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Gauss's Idea: Take a Notion

A grandfather clock in Jodie's home chimes once at one o'clock, twice at two o'clock, and so on. If there are no chimes except at the top of the hour, how many times does the clock chime between one o'clock and ten o'clock?

You might recognize this problem as a basic application of the Gauss formula. Many mathematics teachers enjoy telling the story of the precocious young math wizard who was given this assignment:

> Add all the whole numbers between one and one hundred.

Young Carl Frederick Gauss confounded his teacher's attempt to give him busy work and realized that, instead of performing a tedious string of sums, he could pair numbers and multiply. The sum of the first and last terms is 101, and the same is true for the second smallest and second largest, and so on (see **fig. 1**). Gauss saw that the fifty pairs of numbers each summed to 101, so he quickly multiplied and gave the answer of 5050.

For the clock example above, assuming that we count the chimes at 1 and 10, we have five pairs of numbers that sum to 11. Jodie would hear 55 total chimes. To help students visualize the pairing of terms, we suggest using a strip of 2 cm grid paper, writing each term of the sequence in a separate box, and cutting the *sequence strip* to the correct length (see **fig. 2a**). To find the middle of the sequence, fold the ends together and cut along the fold. The two pieces of grid paper can then be placed as shown in **figure 2b**. When arranged correctly, with one set of numbers upside down, the pattern will be clear.

Gauss once said, "What we need are notions, not notations." He produced an idea of exquisite quality and extreme flexibility. The formula, however, is narrow and unyielding. Using Gauss's idea, a series of mathematical discoveries await us. The formula, just to remind you, is often presented as the sum of the first n positive integers:

$$1+2+3+\cdots+n=\sum_{k=1}^{n}k=\frac{n}{2}(n+1)$$

Edited by **Gwen Johnson**, gwendolyn.johnson@unt.edu, University of North Texas, Dallas, and **James Dogbey**, jdogbey@clemson.edu, Clemson University, South Carolina. This department's classroom-ready activities may be reproduced by teachers. Teachers are encouraged to submit manuscripts in a format similar to this department based on successful activities from their own classroom. Of particular interest are activities focusing on NCTM's Content and Process Standards and Curriculum Focal Points as well as problems with a historical foundation. Send submissions by accessing **mtms.submit.net**.

We heartily recommend pursuing the *notion*, rather than trying to help students decipher this difficult *notation*.

Consider these problems:

1. How many times does Jodie's clock chime between 3:30 and 11:30?
2. Jeremy's mother puts 7 candles on his birthday cake for his seventh birthday, uses 8 candles for his eighth birthday, and so on. How many candles does Jeremy blow out, starting on his seventh birthday and finishing with his thirteenth?

We ask students for the following sums:

Problem 1: $4 + 5 + \cdots + 11 = ?$
Problem 2: $7 + 8 + \cdots + 13 = ?$

Using Gauss's approach, we find pairs of numbers with equal sums, leveraging the speed of multiplication to avoid repeated addition.

For problem 1, 4 + 11 = 15, as does 5 + 10, and so on. We have four pairs of numbers whose sums are 15; therefore, the answer is 60. The process works on a wide range of patterns, so we need not start at one or count by ones. For example, we could also ask:

$$2 + 4 + 6 + \cdots + 98 + 100 = ?$$

We have 25 pairs of numbers whose sums are 102, so the answer is 2550.

In problem 2, the sum of the listed numbers $7 + 8 + \cdots + 13$ has an odd number of terms, so an extra half pair remains. Paper folding can be used here as well to help spark a discussion about whether Gauss's idea still works. With this problem, we must cut one of the squares in half (see **fig. 3**). We have three pairs of numbers whose sum is 20, plus the leftover 10. In other words, we have three pairs of 20 plus a half pair of 20, or 3.5 × 20, which gives the correct answer of 70.

Fig. 1 To find the sum of the first 100 numbers, Gauss paired numbers in the series to establish a formula for the sum.

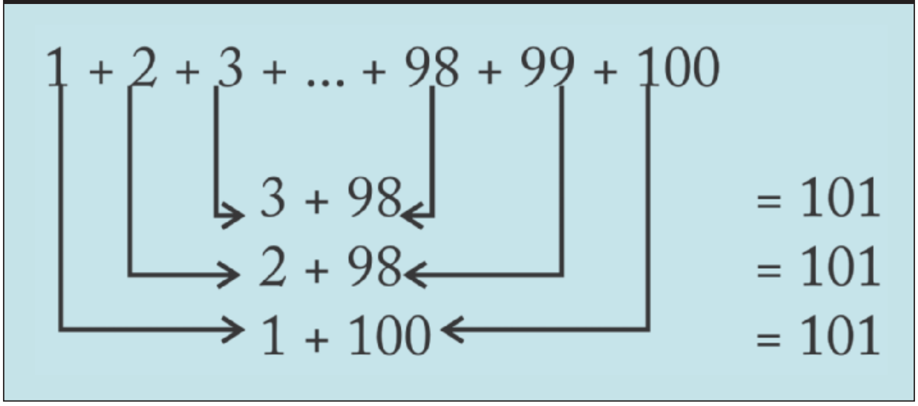


Fig. 2 A strip of paper with consecutively numbered sections that is then folded and cut illustrates Gauss's idea.

