Natalya Vinogradova

What Do You See?


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The idea of using multiple representations of mathematical concepts as a pedagogical strategy has received considerable attention (Cuoco and Curcio 2001). But, as stated by Zimmerman and Cunningham, it appears that "a concerted effort seems to be underway to bring visualization into the school and collegiate curriculum" (1991, p. 34).

Students need to develop concrete visualizations of certain common numerical and algebraic expressions. Zimmerman and Cunningham (1991) describe the word *visualization* in mathematics as "the student's ability to draw an appropriate diagram (with pencil and paper, or in some cases, with a computer) to represent a mathematical concept or problem and to use the diagram to achieve understanding, and as an aid in problem solving" (p. 3). Consequently, we need to teach our students to "read" diagrams and models and see connections that are not apparent at first sight. For example, in many textbooks the algebraic equality

$$(x + 1)^2 = x^2 + 2x + 1$$

is illustrated as shown in **figure 1**.

When analyzing this diagram, it is important to focus on the decomposi-

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tion of a large square into two smaller squares and two rectangles. The area of the square with side $(x + 1)$ can be presented in two different ways: (1) as $(x + 1)(x + 1)$ and (2) as the sum of the areas of all four quadrilaterals in its decomposition, $x^2 + x + x + 1$. This illustrates $(x + 1)^2 = x^2 + 2x + 1$.

CAN WE SEE ANYTHING ELSE HERE?

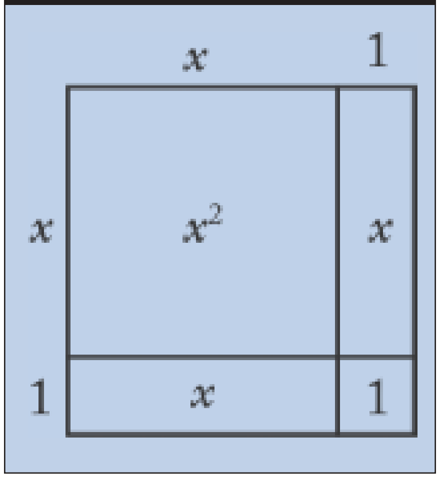
The diagram does not have to be static. This design can be created using Algeblocks®, or the sections shown in **figure 1** can be cut out and rearranged in various ways. As long as all parts are used, no new ones are added, and there is no overlap, the area of the total configuration stays constant. Therefore, algebraic expressions that describe each area will be algebraically equivalent even though they may look very different. This experience of rearranging the geometric shapes and reading the algebraic quantities they represent in different ways may help students become more fluent in algebraic language. **Figure 2** shows three geometric representations of the expansion of $(x + 1)^2$:

- $x(x + 2) + 1 = x^2 + 2x + 1$
- $x(x + 3) - 1(x - 1) = x(x + 3) - (x - 1)$
- $x(2x) - x(x - 2) + 1 = 2x^2 + 1 - x(x - 2)$

The other important skill of reading an illustration is the ability to focus on one particular part and analyze its meaning in terms of the whole picture. For example, **figure 3** is similar to **figure 1** but contains shading.

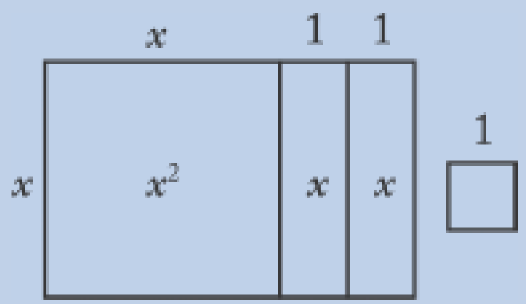
We can read the shaded part in several different ways as $x + x + 1$, $2x + 1$, or $(x + 1) + x$. Note that this shaded area represents an odd number.

Fig. 1 This classic representation of $(x + 1)^2 = x^2 + 2x + 1$ is found in many textbooks.

