palette of problems

David Rock and Mary K. Porter

1. Using the digits 2, 3, and 5 exactly once to form two different factors, find the greatest possible product.

2. Determine the next three numbers in the following sequence:

0, 2, 4, 6, 12, 14, 28, 30, 60, 62, ___, ___, ___

3. Your school has 350 students in attendance. An anthill is on the school grounds. The number of ants outside the anthill is 1/20 the number of ants inside the anthill. The number of ants outside the anthill is also 1/10 the number of students who attend your school. How many ants are inside, and how many are outside the anthill?

4. Determine the missing value in the following list of numbers. (Hint: Consider sums of pairs of numbers.)

1, 6, 12, 4, 7, 9, 15, 2, ___, 10, 13, 5, 11, 16

5. Examine the pattern of digits constructed by stringing together the counting numbers beginning with 1 and continuing without spaces or commas:

1234567891011121314151617181920212223. . .

Determine the 3003rd digit in the sequence.

Use the following series for **questions 6** and **7**:

$$1 + 11 + 101 + 1001 + 10001 + \cdots + 1000\ldots0001$$

The last term has 20 zeroes.

6. When the sum is calculated and written as a single positive integer, how many places in the sum will be filled with the digit 0?

7. When the sum is calculated and written as a single positive integer, what will be the sum of the digits?

8. Two cyclists begin a race against each other at opposite ends of a racecourse. They are travelling along the same route and will pass each other during the race. If cyclist 1 is traveling at an average rate of 23 miles per hour (mph) and cyclist 2 is traveling at an average rate of 25 mph, how far apart will the two cyclists be 3 minutes before they meet each other?

9. Jon has a piece of paper 22 in. long and 16 in. wide. He cuts a 2 in. × 2 in. square from each corner. He then folds up the sides of the paper and forms a box with an open top. What is the volume of this box?

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10. In the following multiplication problem, each of the letters A, B, N, and T represents a different digit from this list: 1, 3, 5, and 7. Determine the value of each letter.

$$\begin{array}{r} AN \\ \underline{\times \quad B} \\ TNT \end{array}$$

11. At an office supply store, 3 pencils cost $0.03 more than 2 pens, and 1 pen and 10 pencils together cost $2.40. Determine the cost of each writing tool.

12. Alex can paint a 10 ft. × 14 ft. wall in 20 minutes. Kristen can paint the same-sized wall in 14 minutes. If Alex and Kristen are both painting at the same time, how long will it take them together to paint the same-sized wall? Round your answer to the nearest tenth of a minute.

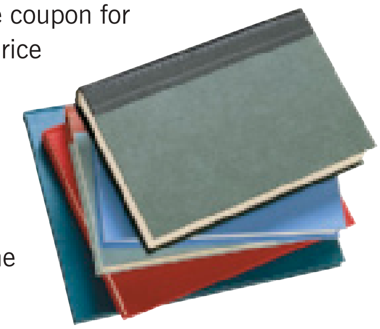
13. Find the area of a quadrilateral enclosed by the following ordered pairs:

- $A(-3, -2)$
- $B(-6, 2)$
- $C(2, 8)$
- $D(1, 1)$

14. In the sequence of numbers listed below, what is the largest term that is a two-digit prime number?

3, 10, 17, 24, 31, 38, . . .

15. Mary has two coupons. The coupon for bookstore A is for 25% off the price of one book; the coupon for bookstore B is for 10% off the price of one book. Mary uses the coupon at bookstore A to buy a book whose original price is $45. If she wants to spend the same amount of money at both bookstores, what should be the original price of the book she buys at bookstore B (before she uses the 10% off coupon)?



16. In a survey, 726 people were asked which of these three drinks they like the most: cola, milk, or water. Of those surveyed, 30% liked cola more than water. Equal numbers of people liked milk and water. How many people liked cola best?

(*Answers on page 382*)

The solutions are appended to the online version of the "Palette of Problems" at **www.nctm.org/mtms**.

solutions to palette

(Continued from pages 326–27)

(Alternative approaches to those suggested here are encouraged.)

