Students' Can Take Us Off Guard

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These students produced incorrect answers, but their teacher helped them ultimately become more proficient with computation by skillfully leveraging their prior understanding.

An interesting change has been occurring in our middle school mathematics classes. Some students have become creative, free-thinking inventors. If they have had little experience with an operation, they invent strategies of their own. Sometimes these strategies have been well informed and supported by strong mental math skills and a good knowledge of mathematical properties. At other times, the strategies are uninformed and lack grounding in good number sense.

When mathematics education changed direction to support sense making—from teaching standard algorithms as rote procedures to having students explore conceptually based strategies—we noticed that students' errors changed, too. In place of procedural mistakes, we found a wider range of errors that required us to more closely analyze student work to determine how best to further their learning.

Strategies

KAARSTEN/VEER

Procedural Instruction Doesn't Always Lead to Good Performance

The low proportion of procedural fluency among middle school students is not new to U.S. mathematics educators. A division problem on the 1996 eighth-grade National Assessment of Educational Progress (NAEP) showed that only 35% of students across the nation could correctly answer the question below, and calculators were available. However, research does not support a return to the other extreme in which instruction simply drills students on procedures.

Anita is making bags of treats for her sister's birthday party. She divides 65 pieces of candy equally among 15 bags so that each bag contains as many pieces as possible. How many pieces will she have left?

a. 33
b. 5
c. 4
d. 3
e. 0.33

Kaasila, Pehkonen, and Hellinen (2010) studied Finnish preservice teachers and high school students whose computational education consisted of an early presentation of the standard algorithms from second grade on. They reported in 2010 that only 30 percent of each group could correctly find 491 ÷ 6 if told that 498 ÷ 6 = 83 and told *not* to use the standard long division algorithm. Years of procedural instruction and relative success in the school system failed to provide these students with the conceptual understanding necessary to solve such a nonroutine problem.

These changes have produced both benefits and challenges. Students are thinking independently and willing to be creative in all areas of problem solving. Middle-grades students are also much better at mental math than they used to be. However, from a pedagogical perspective, following these same students' thinking has become more difficult because they have learned a variety of strategies for each operation. Because some students continue to use inefficient, cumbersome, and time-consuming strategies, teachers struggle to get them to accept more efficient mathematics.

This article shares examples of division computations completed early in the school year by sixth, seventh, and eighth graders. As a university partner, I was able to lead all but one of our local middle-grades mathematics teachers in a professional development workshop that focused solely on whole-number computation. We began the workshop by asking students in all three grade levels to take the same short assessments of multiplication (see Keiser 2010) and division so that we could get an idea of their level of proficiency. The teachers and I used this student work to plan the teachers' future computational interventions in the classroom for that academic year. An early inspection of students' work gave us much insight concerning ways in which these students' K–5 preparation had changed.

The division assessments given to the students in the sixth through eighth grades consisted of two problems. One was a "naked" problem, meaning *without context*, in which the division symbol was already written in the problem; the other was a story problem involving division (see **fig. 1**). Both problems asked first, for an estimate, and second, for a solution. In analyzing these preassessments, we saw primarily two division strategies:

1. Repeated subtraction of the divisor in groups (see **fig. 2a**)
2. Using multiplication and adding up to the dividend (see **fig. 2b**)

Very few students used the long-division algorithm. In fact, out of one teacher's 91 sixth-grade students, only 4 used the standard long-division algorithm, and only 2 used it correctly.

When this same group of students was asked to solve 965 ÷ 16, only 30

Fig. 1 Despite the fact that both these problems involved division, the estimates and approaches that students used varied.

A "NAKED" DIVISION PROBLEM
What is 1072 ÷ 26? First estimate and show how you found your estimate, then solve.

A CONTEXTUAL DIVISION PROBLEM
The FFA group at Greendale High School just had its fruit sale. The group receives a shipment of 965 navel oranges in several large crates. Students need to repackage the oranges in smaller boxes that will hold 16 oranges each. How many boxes will the FFA group need to be able to package all the oranges?