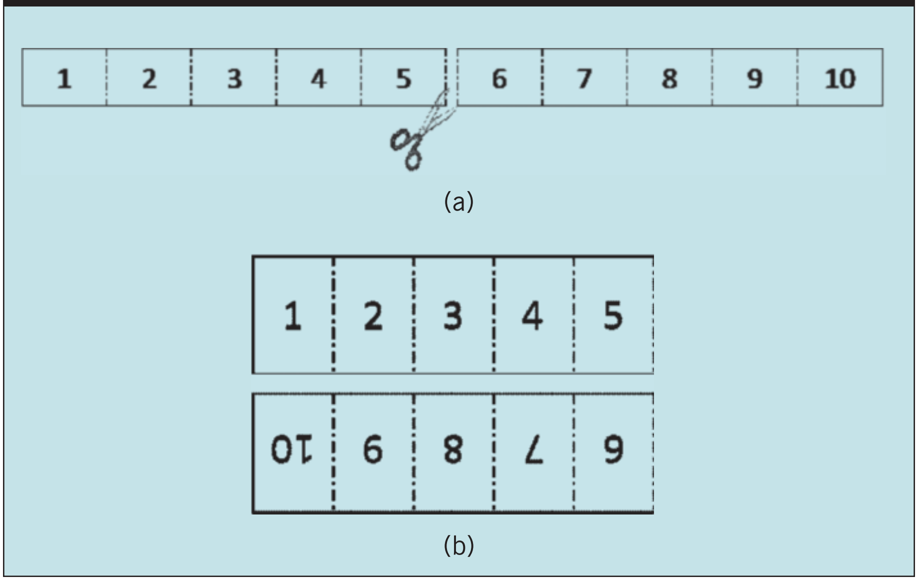
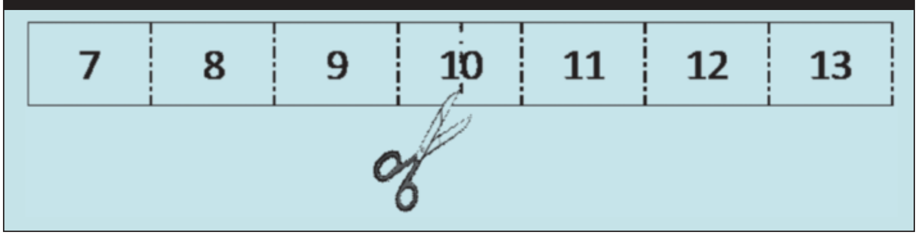


Fig. 3 When the series has an odd number of terms, the middle number is divided in half.



Sixth-grade students can use Gauss's idea with the following:

1. $21 + 24 + 27 + \cdots + 87 + 90$
2. $25 + 30 + 35 + \cdots + 70 + 75$
3. $9 + 13 + 17 + \cdots + 33 + 37$
4. $28 + 26 + 24 + \cdots + (-6) + (-8)$

Although the formula works in some settings, Gauss's idea is more robust, and we explore it in **activity sheets 1**, **2**, and **3**. **Activity sheet 1** uses paper-folding activities to help students investigate this approach. **Activity sheet 2** has similar problems that involve story contexts. **Activity sheet 3** demonstrates several short discovery problems that extend the idea to noninteger settings. Other interesting examples can be written using Gauss's idea to develop number sense and computational fluency. We suggest using short discovery problems based on this approach throughout the curriculum to introduce new topics and to review and practice other topics. All investigations described can be solved by sixth-grade students who use Gauss's idea creatively.

TEACHING NOTES

What is Gauss's idea? A sequence of numbers with a fixed, or common, difference between terms is called an *arithmetic sequence*. The first term can be any real number including a negative integer or a rational number. The only requirement is that the difference between any two adjacent terms is constant.

Gauss's approach provides an easy way to calculate the sum of a finite number of consecutive terms in any arithmetic series, with one caveat: Discovering the *exact* number of pairs can sometimes be tricky. However, that trickiness has led two of our classes of in-service teachers (and their classes of fourth-grade and fifth-grade students) into some exciting mathematical explorations. By using this idea, we can structure quality investigations for students at almost any level.

