palette of problems

David Rock and Mary K. Porter

1. Find the four unique prime factors of 6555. *Hint*: They are each less than 40.

2. Ethan is playing in a soccer league that has 6 teams (including his team). Each team plays every other team twice during the regular season. The top two teams play in a final championship game after the regular season. In this league, how many soccer games will be played in all?

3. Katelyn's school building opens 1 hour before school begins. Students arrive in groups. Katelyn and her brother, the first 2 students to arrive, are considered the first group. The second group to arrive has 1 more student than the first group. The third group has 1 more student than the second group. If 405 students are in school, how many groups will arrive, assuming that each group has 1 more member than the group that just arrived?

4. Your teacher has a bag that contains fewer than 100 pieces of candy. If your teacher makes groups with 2, 3, or 4 pieces of candy, she will have 1 piece left over. If she makes groups with 5 pieces of candy, she will have none left over. How many pieces of candy could be in the bag? If there is more than one answer, provide all the answers that you can.

5. Observe the numbers in each of the following sets. Determine the missing number *a* in the last set.

$$\{29, 2, 27\}, \{28, -3, 33\}, \{29, 7, 22\}, \{a, 3, 39\}$$

6. If four hens lay 9 eggs in 7 days, how many days will it take four dozen hens to lay 27 dozen eggs?

7. Two different sets of two or more consecutive positive integers can be added that sum to 100.

a. Find the set consisting of 5 numbers.

b. Find the set consisting of 8 numbers.

8. Observe the new operation \$ on the following integers:

$$\begin{aligned} 7 \,\$\, 2 &= 16 \\ 7 \,\$\, 5 &= 19 \\ 3 \,\$\, 9 &= 15 \\ 4 \,\$\, 9 &= 17 \\ 5 \,\$\, 9 &= 19 \\ 15 \,\$\, 5 &= 35 \end{aligned}$$

Find the value of 75 \$ −30.

9. Each year, the value of the Koenigs' house increases by the same amount of money. When they bought the house, it was worth \$136,000. Five years later, it was worth \$167,040. How much will the house be worth 8 years after they bought it?

10. Refer to problem 9. How many years after buying the house will it surpass \$250,000 in value?

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11. Haley has a wooden board that is 13 feet long. She uses an electric saw to cut the board into five pieces. Every cut made with this saw results in a loss of 1/8 inch in length. Haley cuts a piece that is 5 1/2 feet long. She then cuts the remaining piece of board into four pieces of equal length. What will be the length of each of those remaining four pieces?

12. Randy is mowing his lawn. The front wheels of the mower are 5 inches in diameter, and the back wheels are 8 inches in diameter. Randy begins by mowing a path along the front edge of his lawn, which is 48 feet long. In this path of length 48 feet, how many more rotations will each front wheel make than each back wheel? Round your answer to the nearest tenth of a rotation.

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Use the following clues about Consonant Land to answer problems 13–15.

- 1 *penny* is worth 7 cents: 2 cents for each consonant (p, n, n, y) minus 1 cent for the vowel (e).
- 1 *nickel* is worth 6 cents: 2 cents for each consonant (n, c, k, l) minus 1 cent for each vowel (i, e).
- 1 *dime* is worth 2 cents: 2 cents for each consonant (d, m) minus 1 cent for each vowel (i, e).

13. How much is 1 quarter worth?

14. If someone asks you for change for a penny, what coins will you give that person?

15. If you have twice as many pennies as nickels, and the total value of these coins is the same as the value of 32 quarters, how many pennies do you have?

16. At the Fresh Fruit Market, a box of 16 peaches costs $2.01 more than a bag of 12 apples. Bobby bought two boxes of peaches and five bags of apples. He paid $46.11, which included 6% sales tax. What is the price of each box of 16 peaches, and what is the price of each bag of 12 apples?