1. Give an estimate. Show how you found your estimate.
2. Solve.

percent (27 of 91) found a reasonably correct answer, after giving responses of 60, 61, and 60 r5. Some students answered the question and rounded up to 61; others just stopped with an answer to the computation.

Although we were discouraged by our students' lack of proficiency, we also found evidence in our preassessments that our students had more conceptual understanding than past students. Many used interesting strategies and exhibited much creativity, and we wanted to draw on these strengths to improve their computational fluency. Because the large majority of our *Connected Mathematics Project* (CMP) curriculum does not focus on whole-number computation, we needed to consider the amount of time that we would spend helping our students improve their computational skills.

Some of our middle-grades teachers had been teaching the long division algorithm to a majority of students who either said they had never seen it or who preferred to use a different strategy. Our preassessment revealed that few of our sixth-grade students used the long-division algorithm correctly and efficiently, so we decided that direct instruction of this algorithm would not be the best approach.

> ***When students progress to the middle grades with a conceptual grounding rather than with a skills-based background, it is up to us, their teachers, to use their strengths to help them with their weaknesses.***

The students were willing to engage in problem solving and sense making, and we decided that this was a strength to build on. We asked them to examine samples of division computation, both correct and incorrect, to help them think about the mathematical soundness of their own strategies.

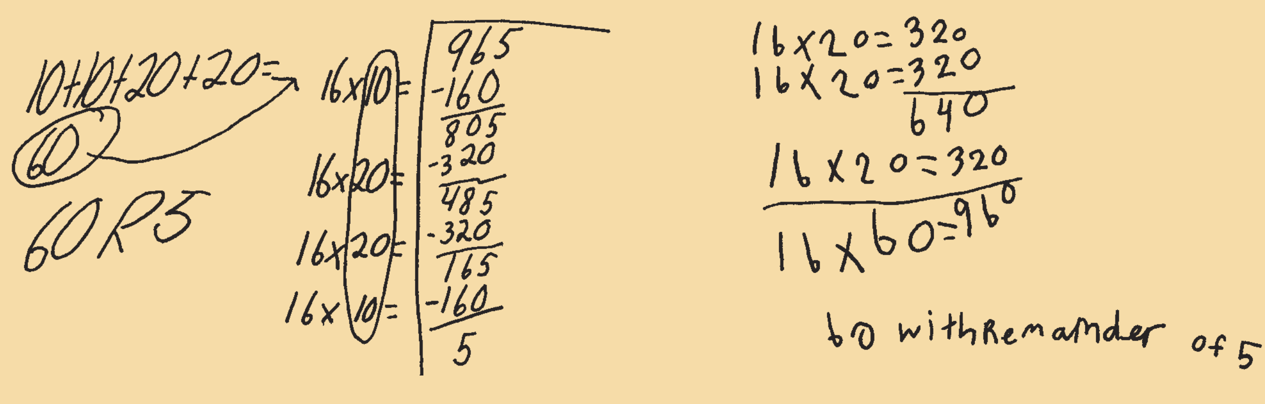
The teachers agreed to be on the lookout for instructional moments that occurred within the CMP curriculum to have students analyze others' methods and compare their ideas. Perhaps observing others' efficiency in using the long-division algorithm or more efficient approaches would encourage students to make a shift in their use of strategies.

It was hoped that discussions on correct and incorrect approaches could benefit students and help them learn strategies and be able to either discredit or confirm an approach. Two approaches to solving division problems follow, having been drawn from the students' preassessment. It is hoped that these approaches will provide fruitful discussions.