ANSWERS

1. 160
2. 124, 126, 252
3. 700 ants inside; 35 ants outside
4. 8
5. 0
6. None
7. 25
8. 2.4 miles apart
9. 432 in.^{3}
10. A = 5, B = 3, N = 7, and T = 1
11. Pen: $0.30; pencil: $0.21
12. Approximately 8.2 minutes
13. 37.5 square units
14. 73
15. $37.50
16. 286

SOLUTIONS

1. One factor must be a single-digit number and the other must have two digits. Of the possible solutions, 160 is the largest.

$$\begin{aligned} 2 \times 35 &= 70 \\ 2 \times 53 &= 106 \\ 3 \times 25 &= 75 \\ 3 \times 52 &= 156 \\ 5 \times 23 &= 115 \\ 5 \times 32 &= 160 \end{aligned}$$

2. This sequence is formed by alternately adding 2 to the previous number and then doubling the result. Start by adding 2 to 0 to obtain 2, then doubling to obtain 4, and so on.

0 (+ 2), 2 (× 2), 4 (+ 2), 6 (× 2),
12 (+ 2), 14 (× 2), 28 (+ 2),
30 (× 2), 60 (+ 2), 62 (× 2),
124 (+ 2), **126** (× 2), **252**

A second way to see the sequence is to observe differences between every other term, beginning first with 0, 4, 12, 28, 60, ... Notice that the differences are increasing powers of 2. Therefore, the next difference will be 64; the next term in the sequence after 62 will be 60 + 64 = 124. Notice also that the differences between terms of the remaining sequence 2, 6, 14, 30, 32, . . . are also increasing powers of 2. Therefore, the next difference will be 64, and the next term in the sequence after 124 will be 62 + 64 = 126. Following this pattern, the third missing number will be 128 greater than 124, or 124 + 128 = 252.

3. Outside the mound, there are (1/10)(350) = 35 ants. The number of ants inside the mound is 20 times greater, or 20 × 35 = 700.

4. The sum of the first and last numbers is 17. The sum of the second and second to last is 17, and so on, so that 9 + ? = 17. The missing number is 8.

5. There are 9 one-digit numbers from 1–9, which account for the first 9 digits; 90 two-digit numbers from 10–99 account for 90 × 2 = 180 digits; 900 three-digit numbers from 100–999 account for 900 × 3 = 2700 digits. Therefore, the one-, two-, and three-digit numbers account for 9 + 180 + 2700 = 2889 digits in the sequence. The first 28 four-digit numbers (1000–1027) will account for an additional 112 digits, or 2889 + 112 = 3001, showing that 7 is the 3001st digit, which is the last digit of 1027. The next four digits will be 1028, yielding 0 as the 3003rd digit.

6. Try a simpler problem where the last term has 4 zeros.

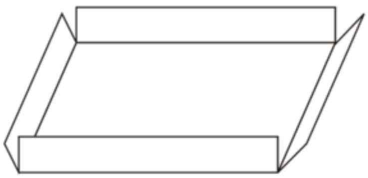
$$\begin{array}{r} 1 \\ 11 \\ 101 \\ 1001 \\ 10001 \\ \underline{+\ 100001} \end{array}$$

The ones digit of the sum will be 6, which is 2 more than the number of zeros in the last number. The 1 and 11 are included, but they have no zeros. The five decimal places are filled with ones. Therefore, if the last addend has 20 zeros, the units digit will be the sum of 20 + 2 ones, or 22 ones, which creates a 2 in the units digit, and 2 is regrouped to the tens column. The tens column will have only 1 nonzero digit, specifically from the number 11. Therefore, the tens column is 2 + 1 = 3. In each of the positions from the right numbered 3 to 22, there is only 1 nonzero digit, a 1. Therefore, the digits in the 3rd to 22nd positions will be a 1, creating 22 nonzero digits for the sum, and no zeros.

