
YELLOW TAPE: CAT LONDON/ISTOCK; DETECTIVE: POWER OF FOREVER PHOTOGRAPHY/ISTOCK; DNA TESTING: PGIAM/ISTOCK IMAGES

# Using Tiered Explorations to Promote Reasoning

*Address the needs of diverse learners with a class structure that is designed around a crime scene theme and based on student choice and perceptions of the math being studied.*

Cory A. Bennett

When I was a first-year teacher, creating an effective environment for differentiation in a full-inclusion classroom that combined special and regular education students was a challenge. Traditional approaches to understanding student thinking, such as reading homework and taking summative assessments, provided little meaningful information on which to gauge the effectiveness of my instruction or my students' actual progress in learning mathematics. As my second year of teaching began, I was also struggling with the practical aspects of differentiating instruction; I understood the theory but lacked an understanding of how to implement it in my classroom, with my students.

I understood that mathematical reasoning is one hallmark of mathematical understanding (CCSSI 2010; NCTM 2000). Skemp (1997) defines mathematical understanding in two ways: *instrumental* and *relational*. Instrumental understanding, the ability to perform specific procedural tasks, is important. However, relational understanding represents the fluid, dynamic, and creative nature of mathematics. Relational understanding can be thought of as the ability to apply conceptual and procedural understandings along with adaptive reasoning to find a solution (Kilpatrick, Swafford, and Findell 2001), even if a solution strategy is not immediately apparent.

Helping all students develop flexible, effective, and efficient reasoning strategies can be supported through classroom structures that focus on differentiated learning experiences. That said, differentiation alone, while focused on students' individual needs, is not enough to ensure that students are developing reasoning strategies that support relational understanding. Teachers need to create classroom cultures and structures that promote thinking (Cobb and Yackel 1998; Hull, Balka, and Harbin Miles 2011).

Providing this environment in my classroom remained a challenge despite my theoretical knowledge. One day during students' recess, I overheard some of my students talking about a police drama on television. An idea struck me on how to create a differentiated classroom structure that supported student reasoning: My class would become a Crime Scene for Mathematics Investigation (CSMI). Shortly thereafter, I introduced CSMI to my students. One year later, I had revised and refined the process to work smoothly within my classroom.

## THE CSMI FRAMEWORK

CSMI was created as a small group, cooperative, self-selective grouping strategy to allow students to explore mathematics based on their own understanding and perceived readiness, much like the concept and application of literature circles in language arts. In essence, these self-chosen small groups met as mini-learning communities to support one another in developing mathematical reasoning strategies through open discussion while problem solving.

Three tiered sections of CSMI dealt with similar mathematical content or themes; these sections were called Yellow Tape, Detectives, and Forensics. I described the sections to the students from the perspective of police officers who have different duties but must still work together. Just as there are different types of police officers who work to solve crimes, different groups of mathematicians in the class must work together to learn math based on a common theme. Although CSMI comprised three sections, several small groups would often work at the same level and in the same section.

- The Yellow Tape section was described as being similar to police officers who use fundamental but necessary tools to solve a crime.
- The Detectives section focused on slightly more difficult problems.
- The Forensics section engaged in more complex explorations that often required a combination of mathematical skills to solve the problem.

The topic of the mathematics did not affect the implementation of CSMI; however, it was most effective when I had several rich problems from which students could choose. The diversity of meaningful problems is an important part of effectively implementing CSMI.

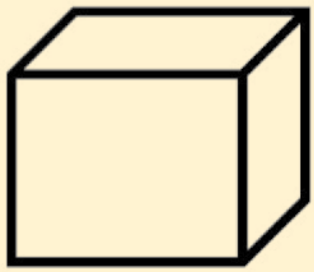
All sections worked on grade-level state and national standards. All students, regardless of the section, were engaged in problem solving appropriate for their perceived readiness. They were also responsible for justifying, modeling, and illustrating their thinking or a group member's thinking.