AN INCORRECT APPROACH FOR 965 ÷ 16

The student work in **figure 3** shows the same misapplication of the distributive property to the division problem 965 ÷ 16. In **figure 3a**, the student incorrectly used the array (or area) model that was normally reserved for multiplication, with the placement of both factors along the sides and the product in the center. Had she tried to use the array backward, to find the length of

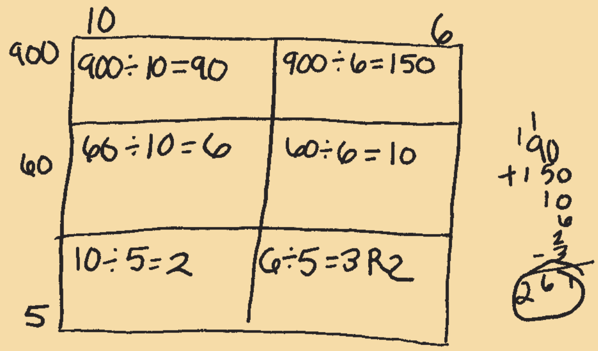
Fig. 2 Sixth-grade students solved 965 ÷ 16 from the contextualized problem in **figure 1** using different methods.



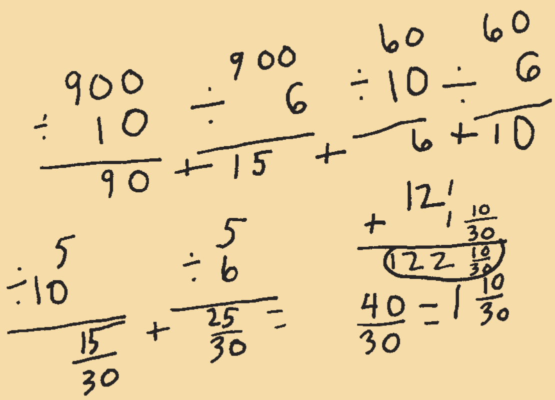
(a) Using repeated subtraction

(b) Summing several groups of 16 to get as close to 965 as possible

Fig. 3 Incorrect solutions by sixth graders when evaluating 965 ÷ 16 revealed much about their past math experiences and understanding of the distributive property.

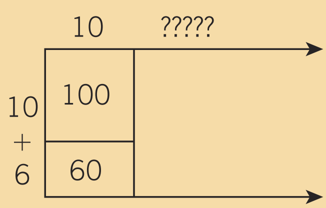


(a) Array model

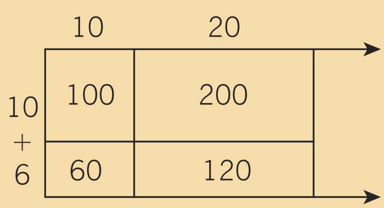


(b) Division algorithm

Fig. 4 One possible way to use the array model "backward" allowed the missing factor to be found.



The student asks, "What does the length need to be so that 965 is the area of the rectangle?" If it's only 10, I've only used up 160 of the 965 and still need 805 more.



The student notices that even with a length of 30, the area is only 480. Doubling the length (60) would create an area of 960. The student then tries a length of 61 and sees that the area would be 976, which is too much, so the answer must be 60 r5.

one side of the rectangle when the area was 965 and the other side was 10 + 6, she could have been quite successful. **Figure 4** shows a series of steps that would have helped the student use the array model to find the missing factor (division), rather than the product (multiplication).

Both examples in **figure 3** contained misconceptions:

$$\begin{aligned}965 \div 16 &= \frac{900+60+5}{10+6}\\ &= \frac{900}{10}+\frac{900}{6}+\frac{60}{10}+\frac{60}{6}\\ &\quad +\frac{5}{10}+\frac{5}{6}\end{aligned}$$

The second student's work—had the approach been correct—actually showed some promise because the fraction addition was accurate. Many of our students produced similar work, which begged the question: "Why is this wrong?"

A seventh-grade teacher and I discussed the many instructional routes we could take. The worst approach would be to tell the class that the distributive property is only for multiplication and that it does not work for division. We thought that students would continue to wonder *why* it does not work in the same way. Instead, we returned the incorrect solution to the entire class and ask them to analyze the method for themselves. Does it give the correct solution? If not, does it ever work to rewrite the numerator *and* the denominator of the division fraction as a sum and then split it into separate fractions?