7. From the previous question, we know that 20 digits are 1 (positions numbered 3 to 22, counting from the right), and that a 3 is in the tens place and a 2 is in the ones place. The sum of the digits is 20(1) + 2 + 3 = 25.

8. They travel toward each other at a combined rate of 48, or 23 + 25, mph. Therefore, 48 miles per 60 minutes = 0.8 miles per minute. Therefore, they will be 3(0.8) = 2.4 miles apart 3 minutes before they meet.

9. Begin by determining the length (l), width (w), and height (h) of the box. The length of the paper was 22 inches; when Jon cut off the corner squares, 2 in + 2 in. = 4 in. was removed from each dimension, making l, the length of the box, 22 in. – 4 in. = 18 in. It also made w, the width of the box, 16 in. – 4 in. = 12 in. The height of the box, h, when the paper is folded, is 2 inches. The volume of the box is $l \times w \times h$, or 18 in. × 12 in. × 2 in. = 432 in.^{3}.



10. Note that neither B nor N is 1; if either were 1, T would equal B or N, which is impossible, since each letter represents a different digit. Also note that neither B nor N is 5; if either were 5, then T would either be 0 (which is not a choice) or 5 (which is impossible, since no two letters have the same value). Thus, either N = 3 and B = 7, or N = 7 and B = 3. Suppose that N = 3 and B = 7. Then T = 1, because 3 × 7 = 21. This leaves 5 to be the value of A. Replace A, N, B, and T in the given multiplication problem with the values 5, 3, 7, and 1, respectively, for:

$$\begin{array}{r} 53 \\ \underline{\times\ 7} \\ 131 \end{array}$$

But this is incorrect, because 53 × 7 = 371, not 131. Therefore, N must equal 7 and B must equal 3. Then T = 1, because 7 × 3 = 21. This leaves 5 to be the value of A. Replacing A, N, B, and T in the given multiplication problem with the values 5, 7, 3, and 1, respectively, we get this correct multiplication problem:

$$\begin{array}{r} 57 \\ \underline{\times\ 3} \\ 171 \end{array}$$

Thus, A = 5, B = 3, N = 7, and T = 1.

11. Let x represent the cost of 1 pencil (in cents), and let y represent the cost of 1 pen (in cents). Since 3 pencils cost \$0.03 more than 2 pens, then $3x = 2y + 3$. Since 1 pen and 10 pencils cost \$2.40, then $y + 10x = 240$, or $y = 240 - 10x$. Substitute this value for y in the equation $3x = 2y + 3$ and solve:

$$\begin{aligned} 3x &= 2(240 - 10x) + 3 \\ 3x &= 480 - 20x + 3 \\ 23x &= 483 \\ x &= 483/23 = 21 \end{aligned}$$

Each pencil costs \$0.21. Use this value for x (the pencil cost) to find the cost of 1 pen:

$$\begin{aligned} y &= 240 - 10(21) \\ &= 240 - 210 \\ &= 30 \end{aligned}$$

Each pen costs \$0.30.

12. A 10 ft. × 14 ft. wall has an area of 140 ft.^{2}, so the rate at which Alex paints is

$$\frac{140 \text{ ft.}^2}{20 \text{ min.}} = 7 \text{ ft.}^2 \text{ per min.}$$

The rate at which Kristen paints is

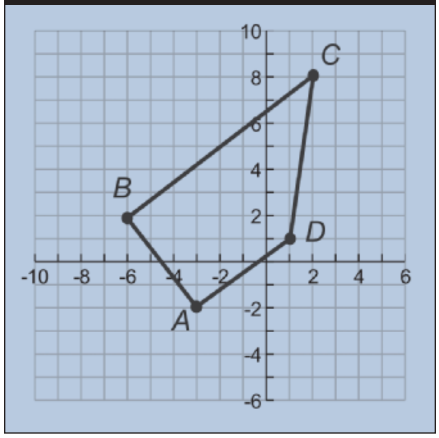
$$\frac{140 \text{ ft.}^2}{14 \text{ min.}} = 10 \text{ ft.}^2 \text{ per min.}$$