The solutions are appended to the online version of the "Palette of Problems" at **www.nctm.org/mtms**.

solutions to palette

(Continued from pages 458–59)

(Alternative approaches to those suggested here are encouraged.)

ANSWERS

1. 3, 5, 19, 23
2. 31
3. 27 groups
4. 25 or 85; students must list both answers
5. 16
6. 21 days
7. a. {18, 19, 20, 21, 22};
7. b. {9, 10, 11, 12, 13, 14, 15, 16}
8. 120
9. $185,664
10. 19 years after they bought the house
11. 1 ft., 10 3/8 in.
12. Approximately 13.8 rotations
13. 5 cents
14. 1 dime and 1 quarter
15. 16 pennies
16. One box of peaches costs $7.65; one bag of apples costs $5.64

SOLUTIONS

1. Since the last digit is a 5, the number is divisible by 5 and 5 is a factor. Therefore, we have to find three prime factors of 6555/5, or 1311. Since the sum of the digits of 1311 is 6, which is divisible by 3, we know that the number has a factor of 3: 1311/3 = 437. Checking other prime numbers greater than 5, we find that 437/19 = 23. Thus, $6555 = 3 \times 5 \times 19 \times 23$.

2. This problem resembles the handshake problem with 6 people. If each team plays every team only once, the number of games can be counted. To establish a process for counting the combinations, count the number of games with fewer than six teams:

- Two teams will play 1 game (AB).
- Three teams will play 3 games (AB, AC, BC).
- Four teams will play 6 games (AB, AC, AD, BC, BD, CD).
- Five teams will play 10 games (AB, AC, AD, AE, BC, BD, BE, CD, CE, DE).

Look for a pattern. Adding a—

- third team adds 2 games.
- fourth team adds 3 games, for a total of 6.
- fifth team adds 4 games, for a total of 10 games.

Each additional team will add the next consecutive counting number of games to the total. Six teams will play a total of 10 + 5 = 15 games. The formula would be

$$\frac{(n)(n-1)}{2},$$

where n stands for the number of teams in the league. See the computation below:

$$\frac{(6)(6-1)}{2} = \frac{(6)(5)}{2} = \frac{30}{2} = 15$$

Since each team plays every other team twice, rather than just once, 2(15) = 30. Add the one championship game, and 31 games will be played overall.

3. Apply Gauss's expression

$$\frac{(n)(n+1)}{2}$$

to determine the sum of consecutive counting numbers $1 + 2 + 3 + \cdots + n$. In the expression, n is the number of counting numbers. To find the value of n that gets us close to 405, we can use guess and check:

$$\frac{(n)(n+1)}{2} \geq 405$$

Look for the smallest value of n that will make the statement true.

If $n = 25$, then (25)(26)/2 = 325.
If $n = 26$, then (26)(27)/2 = 351.
If $n = 27$, then (27)(28)/2 = 378.
If $n = 28$, then (28)(29)/2 = 406.

If we begin with a group with 1 student, we have 28 groups and 406 students. But we must begin with a group having 2 students. Therefore, the first group of 1 is eliminated. We therefore have 27 groups for a total of 405 students.

To better understand, try a simpler problem. If the school had 14 students, $n = 5$ would yield (5)(6)/2 = 15, or

$$1 + 2 + 3 + 4 + 5 = 15.$$

Without the first group of 1, there would be 4 groups and 2 + 3 + 4 + 5 = 14 students. Returning to the school problem, since $n = 28$, then removing the first group of 1 yields 27 groups for 406 – 1 = 405 students.

4. The number of pieces of candy must be a multiple of 5 and not a multiple of 2, 3, or 4. Determine an odd multiple of 5 that will also result in a remainder of 1 when divided by 2, 3, and 4. If there were no leftovers when dividing by 2, 3, or 4, we would know that we are looking for multiples of 12 (from 3×4). We are looking for numbers that are 1 greater than a multiple of 12, since we have a remainder of 1. The multiples of 12 less than 100 are 12, 24, 36, 48, 60, 72, and 84. Of these, 24 and 84 are of interest because they are 1 less than a multiple of 5; 25 and 85 must be the possible number of pieces.