We hoped that students would discover situations that do not work as well, such as:

$$\begin{aligned}4 &= \frac{20}{5}\\ &= \frac{20}{1+1+1+1+1}\\ &\neq \frac{20}{1}+\frac{20}{1}+\frac{20}{1}+\frac{20}{1}+\frac{20}{1}=100\end{aligned}$$

However:

$$\begin{aligned}4&=\frac{20}{5}\\&=\frac{1+1+8+10}{5}\\&=\frac{1}{5}+\frac{1}{5}+\frac{8}{5}+\frac{10}{5}=4\end{aligned}$$

From these two examples, breaking up the numerator and not the denominator will work correctly, but rewriting the denominator as a sum will not work. In fact, had the two students in **figure 3** written 965 ÷ 16 as

$$\begin{aligned}\frac{900+60+5}{16}&=\frac{900}{16}+\frac{60}{16}+\frac{5}{16}\\&=56\frac{1}{4}+3\frac{3}{4}+\frac{5}{16}\\&=60\frac{5}{16},\end{aligned}$$

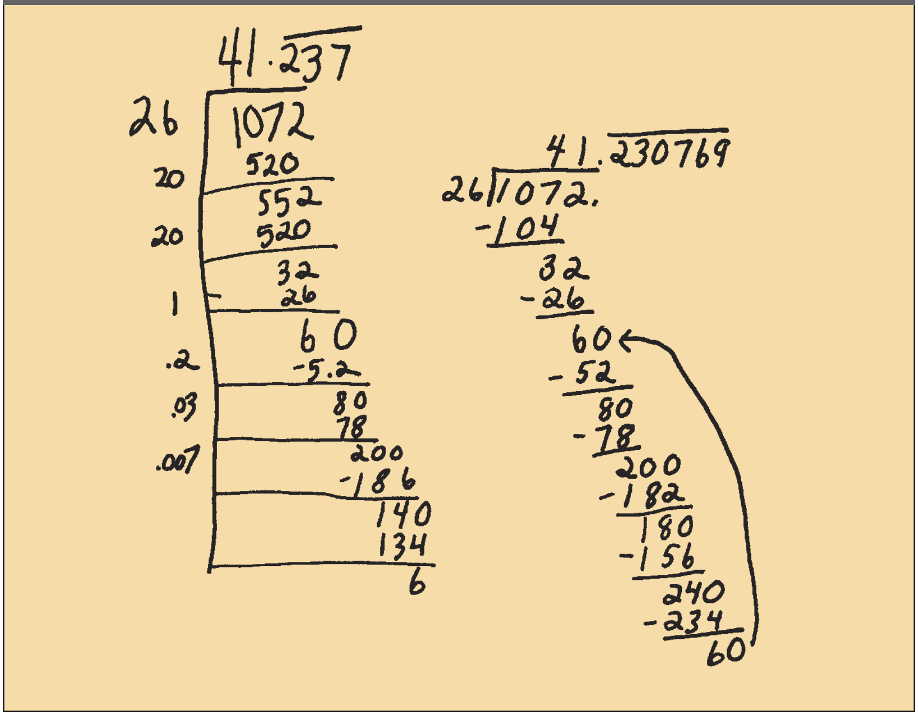
they would have produced the correct answer. It is not a convenient or efficient approach, but it works. See, for example, rewriting the numerator as a sum of convenient numbers:

$$\begin{aligned}\frac{320}{16}+\frac{640}{16}+\frac{5}{16}&=20+40+\frac{5}{16}\\&=60\frac{5}{16}\end{aligned}$$

The division resembles other successful strategies shared earlier in the article.

Allowing students to analyze others' methods requires a trusting and comfortable problem-solving environment. However, the conversations alone help students learn how to test their ideas for themselves with simpler cases so that they will not simply apply mathematical properties to situations that will not work. These conversations help students reflect on the conceptual and notational features of each strategy or algorithm, as well (NRC 2001).