**Figure 1** shows three types of problems that explored the Pythagorean theorem and followed the CSMI strategy. Understanding the theorem and using it proficiently were needed to find

**Fig. 1** These CSMI problems explored the Pythagorean theorem.

**YELLOW TAPE**
Starting from Sunny Harbors, a boat sails due west for 4 miles, then due south for 9 miles, and then due west for 10 miles. How far is the boat directly from the harbor?

**DETECTIVES**
A fly is sitting on one corner of a sugar cube. The cube's volume is 1 cubic inch. If the fly is walking only, what paths might it take to get to the opposite corner? Which is the shortest possible path it can take?



**FORENSICS**
A knife is used to cut off the top of a spherical orange, 4 cm from the center of the orange. The orange has a radius of 5 cm. What is the area of the circle that was created by the cut?

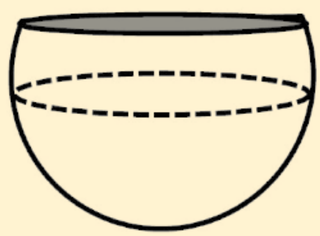
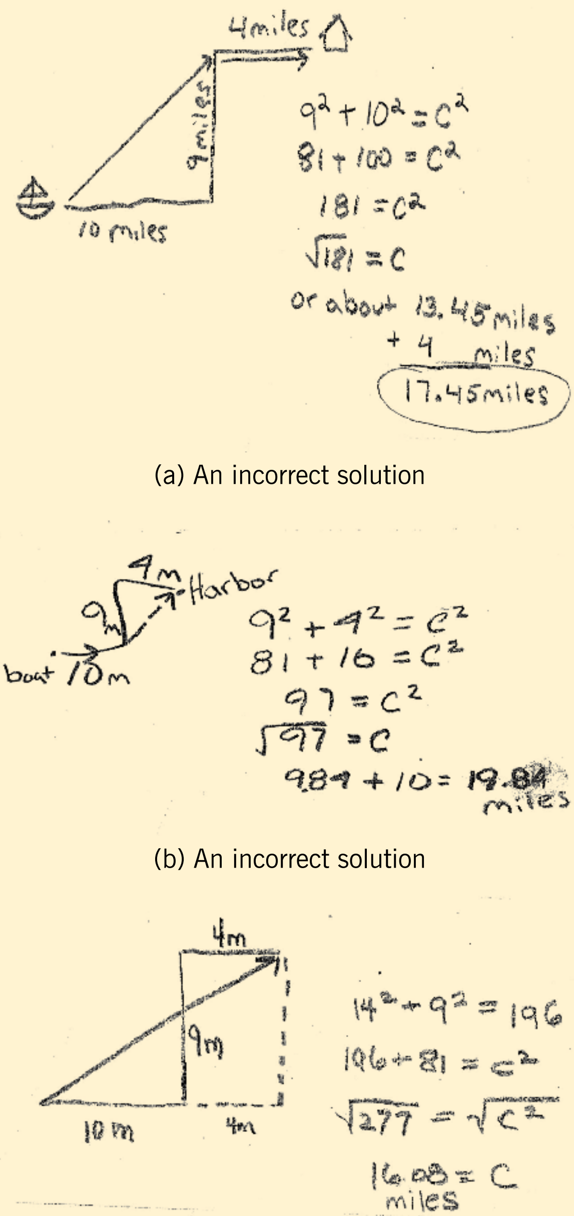


**Fig. 2** Students who chose the Yellow Tape, or least analytical, section for the Pythagorean theorem question from **figure 1** produced these solutions.




(a) An incorrect solution

(b) An incorrect solution

(c) A correct solution

a reasonable solution, yet the complexity of each problem was slightly different in each section. **Figure 2** shows three common responses from the Yellow Tape groups. Each representation contained a solution, but it was the discussion justifying or refuting students' solutions that strengthened their reasoning.