See, for instance, the following series:

Example 1

$21 + 24 + 27 + \cdots + 87 + 90 = ?$

We can see that the sum of each pair is 111. Most students can routinely find that part of the solution. The more interesting issue involves determining how many pairs of numbers are in the sequence. Most students

Fig. 4 Students may easily determine the sum of each pair but not how many pairs are found. This arrangement can easily be explored using arrays.

21 24 27 30
33 36 39
42 45 48
51 54 57 60
63 56 69
72 75 78
81 84 87 90

(a)

28	26	24	22	20
18	16	14	12	10
8	6	4	2	0
−2	−4	−6	−8	

(b)

try an ordered list of some sort. With experience, they often think of quick ways to speed the process. If you have a hundred chart in your classroom, your students may think of an array. The approach shown in **figure 4a** is typical.

Of the seven rows of numbers, four rows have three numbers (12 total), and three rows have four numbers (12 total). Since 24 numbers are in the sequence, 12 pairs are found. The sum of the sequence is 1332. Short discovery investigations like this promote the use of ordered lists and arrays, which are good problem-solving tools for all levels of mathematics to ensure that each item in a group is counted.

What happens with an odd number of terms?

Example 2

$25 + 30 + 35 + \cdots + 70 + 75 = ?$

We have sums of 100, but eleven terms are in the series. How many pairs do we have? Initially, our in-service teachers worried that having 5.5 pairs might lead to a noninteger answer when the sum must clearly be a whole number. In this case, the sum is 550. But will the sum of an odd number of integers always be an integer? The answer is yes, but asking students to verify the result will spark a good mathematical discussion.

We asked a class of in-service teachers if Gauss's approach would work if the series included negative terms:

Example 3

$28 + 26 + 24 + \cdots + (-6) + (-8) = ?$

These teachers were concerned because the first and last sum seemed very different. We asked how we could investigate whether it would work for a series such as this. They decided to use another problem-solving tool by testing several simpler but related problems. They used the sums of several short sequences to easily perform the additions and verify the result using Gauss's approach. See these examples:

- $(-1) + 0 + 1 + 2 + 3 + 4$
- $(-1) + 1 + 3 + 5$

Returning to the original sum, we attempted the same type of arrangement as found in example 3 (see **fig. 4b**).

Counting the pairs posed no problem when using this 4×5 array. With 20 cells in the array and one "hole," our students quickly counted the 19 numbers. That means that there are 9.5 pairs, and the sum of each pair is 20. Therefore, the sum of the sequence is given by $9.5 \times 20 = 190$.

DISCUSSION

Choosing the right arithmetic sequence can reinforce arithmetic operations with fractions, mixed numbers, and integers. The approach that was discovered by young Gauss also works on sums such as these:

- $2\frac{1}{5}+2\frac{3}{5}+3+3\frac{2}{5}+\cdots+6\frac{3}{5}+7$
- $3.20 + 3.32 + 3.44 + \cdots + 4.28 + 4.40$

Short discovery problems using this idea can introduce many topics in the middle-grades curriculum. Many different problem-solving techniques arise as we attempt to find the patterns and count pairs. In so doing, operational fluency with different types of rational numbers can be improved. In place of repetitive practice activity sheets, we suggest strategically placing problems involving Gauss's idea throughout the course to help students review operations with integers and rational numbers.

For younger students, we make the sequences shorter and use simpler fractions, mixed numbers, or decimals. For older or more advanced students, we can make the sequences longer, use more difficult quantities, or both.

Allowing students to tackle problems without being confined to a formula may present them with an opportunity to develop habits of mind that are implicit in good problem solving. For instance, many students may actually try to develop a formula. They will notice patterns in their work and want to summarize and simplify. This desire contributes to an appreciation for algebraic notation and a clearer understanding of symbols. These types of short discovery activities presented on a regular basis allow students to progress naturally into successful problem solving and to move along the problem-solving continuum from novice to expert.

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Solutions to **activity sheets 1**, **2**, and **3** are appended online to "Mathematical Explorations" at **www.nctm.org/mtms.**

activity sheet 1

Name ______________________________

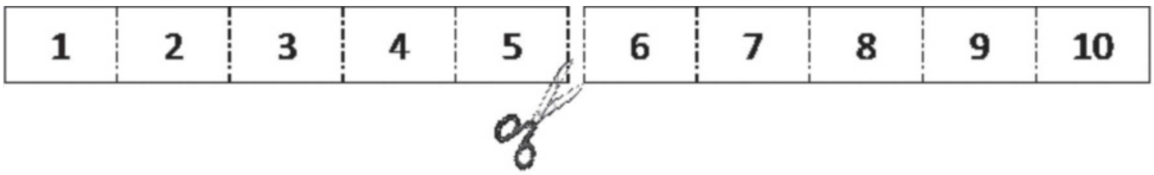
THE GRANDFATHER CLOCK PROBLEM

1. A grandfather clock in Jodie's home chimes once at one o'clock, twice at two o'clock, three times at three o'clock, and so on. If there are no chimes except at the top of the hour, we want to determine:

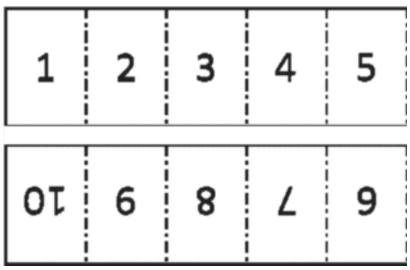
How many times does the clock chime between one o'clock and ten o'clock?