Fig. 5 Seth's work (at left) was modeled (at right) with long division.



A CORRECT APPROACH FOR 1072 ÷ 26

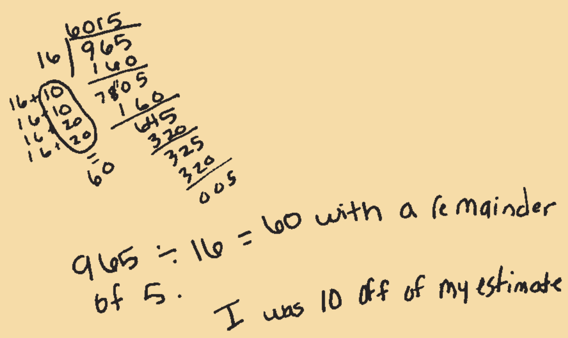
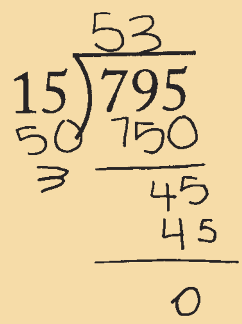
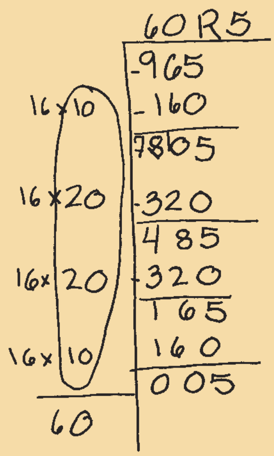
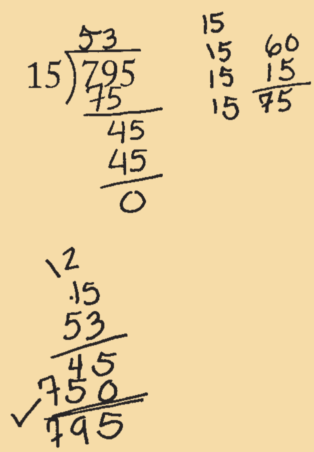
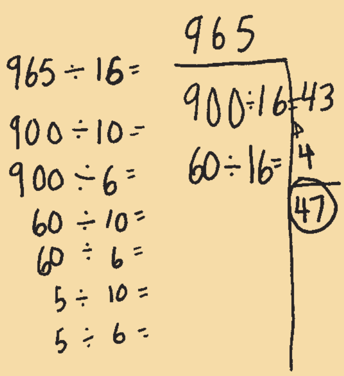
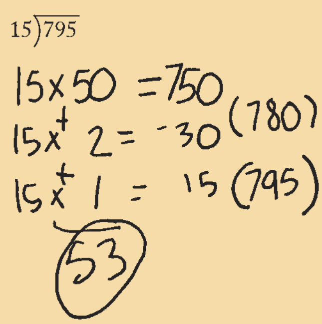
The approach used on the left in **figure 5** was "invented" by an eighth grader we will call Seth. When discussing it with the teachers, we found that a sixth-grade student had produced the same idea when her class was exploring division past the decimal point. Both students must have felt very comfortable with the concept of "division as repeated subtractions of groups of the divisor."

When Seth got to the "remainder" of 6, he added a 0, but he might have been thinking of it as a 6 when he recorded the 0.2 at left. Rather than thinking, "How many 26s can I subtract from 6?" he may have thought, "What fraction of 26 can I take away from 6?" If he took 1/10 of 26, that would be only 2.6, but 2/10, or 0.2, would be 5.2, which would be much closer. This student did remarkably well until he reached this point: "What fraction of 26 can I take away from 0.020?" Because he could not take 1/1000 of 26 away, which would be 0.026, he brought down a second 0 so that he could take 0.0007 26s away from 0.0200. However, he recorded the place value incorrectly as 0.007 *and* multiplied 7 × 26 and found 186 instead of 182.

