palette of problems

David Rock and Mary K. Porter

1. The median and mode are equal for this set of seven whole numbers:

$$\{1, 3, 4, 5, 6, 11, x\}$$

The mean is also one of the six known members of the set. What is the value of x?

2. Alex and Monte are shopping for new clothes. Alex says to Monte, "If you give me $12, we will have an equal amount of money." Monte responds, "That may be true, but if you give me $6, I will have twice as much money as you." How much money do Alex and Monte each have?

3. A standard deck of 52 playing cards is shuffled. The top card in the deck is turned over, and a red 3 is observed. When the next card is turned over, the 3 of hearts appears. What is the probability that the third card turned over will be—

a. black?
b. red?
c. a 3?
d. the 3 of diamonds?

4. A cell phone company has 25 towers in a city. Each tower can handle 70,000 calls an hour. Each tower successfully connects 95% of the incoming calls sent to that tower. How many calls *would not be* connected during one 24-hour period, assuming that each of the 25 towers was running at maximum capacity at all times?

5. Tamara collects rocks. She has fewer than six dozen in her collection. If she groups the rocks by 2s, 1 rock is left over. If she groups the rocks by 3s, 2 are left over. If she groups the rocks by 4s, 3 are left over. If she groups the rocks by 5s, 4 are left over. How many rocks are in Tamara's collection?

6. A container holds 48 marbles (yellow, red, blue, and white). There are twice as many yellow as red marbles; twice as many blue as white marbles; and six more white than red marbles. What is the probability of randomly drawing a white marble out of the container?

7. Two red marbles are added to the container in problem 6; it now holds 50 marbles. What is the probability of randomly drawing a red marble out of the container?

Use the following poem for questions 8 and 9:

Roses are red
Violets are blue
Math is sweet
And so fun to do.

8. Determine the mean number of vowels per line of poetry.

9. Using the vowels only, which is the mode?

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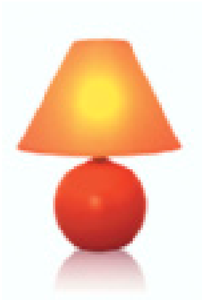


10. Seth withdrew some money from his bank account. After giving $15 to his cousin, he spent 1/5 of what was left on a lamp, then used the remaining $128 to buy a calculator. What percent of Seth's money did he give to his cousin? Round to the nearest tenth of a percent.

11. On a trivia quiz, a player earns 13 points for every correct answer and loses 5 points for every incorrect answer. If Marla's score is 192 after answering all 30 questions, how many questions did she answer correctly?

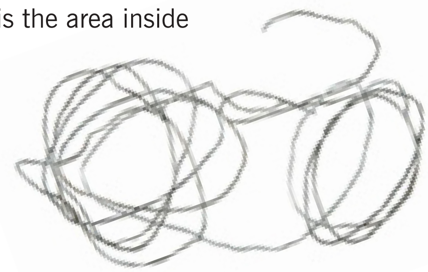
12. Marcus is playing a different trivia quiz and answered all the questions. This time, a player earns 9 points for every correct answer and loses a certain number of points for every incorrect answer. If Marcus's score is 230 after answering 2/3 of the 60 questions correctly, how many points does a player lose on this quiz for every wrong answer?

13. Three cats are fed 4 small meals each day. If the cats share the food equally, each cat gets a total of 110 grams of food a day. However, the largest of the 3 cats always eats 10% more than his share. How much food does 1 of the 2 smaller cats eat each day, assuming that they share the remaining food equally?

14. Christopher buys a length of wire. He bends it to form a rectangle that is four times as long as it is wide. If he wants the rectangle to have an area of 100 square inches, what should be the length of the original piece of wire?

15. Christopher's sister, Jessica, cuts a 36-inch piece of wire into two pieces. Each piece will be bent to form a square. The area of the larger square will be four times the area of the smaller square. What will be the area of the smaller square?

16. Christopher and Jessica cut a piece of wire into 2 pieces of equal length and bend each to form an equilateral triangle. If the original piece of wire was 24 inches long, what is the area inside one of the equilateral triangles? Round your answer to the nearest tenth of a square inch.



The solutions are appended to the online version of the "Palette of Problems" at **www.nctm.org/mtms**.

solutions to palette

(Continued from pages 398–99)

(Alternative approaches to those suggested here are encouraged.)

ANSWERS

1. 5
2. Alex has \$42; Monte has \$66.
3. (a) 26/50 = .52; **(b)** 24/50 = .48; **(c)** 2/50 = .04; **(d)** 0
4. 2,100,000 calls
5. 59 rocks
6. 11/48 = .229
7. .14
8. 5.25 vowels
9. e
10. Approximately 8.6%
11. 19 questions
12. 6.5 points
13. 104.5 grams
14. 50 inches
15. 9 square inches
16. Approximately 6.9 square inches

SOLUTIONS

1. Since the mode is the number that occurs most frequently, and each given value is only listed once, then x is one of the given values and is also both the mode and the median. Since the mean is one of the 6 known values, it must be an integer, so the sum of the 7 numbers must be divisible by 7. The sum without x is 30. The only number out of the set that can be added to 30 that will give a new sum that is divisible by 7 is 5. Therefore, the set is {1, 3, 4, 5, 5, 6, 11}. We see that the mode and median are 5. The mean is $(1 + 3 + 4 + 5 + 5 + 6 + 11)/7 = 35/7 = 5$.