To find the answer, follow these steps:

- Use grid paper and cut one row 10 blocks long (we call this the *sequence strip*).
- Write "1" in the first box, "2" in the second box, and so on, up to "10."
- Fold the sequence strip in half, and cut along the crease.



- Take the second strip and place it below the first, so that the sum in the columns is 11.



a. How many pairs of 11 are found?

b. What is the sum of the whole numbers from 1 to 10?

c. Using your answers from **(a)** and **(b)**, how many times does the clock chime between one o'clock and ten o'clock?

2. Use the same paper-cutting method to find the following sums. Draw a picture and write an equation (using multiplication) to show how you found the sum.

a. $25 + 30 + 35 + \cdots + 75 + 80 =$

b. $7 + 8 + \cdots + 13 =$

c. $26 + 23 + 20 + \cdots + (-1) + (-4) =$

activity sheet 2

Name ______________________________

GAUSS'S IDEA AT WORK

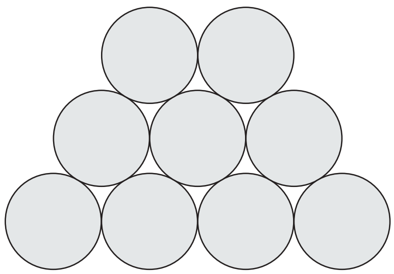
Use the same approach you applied to solve the Grandfather Clock problem, or present another approach, to answer the following questions. Show or describe your calculations.

1. A grandfather clock chimes once at one o'clock, twice at two o'clock, three times at three o'clock, and so on.

a. Jessica eats supper at 6:30 p.m. and breakfast at 6:30 a.m. How many times does the clock chime between supper and breakfast?

b. Jessica is in school from 7:45 a.m. until 2:45 p.m. How many times does the clock chime while she is at school?

2. A stack of logs always has 1 fewer log in the row on top than the row directly below it. If the stack has 15 rows of logs and the top row contains 8 logs, how many logs are in the stack?



3. Jeremy's mother puts 7 candles on his birthday cake for his seventh birthday, uses 8 candles for his eighth birthday, and so on. How many total candles does Jeremy blow out, starting on his seventh birthday and finishing with his fifteenth birthday?

activity sheet 3

Name ______________________________

COMPLEX PROBLEMS USING GAUSS'S IDEA

Use the same approach you applied to solve the Grandfather Clock problem, or present another approach, to answer the following questions. Show or describe your calculations.

1. All middle school students were to report to the gym for a meeting. The students were seated in the stands. Six people were seated in the first row. Each row after the first contained 2 more students than the one before it. A total of 17 rows were filled in this manner.

a. How many students were seated in row 17?

b. How many total students were at the meeting?

2. Joe began an exercise program. He mapped out a loop through his neighborhood, which he followed while walking. On the first day, he walked 2/5 of the loop. Each day, he added 2/5 of the loop to his walk. Joe continued adding until he walked 3 3/5 of the loop.

a. How many days did it take him to walk that far (in a single day)?

b. How many loops did Joe walk?

3. Find the sum.

a. $0.12 + 0.34 + 0.56 + \cdots + 2.98 = ?$

b. $28 + 26 + 24 + \cdots + (-6) + (-8) = ?$

c. Use the numbers and the corresponding sum from either problem 3a or 3b and write a word problem to match the context.

Gauss's Idea: Take a Notion

SOLUTIONS

Activity Sheet 1

1. a. 5 pairs
1. b. 55
1. c. 55 times
2. a. $6 \times 105 = 630$
2. b.

$$3\frac{1}{2}\times 20=(3\times 20)+\left(\frac{1}{2}\times 20\right)=70$$

2. c.

$$5\frac{1}{2}\times 22=(5\times 22)+\left(\frac{1}{2}\times 22\right)=121$$

Activity Sheet 2

1. a. 78 times
1. b. 158 times
2. 225 logs
3. 99 candles

Activity Sheet 3

1. a. 38 students
1. b. 374 students
2. a. 9 days
2. b. 18 loops
3. a. 21.7
3. b. 190
3. c. Answers will vary.