5. The numbers in each of the sets sum to 58; $a + 3 + 39 = 58$, or $a = 16$.

6. If four hens lay 9 eggs, then a dozen hens (4×3) hens will lay $9 \times 3 = 27$ eggs in 7 days. Therefore, four dozen hens will lay $27 \times 4 = 108$ eggs in 7 days. Twenty-seven dozen eggs is $12 \times 27 = 324$ eggs. Since 324 is three times as many as 108 eggs, it would take three times as many days, or 21 days.

7. a. Since we know there are 5 integers, $100 \div 5 = 20$. Two of the integers are less than 20 (18 and 19) and two are greater than 20 (21 and 22).

7. b. Since we know there are 8 integers, $100 \div 8 = 12.5$. Therefore, four integers will be less than 12.5 (9, 10, 11, 12) and four will be greater than 12.5 (13, 14, 15, 16).

8. Examine the operations given:

7 \$ 2 = 16
7 \$ 5 = 19
3 \$ 9 = 15
4 \$ 9 = 17
5 \$ 9 = 19
15 \$ 5 = 35

Since 3 \$ 9 = 15, 4 \$ 9 = 17, and 5 \$ 9 = 19, we see that as the first operand increases by 1, the result increases by 2; therefore, the rate of change is 2. Since 7 \$ 2 = 16 and 7 \$ 5 = 19, we see that as the second operand increases by 3, the result is also increased by 3. Therefore, we can assume that the second operand is added for each result. If we try a \$ $b = (2a + b)$ for all examples given, we find that it works. Therefore,

75 \$ 30 = 2(75) + (-30) = 120.

Table 1 The solution to problem 9

Years after They Bought the House	The Value of the Koenigs' House That Year
0	\$136,000
1	\$136,000 + d
2	\$136,000 + d + d = \$136,000 + 2d
3	\$136,000 + d + d + d = \$136,000 + 3d
4	\$136,000 + 4d
5	\$167,040 = \$136,000 + 5d
. . .	. . .
8	\$136,000 + 8d = ?
. . .	. . .
n	\$136,000 + nd

9. Let d represent the yearly increase (in dollars) in the value of the Koenigs' house and notice the pattern, as shown in **table 1**. Five years after they bought the house, it was worth \$167,040:

$$\begin{aligned} \$136{,}000 + 5d &= \$167{,}040 \\ 5d &= \$167{,}040 - \$136{,}000 \\ 5d &= \$31{,}040 \\ d &= \frac{\$31{,}040}{5} = \$6208 \end{aligned}$$

Thus, each year the house value increased by \$6208. Eight years after buying the house, it will be worth \$185,664.

10. As determined in question 9, in year number n, the value of the house will be worth \$136,000 + \$6208n. We must determine what is the smallest integer value of n that will make

$$\$136{,}000 + \$6208n \geq \$250{,}000,$$

which we solve as follows:

$$\begin{aligned} \$136{,}000 + \$6208n &\geq \$250{,}000 \\ \$6208n &\geq \$250{,}000 - \$136{,}000 \\ \$6208n &\geq \$114{,}000 \\ n &\geq \frac{\$114{,}000}{\$6208} \geq 18.3634\ldots \end{aligned}$$

The smallest integer n that will satisfy this problem is $n = 19$. As an additional check, note that $n = 19$ years after buying the house, it will be worth:

$$\begin{aligned} &\$136{,}000 + (\$6208)(19) \\ &= \$136{,}000 + \$117{,}952 \\ &= \$253{,}952 \end{aligned}$$

This figure surpasses the \$250,000 mark, whereas at $n = 18$, the house is worth less than \$250,000.