2. Let A = Alex's amount and M = Monte's amount, in dollars. From Alex's statement, $A + 12 = M - 12$. If we subtract 12 from each side, then subtract M from each side, we get $A - M = -24$. From Monte's statement, we can write and simplify:

$$\begin{aligned} 2(A - 6) &= M + 6 \\ 2A - 12 &= M + 6 \\ 2A - M &= 18 \end{aligned}$$

Subtracting these two equations:

$$\begin{array}{r} 2A - M = 18 \\ -\underline{(A - M = -24)} \\ A = 42 \end{array}$$

Since $A = 42$, we find that $(42) + 12 = M - 12$ and that $M = 66$.

3. A standard deck of 52 playing cards has 13 red hearts, 13 red diamonds, 13 black clubs, and 13 black spades. Since the second card turned over is the 3 of hearts and the first card is a red 3, the first card must be the 3 of diamonds. Therefore, 50 cards remain, with 24 being red and 26 being black. The probability that the third card turned over will be—

a. black is 26/50 = .52.
b. red is 24/50 = .48.
c. a 3 is 2/50 = .04.
d. the 3 of diamonds is 0, since it was the first card turned over.

4. The total number of calls that can be successfully connected in one hour is $70,000 \times 25$, or 1,750,000. A total of 5%, or 87,500, of those calls are not successfully connected each hour: $87,500 \times 24 = 2,100,000$.

5. Tamara has fewer than 72 rocks, so list the possible values for each grouping less than 72. When grouping by 2s, 1 rock is left over, so the number must be odd. Therefore, the listings below are for odd numbers only:

By 3s, we begin with 5 (from 3 + 2); odd values are 5, 11, 17, 23, 29, 35, 41, 47, 53, 59, 65, 71.
By 4s, we begin with 7 (from 4 + 3); odd value are 7, 11, 15, 19, 23, 27, 31, 35, 39, 43, 47, 51, 55, 59, 63, 67, 71.
By 5s, we begin with 9 (from 5 + 4); odd values are 9, 19, 29, 39, 49, 59, 69. Notice that 59 is the only value in each set of multiples.

Another method is to group the rocks by 5s and note that there are 4 left over. Therefore, we know that the number of rocks in the collection must end in either 4 or 9 (four more than multiples of 5). We can eliminate those possibilities ending in 4 because the grouping by 2s tells us that there are an odd number of rocks. Check 69, 59, 49, . . . to see which leaves a remainder of 2 when divided by 3, and we find that 59 is the correct answer.

6. Let r = red marbles, y = yellow marbles, w = white marbles, and b = blue marbles. Then:

(1) $2r = y$
(2) $2w = b$
(3) $r + 6 = w$

Equation (3) can be written as $r = w - 6$ and substituted into (2) to obtain

$$2(r + 6) = b \text{ or } 2r + 12 = b.$$

For the total, we know

$$r + b + w + y = 48.$$

Substituting expressions involving r throughout, we can write and solve:

$$\begin{aligned} r + (2r + 12) + (r + 6) + (2r) &= 48 \\ 6r + 18 &= 48 \\ 6r &= 30 \\ r &= 5 \end{aligned}$$

Since we know that $r + 6 = w$, we see that $(5) + 6 = w$, so $w = 11$. Since there are 11 white marbles, the probability of drawing a white marble is 11/48 = .229. (Note that $y = 2(5) = 10$ and $b = 2(11) = 22$.)

7. Adding 2 red marbles to the 5 determined in problem 6 means that there are now 5 + 2 = 7 red marbles and a total of 50 marbles. The probability of randomly drawing a red marble is 7/50 = .14.

8. Determine the number of vowels for each line:

Roses are red:
 5 vowels (a: 1, e: 3, o: 1)
Violets are blue:
 7 vowels (a: 1, e: 3, i: 1, o: 1, u: 1)
Math is sweet:
 4 vowels (a: 1, e: 2, i: 1)
And so fun to do:
 5 vowels (a: 1, o: 3, u: 1)

The average number of vowels per line is 5 + 7 + 4 + 5 = 21/4 = 5.25.

9. To determine the mode of vowels for the poem, we have a: 4, e: 8, i: 2, o: 5, u: 2. Therefore, e is the mode.

10. Let x represent the amount of money Seth took out of his bank account. After he gave \$15 to his cousin, he had $x - 15$ dollars. After he spent 1/5 of the remaining money on a lamp, 4/5 was left, and can be written as $(4/5)(x - 15)$. This is \$128, so we write and solve:

$$\begin{aligned}
\frac{4}{5}(x-15)&=128\\
5\cdot\left[\frac{4}{5}(x-15)\right]&=128\cdot 5\\
4(x-15)&=640\\
4x-60&=640\\
4x&=700\\
x&=175
\end{aligned}$$

Since Seth originally had \$175 in his bank account and gave \$15 to his cousin, he gave 15/175 ≈ 0.085714 or approximately 8.6% of his money to his cousin.