**Figures 2a** and **2b** illustrated students' solutions, which were similar. Students realized, however, that the best solution strategy would be to calculate the shortest distance from the boat to the harbor. Thus, they believed that **figure 2a** was the best solution. However, a student from another Yellow Tape group supplied the example in **figure 2c** and shared the solution and reasoning during a discussion with all Yellow Tape groups. During this group talk, other Yellow Tape students made the connection that a solution need not be limited to the path the boat originally followed. The term *directly* meant a line from the boat to the harbor, the shortest distance between two points. Once students recognized that this line could represent a hypotenuse of a right triangle, they understood that **figure 2c** was a better solution.

CSMI emphasizes students' being self-reliant. They are to ask questions of one another and initiate and engage in meaningful discourse; the teacher was not supposed to be the only source of knowledge. Throughout this process, students were able to construct logical arguments and engage in justifying or refuting arguments, which are traits that promote thinking and mathematical reasoning skills (Lannin et al. 2011). As such, there were only two rules for the students:

1. Choose the section that best suits your individual understanding and readiness.
2. Be accountable to one another.

## STUDENT CHOICE

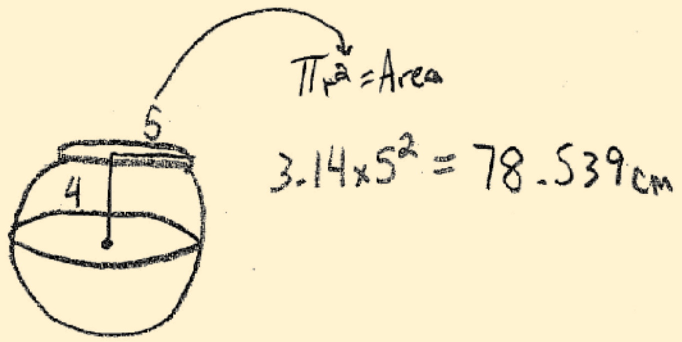
Adolescence is a period of substantial social, emotional, and cognitive development during which students seek opportunities for more autonomy. Allowing students to choose learning experiences that are appropriate for them is one way to create such an opportunity. Choice is important in helping students develop as competent and proficient mathematicians because students become more motivated, assume greater responsibility for their learning (Patall, Cooper, and Robinson 2008), and become more engaged (Jackson and Davis 2000).

I often knew which tiered sections were most appropriate for my students, but ability and readiness are not the same thing. Some students could have tackled a more difficult section but may not have felt ready to do so. Requiring all students to attempt mathematics for which they did not feel prepared or did not believe they could be successful does not instill confidence or increase their self-efficacy in mathematics, both of which matter in promoting student achievement (Klassen 2010; Lent, Brown, and Larkin 1986).

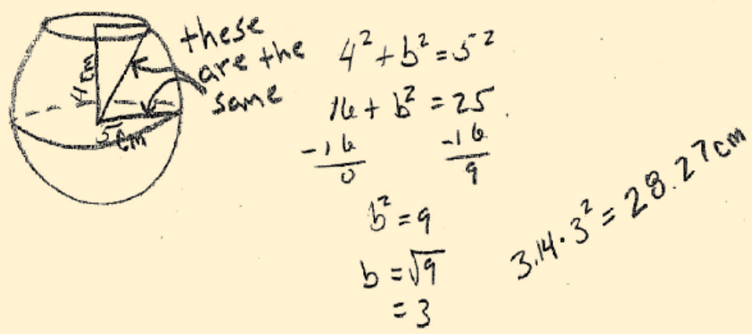
Throughout the unit, most students migrated to more challenging sections, and I continued to introduce different problems for all levels. The number of problems that students worked on depended on the length of the unit; they either tackled a few tasks within one or two tiers in a smaller unit or worked on several items in multiple tiers for longer units. In so doing, students who were able to solve a problem and justify their thinking either moved to a more challenging section if they felt prepared, or they tried another problem within the same section. Occasionally I had to challenge students to make the move, but more often than not, they moved on their own accord. At times, students began work in one section only to realize that it was either too hard or too easy for them, so they changed to a different section.