Working together, they can paint 7 + 10 = 17 ft.^{2} per min. To determine the time it will take them together to paint the wall, take the area of the wall (in square feet) and divide it by their rate of painting (in square feet per minute), which gives the number of minutes the painting will take them together:

$$\begin{aligned} \frac{140 \text{ ft.}^2}{17 \text{ ft.}^2 \text{ per min.}} &= 8.235294\ldots \\ &\approx 8.2 \text{ min.} \end{aligned}$$

13. After plotting and connecting the points, the quadrilateral appears to be a trapezoid, whose vertices are labeled in **figure 1**. Check our hypothesis: The slope of segment BC is 3/4; the slope of AD is also 3/4. Therefore, these sides are parallel and the quadrilateral is a trapezoid. Note that the slope of side AB is –4/3, the negative reciprocal of 3/4, so this segment is perpendicular to the parallel sides and is the height of the trapezoid. Using the distance formula,

Fig. 1 The quadrilateral in problem 13



$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2},$$

the length of this side is 5. Apply the distance formula to each of the parallel sides. For BC, use $B(-6, 2)$ and $C(2, 8)$ as follows:

$$\begin{aligned} d &= \sqrt{((-6)-(2))^2 + ((2)-(8))^2} \\ &= \sqrt{(-8)^2 + (-6)^2} \\ &= \sqrt{64+36} \\ &= \sqrt{100} \\ &= 10 \end{aligned}$$

For AD, use $A(-3, -2)$ and $D(1, 1)$ as follows:

$$\begin{aligned} d &= \sqrt{((-3)-(1))^2 + ((-2)-(1))^2} \\ &= \sqrt{(-4)^2 + (-3)^2} \\ &= \sqrt{16+9} \\ &= \sqrt{25} \\ &= 5 \end{aligned}$$

So the area of the trapezoid is:

$$A = \left(\frac{1}{2}\right)(5)(5 + 10)$$
$$= \frac{75}{2} = 37.5 \text{ square units}$$

14. The pattern of numbers is three more than multiples of 7, so the numbers take the algebraic form 3 + 7(n – 1), where n is the term number. Working backward from the largest 2-digit prime (see **table 1**), 73 is the largest two-digit prime number that is a term in this sequence.

15. At bookstore A, Mary's $45.00 book costs ($45.00)(1 – 0.25) – ($45)(0.75) = $33.75. Let x represent the original price of the book she should buy at bookstore B. Using her 10% off coupon, she will pay 90% of the original price. So at bookstore B, Mary will spend 0.90x. We want this to equal $33.75:

$$0.90x = \$33.75$$
$$x = \frac{\$33.75}{0.90} = \$37.50$$

Thus, the original price of the book Mary should buy at bookstore B is $37.50.

16. Let c be the number of people who liked cola; m, for milk; and w, for water. Since the number of people liking water and milk best is the same, let x represent both of those quantities. The number of people liking cola best is 30 percent more than the number liking water best; this is represented by 1.3x. Since 726 people participated in the survey, we write and solve:

Table 1 The solution to question 14

Two-Digit Prime Number	Determine If There Is an Integer n for which 3 + (n – 1)(7) Equals That Prime Number	A Term in That Sequence?
97	3 + (n – 1)(7) = 7n – 4 = 97 7n = 101 n = 101/7 = 14.42 . . . (Not an integer)	No
89	7n – 4 = 89 7n = 93 n = 93/7 = 13.28 . . . (Not an integer)	No
83	7n – 4 = 83 7n = 87 n = 87/7 = 12.42 . . . (Not an integer)	No
79	7n – 4 = 79 7n = 83 = 83/7 = 11.85 . . . (Not an integer)	No
73	7n – 4 = 73 n = 77 = 77/7 = 11 (An integer)	Yes

$$1.3w + w + w = 726$$
$$1.3w + 2w = 726$$
$$3.3w = 726$$
$$w = 726/3.3$$
$$= 220$$

So 220 people liked water, and 220 people liked milk. Also, 1.3(220) = 286 people liked cola.