11. Let x be the number of questions Marla answered correctly. Then, $30 - x$ is the number of questions she answered incorrectly. She earns 13 points for every correct answer, so she will earn $13x$ points for answering the x questions correctly. Since she loses 5 points for every incorrect answer, her point total will be $13x - 5(30 - x)$, which we set equal to 192 and solve:

$$\begin{aligned}
13x-5(30-x)&=192\\
13x-150+5x&=192\\
18x-150&=192\\
18x&=192+150\\
18x&=342\\
x&=19
\end{aligned}$$

Marla answered $x = 19$ questions correctly.

12. Marcus answered 2/3 of the 60 questions correctly, so he answered 2/3 × 60 = 40 questions correctly and answered the other 20 questions incorrectly. He earned 9 points for every correct answer, or (9)(40) = 360 points, for 40 correct answers. Let x be the number of points that a player loses for every wrong answer on this quiz. Marcus lost $20x$ points for answering the 20 questions incorrectly. Marcus's score on this quiz can be represented as $360 - 20x$, which must equal 230, so we solve:

$$\begin{aligned}
360-20x&=230\\
360-230&=20x\\
130&=20x\\
x&=\frac{130}{20}=6.5
\end{aligned}$$

A player loses 6.5 points for every wrong answer on this quiz.

13. The fact that the cats are fed 4 meals a day is not relevant in finding the solution to this problem. We only need to use the fact that the total per cat is 110 grams of food per day. The 3 cats together consume 330 grams of food per day. The largest cat eats 10% more than his share, so he eats 110 grams plus an additional 10%, which is an additional 0.10(110) = 11 grams. Thus, the largest cat eats 110 + 11 = 121 grams. This leaves 330 – 121 = 209 grams for the 2 smaller cats to share; 209/2 = 104.5 grams of food per day that each of the 2 smaller cats consumes.

14. Let w represent the width of Christopher's rectangle: The rectangle's length is $4w$, and the area of the rectangle will be

$$(\text{length})(\text{width}) = (4w)(w) = 4w^2.$$

This area equals 100 square inches, so $4w^2 = 100$, which means that $w^2 = 25$. Thus, $w = 5$ inches (we ignore the negative solution). The length of the wire is the perimeter, P, of the rectangle, which can be written as $2(w) + 2(4w) = 10w$. So,

$$P = 10(5) = 50 \text{ in.}$$

15. Let x be the length of one side of the smaller square. Then x^2 is the area of the smaller square, so $4x^2$ will be the area of the larger square. To find the length of one side of the larger square, take the square root of this square's area: The positive square root of $4x^2$ is $2x$. The perimeter of each square is found by adding the lengths of the square's four sides, so the perimeter of the smaller square is $x + x + x + x = 4x$ and the perimeter of the larger square is

$$2x + 2x + 2x + 2x = 8x.$$

The total of these two perimeters is

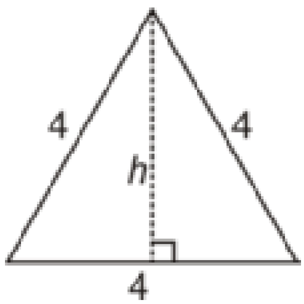
$12x$, which is also the length of the piece of wire, so $12x = 36$ inches, or

$$x = 36/12 = 3 \text{ inches.}$$

Hence, the area of the smaller square is

$$x^2 = (3)^2 = 9 \text{ square inches.}$$

16. The original piece of wire was 24 inches long, so each half is 12 inches. Each 12-inch wire is bent to form an equilateral triangle, and each side of an equilateral triangle must be the same length, which is 1/3 of the 12 inches, or 4 inches, long. The area of any triangle is (1/2)(base)(height), and the base has length 4 inches. (See the diagram below.)



We now need to find the height, h, of the equilateral triangle, which is the distance from one vertex of the triangle to the side of the triangle that is opposite that vertex. The height, h, divides the triangle into two congruent right triangles. Each triangle has height h and base (1/2)(4 in.) = 2 in., and the hypotenuse is 4 inches (because it is one side of the equilateral triangle). Use the Pythagorean theorem, $h^2 + (2)^2 = 4^2$, so $h^2 + 4 = 16$, which means that $h^2 = 12$. Thus, $h = \sqrt{12}$, or approximately 3.4641 . . . inches. Hence, the area of one of the equilateral triangles is

$$\left(\frac{1}{2}\right)(\text{base})(\text{height}) =$$

$$\left(\frac{1}{2}\right)(4)(3.4641...) = 6.9282...,$$

or approximately 6.9 square inches.