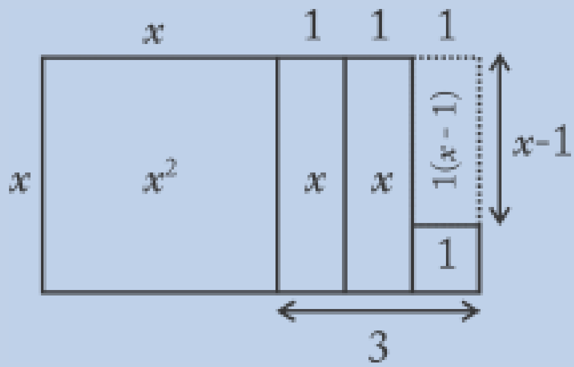


What is the meaning of the shaded part in terms of the whole picture? We can say that it is the difference of two squares: the larger square with side $(x + 1)$ and the smaller square

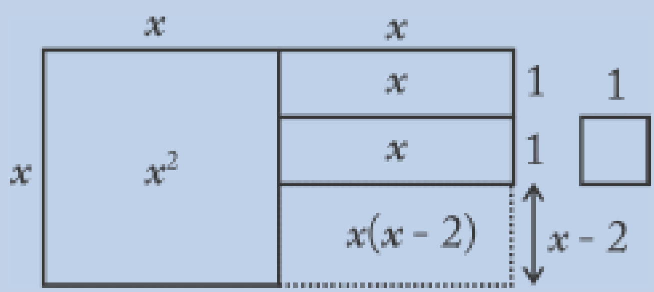
Fig. 2 By rearranging the rectangles and imagining others, these area representations enforce different expansions of $(x + 1)^2$.



(a)
$x(x + 2) + 1$, or $x^2 + 2x + 1$

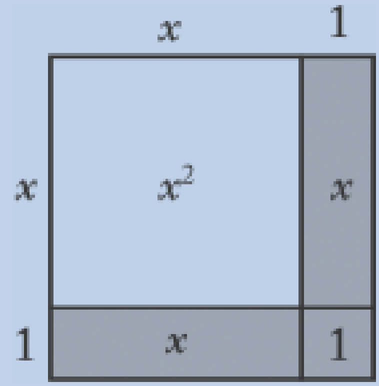


(b)
$x(x + 3) - 1(x - 1)$, or $x(x + 3) - (x - 1)$



(c)
$x(2x) - x(x - 2) + 1$, or $2x^2 + 1 - x(x - 2)$

Fig. 3 Interesting mathematical relationships can be drawn when parts of the area model for $(x + 1)^2$ are shaded.



with side x. So $2x + 1 = (x + 1)^2 - x^2$. Of course, it can be proven algebraically, but it is interesting to note that **figure 3** shows this without any symbolic manipulations. This observation can be expressed in words:

> Any odd number is the difference of two consecutive perfect squares.

A numerical example is helpful: 27 is an odd number, and $27 = 2(13) + 1$. Therefore, $27 = 14^2 - 13^2$. On the other hand, the meaning of **figure 3** can be expressed algebraically as follows:

$$(x + 1)^2 - x^2 = (x + 1) + x.$$

When expressed in words:

> If any two consecutive whole numbers are squared, the difference of the squares is equal to the sum of the original numbers.

For example, $23^2 - 22^2 = 23 + 22 = 45$.

Explorations such as these can help students develop their skills of meaningful visualization. The methods presented here can become convenient tools for helping students make connections among algebraic formalizations, geometric illustrations, and numerical representations of the same idea.

REFERENCES

Cuoco, Al, and Frances R. Curcio, eds. *The Role of Representation in School Mathematics*. Reston, VA: National Council of Teachers of Mathematics, 2001.

Zimmerman, Walter, and Steven Cunningham, eds. *Visualization in Teaching and Learning Mathematics*. Washington, DC: Mathematical Association of America, 1991.

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