When I questioned a sixth grader who had used Seth's approach about the difficulty of keeping the place values straight, she responded that she understood it better this way, so why change? Although my own mental abilities do not readily adjust to taking x tenths, hundredths, or thousandths of a number, I can argue that this student *did* understand her mathematics. In fact, these two students' invented algorithm has meaning attached to the set of steps, whereas long division loses meaning when it fails to acknowledge place value.

Having a student or teacher model the long division algorithm next to Seth's work could produce a rich conversation. It could also include how Seth's method could be made to work every time so that students using it could be more accurate and efficient in

Table 1 Three students shifted strategies when progressing from the sixth grade to the eighth grade.

Sixth Graders Solving 965 ÷ 16	Eighth Graders Solving 795 ÷ 15
Student 1 moved from repeated subtraction with many subtractions to repeated subtraction with very few subtractions.	
	
Student 2 moved from repeated subtraction to the long division algorithm with a check of the multiplication.	
	
Student 3 moved from an incorrect to a correct approach using multiplication and adding up to the dividend.	
	

their calculations. Participants would need to notice that the place value of the recorded side values will always add one digit to the right each time as long as a zero was not in the decimal. This occurred in the answer of

$$41.\overline{230769}.$$

This rich discussion also assumes that the student finds the maximum amount to subtract each time, as done with long division.

If the classroom discussion was successful in placing parameters on a Seth-like approach, it would increase his confidence and help him to be more efficient in the use of his method. The rest of the class would benefit by seeing relationships between their own approach and Seth's; there would be a greater understanding of the operation of division in general; and creative individuals, such as Seth, would know that their thinking was valued in the classroom.

At the same time, students might recognize the beauty of not having to think so carefully about place value at each step, which is nicely provided when we use the long-division algorithm. Students will not move to a more efficient method if they never realize that a better way is possible. In working with independent thinkers, we decided that the side-by-side modeling of students' methods could be a good way to elicit these epiphanies.

CONCLUSION

Our group of middle-grades teachers had decided that having in-class discussions on the basis of students' correct and incorrect strategies was a better use of class time, since these teachers were very committed to fully implementing the CMP curriculum. We followed 84 students from the fall 2008 sixth grade to the fall 2010 eighth grade. We found that these small attempts to integrate side-by-side comparisons of student work into the normal curriculum have helped them in their computational abilities, even though these abilities were clearly not the focus of the CMP curriculum.

The sixth-grade students had a 30 percent accuracy rate on the 965 ÷ 16 problem; this same set of eighth graders had a 54 percent accuracy rate on the 765 ÷ 15 problem. More important, 40 percent of this group of students shifted from a less-efficient strategy to a more-efficient approach. For example, a student shifted from using clusters of multiplication and adding up to the dividend to using repeated subtraction effectively or from repeated subtraction to the long-division algorithm. **Table 1** shows three of these prestrategy and poststrategy shifts from this class of students.

When students progress to the middle grades with a conceptual grounding rather than with a skills-based background, it is up to us, their teachers, to use their strengths to help them with their weaknesses. Since our students are becoming good and confident problem solvers who will invent strategies if they do not know a method, we should promote that confidence by sharing the correct and incorrect thinking with all our students. We should also help them by modeling good ways of testing our inventions to make sure they are mathematically sound.

Our students are no longer open to memorizing an algorithm without understanding it. Therefore, attempting to make all students use the long-division algorithm is a practice that we have decided to eliminate. Instead, we hope that students will learn to value the efficiency of such an approach when it is modeled side by side with other student approaches. Fuson (2003) recommends the following regarding students' computational fluency:

> Clearly, the twenty-first century requires a greater focus on a wider range of problem-solving experiences and a reduced focus on learning and practicing by rote a large body of standard calculation methods. How to use the scarce hours of mathematics learning time in schools is a central issue. (p. 301)

Our hope is that by honoring our students' creative abilities, we will keep them involved and still open to improving their computational skills.

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