in a neighboring college and pull similar problems from it. When they see that everyone can understand calculus, perhaps more students will be willing to pursue advanced studies in both mathematics and science.

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Fig. 2 These middle school mathematics concepts and problems can provide a foundation for calculus.

OPTIMIZATION (MAXIMUMS AND MINIMUMS)

Fixed perimeter—maximum area:

Given 30 meters of fencing, what is the largest area that can be fenced?

Fixed area—maximum and minimum perimeter:

A piece is cut from one side of an index card, slid across the card, and taped in place. What type of piece will result in the greatest increase in perimeter? (Lappan et al. 2006)

Maximum volume:

Squares of various sizes are cut from the corners of a piece of grid paper. The paper is folded into a box. What is the side length of the cut square that results in the box with the greatest volume?

LIMIT (BOUNDARY VALUE: THE VALUE THAT IS APPROACHED WITHOUT BEING REACHED)

Area of a circle:

Look at a polygon inscribed in a circle. If the number of sides of the polygon is increased, what can you say about the inscribed polygon with respect to the area of the circle?

Approximating the area of an irregular figure:

Draw an irregular figure on inch grid. Estimate the area by counting squares. Overlay 1/2-inch grid paper, and estimate the area. Repeat with 1/4-inch grid paper. How do the estimates compare with the actual area of the figure?

EXPONENTIAL CHANGE

Exponential growth:

Place 5 Skittles in a box. Shake the box. Each Skittle with an "S" showing has "reproduced." Add another Skittle for each piece that has reproduced. Record the total and repeat. Graph the total number of Skittles following each shake. What pattern of growth will you see in the total number of Skittles in the box?

Exponential decay:

Mark one side of 100 cubes. The cubes represent atoms of a radioactive element, and the marked sides are used to simulate a decay rate of 1/6 per day. Place the atoms in a cup, shake and dump the cup, and remove any atom with the marked side landing up. Record the new total and repeat. How would you describe the rate of change in the number of atoms? (Lappan et al. 2004a)

GEOMETRIC SEQUENCE CONVERGENCE ($1 + 1/2 + 1/4 + 1/8 + \cdots +$)

Ripping paper:

Start with 2 sheets of paper. Set aside one sheet. Using the second sheet, find 1/2 and set it aside. Now find 1/4 and set it aside. If you continue doing this forever, will you use up the second sheet of paper?

HARMONIC SEQUENCE CONVERGENCE ($1 + 1/2 + 1/3 + 1/4 + \cdots +$)

More paper tearing:

Again, start with 2 sheets of paper. Set aside 1 sheet. Using the second sheet, find 1/2 and set it aside. Now find 1/3 and set it aside. You cannot find 1/4 without using a new sheet of paper. (There is not enough left.)