11. After Haley cuts off the piece of board that is 5 1/2 feet long, the remaining part of the board will have this length:

(13 ft. – (5 ft., 6 in.)) – (1/8 in.)
= (7 ft., 6 in.) – (1/8 in.),

which is 7 feet, 5 7/8 inches. Haley

must now cut this piece into four pieces of equal length, which will require three saw cuts for a loss of 3(1/8 in.) = 3/8 in. of the board. Therefore, the four equal-sized pieces of wood will have a total length of (7 ft., 5 7/8 in.) – (3/8 in.), which is 7 ft., 5 1/2 in. Hence, each of the four pieces will have this length:

(1/4)(7 ft., 5 1/2 in.)
= (1/4)7 ft. + (1/4)5 in. + (1/4)(1/2) in.
= 7/4 ft. + 5/4 in. + 1/8 in.
= (1 ft. + 9 in.) + (1 in. + 1/4 in.) + 1/8 in.
= 1 ft. + 10 in. + 2/8 in. + 1/8 in.
= 1 ft., 10 3/8 in.

12. Note that the circumference of a circle is diameter × π, so the circumferences of the 5-inch and 8-inch wheels are 5π inches and 8π inches, respectively. Therefore, in one complete rotation of the 5-inch wheel, the lawn mower travels 5π inches; with the 8-inch wheel, the lawn mower travels 8π inches. When the lawn mower travels 48 feet, which is (48)(12) = 576 in., the 5-inch wheel will make this number of rotations:

$$\frac{576 \text{ in.}}{5\pi \text{ in.}} \approx 36.669298...$$

The 8-inch wheel will make this number of rotations:

$$\frac{576 \text{ in.}}{8\pi \text{ in.}} \approx 22.918311...$$

The difference between these numbers is

$$36.669298\ldots - 22.918311\ldots = 13.750987\ldots,$$

or approximately 13.8 rotations.

13. Two cents for each consonant (q, r, t, r) is 8; subtract 1 cent for each vowel (u, a, e): 8 – 3 = 5.

14. Since 1 penny is worth 7 cents, you would give 1 dime (worth 2 cents) and 1 quarter (worth 5 cents).

15. The value of 32 quarters is 32 × 5 = 160 cents. Let n be the number of nickels you have. Then $2n$ is the number of pennies you have. Since each penny is worth 7 cents and each nickel is worth 6 cents, then the value of your coins is

$$2n(7) + n(6) = 14n + 6n = 20n \text{ cents.}$$

Thus, $20n = 160$, so $n = (160)/20 = 8$. Therefore, you have $n = 8$ nickels, and, hence, $2n = (2)(8) = 16$ pennies.

16. When Bobby bought fruit, he paid \$46.11, which included a 6% sales tax. If f is the total price of the fruit he bought, then $(1.06)f$ represents the total he paid, including tax, so write and solve:

$$\begin{aligned}(1.06)f &= \$46.11\\ f &= \frac{\$46.11}{1.06}\\ &= \$43.50\end{aligned}$$

We know the total price of the fruit he bought, not including the sales tax, is \$43.50. Let p represent the price of a box of peaches, and let a represent the price of a bag of apples. Since 1 box of peaches costs \$2.01 more than 1 bag of apples, $p = a + \$2.01$. Bobby bought 2 boxes of peaches and 5 bags of apples, which can be represented by $2p + 5a$. So

$$2p + 5a = \$43.50.$$

Substitute $p = a + \$2.01$, and solve:

$$\begin{aligned}2(a+\$2.01)+5a &= \$43.50\\ 2a+\$4.02+5a &= \$43.50\\ 7a+\$4.02 &= \$43.50\\ 7a &= \$43.50-\$4.02\\ 7a &= \$39.48\\ a &= \frac{\$39.48}{7}\\ a &= \$5.64\end{aligned}$$

Find p by substituting $a = 5.64$ into $p = a + \$2.01$ and get

$$p = (\$5.64) + \$2.01 = \$7.65.$$