By choosing to move to a different section, or level, students were self-monitoring their learning, a highly effective trait that is difficult to teach (Schmitz and Perels 2011). Peer relationships are important during adolescence, and these students felt confident they would have support regardless of their section choice. One student commented, "[CSMI] was helpful because you could help each other out. Also, you can choose any group you think would be the best fit for you." Choosing an appropriate section in which to learn math was more important than choosing to be with their friends.

**Fig. 3** Incorrect and correct student work occurred when solving the problem from **figure 1** at the Forensics, or highest, level.




(a) An incorrect solution




(b) A correct solution

## STUDENT ACCOUNTABILITY

Mathematics is a social endeavor. Mathematicians collaborate, ask for assistance, and seek affirmation of their work. This sense of accountability exists within a community of mathematicians. In the classroom, this accountability means that students talk to one another, pose questions, and receive support from peers. Within each section, the expectation was that students would challenge themselves according to their own understanding and readiness of the math under study. They would also support one another without fear of being judged. In short, there was simply no shame in learning.

Developing a classroom culture that emphasizes thinking is important in supporting student learning (Roberts and Billings 2009). When students are accountable to one another, they begin to trust their peers and learn that they, not the teacher, hold the intellectual authority for learning. An example of this stems from the conversations that students had while working in their groups. During the Forensics exploration on the Pythagorean theorem (see **fig. 1**), one student thought a solution had been found (see **fig. 3a**) but did not realize that it was incorrect until another student questioned the solution:

> [The radius of] the slice can't be 5 because then it would be the same size circle as the other one.

Eventually, one student decided to view the problem differently.

> I know. What if we just think of it as a plain circle and not a sphere? Couldn't we just make this part be the hypotenuse?

In the discussion that followed, other students recognized the importance of this idea. The radius of the sphere was the hypotenuse of a right triangle, with the given distance between the two circles as one of the legs. The other leg (the radius of the smaller circle) was an unknown length. In turn, this new thinking allowed students to solve for the radius of the smaller circle and thus the area of the smaller circle (see **fig. 3b**).

## CSMI PROCEDURES

Although CSMI might look different from classroom to classroom, four basic procedures are involved in implementing CSMI, as illustrated in **figure 4**.

### 1. *Identify Learning Goals*

During my initial lesson planning, I established clear learning goals using state and national standards. Next, the learning goals served as a way to anticipate student misconceptions. By anticipating misconceptions, I was also able to develop more focused learning experiences and problems for my students that were better suited to their varying developmental and experiential needs.

### 2. *Purposefully Select Problems*

Although using meaningful problems is critical, creating such problems as a relatively new teacher can be a challenge. Therefore, when implementing CSMI, I relied on several different sources, including available texts, colleagues, or outside resources such as NCTM's Illuminations. I also created a library of good questions on different topics for CSMI explorations. I might modify one question to be more appropriate for different CSMI sections, but I usually used different questions that addressed the same learning goals but at different levels of complexity.

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### 3. *Elicit Evidence of Learning*

Using the CSMI structure, I could hear and see the connections students were making in their learning. These valuable formative data guided my instruction. As I moved from group to group, I listened to students' discussions, paying careful attention to the nature of their conversations, reasoning, questions, and solutions. Discourse is a highly effective tool in supporting students' mathematical understanding (Larsen and Bartlo 2009) and should be encouraged.

When interacting with students, I am generally cautious in voicing my thoughts because I want my students to think independently and enhance their reasoning strategies. For example, in a lesson about slope, students were to describe what a graph would look like if a person with super speed were to run from a starting point. This part of the exploration focused on the difference, and common misconceptions, between zero and undefined slopes. During the conversation, I wanted to challenge students' assumptions until they were able to use mathematics to justify their reasoning.

Some students initially thought that the graph would be a vertical line, but others were unsure of this conjecture (see **fig. 5**). Throughout the students' discourse, I posed questions at crucial moments to either challenge or guide their thinking so they could recognize their misconceptions. Again, working with small groups of students allowed me to assist those who struggled by pressing their thinking to identify misconceptions, providing specific support to students with diverse needs, ensuring equitable access to the teacher, and assessing

**Fig. 4** CSMI procedures followed a four-step cycle.

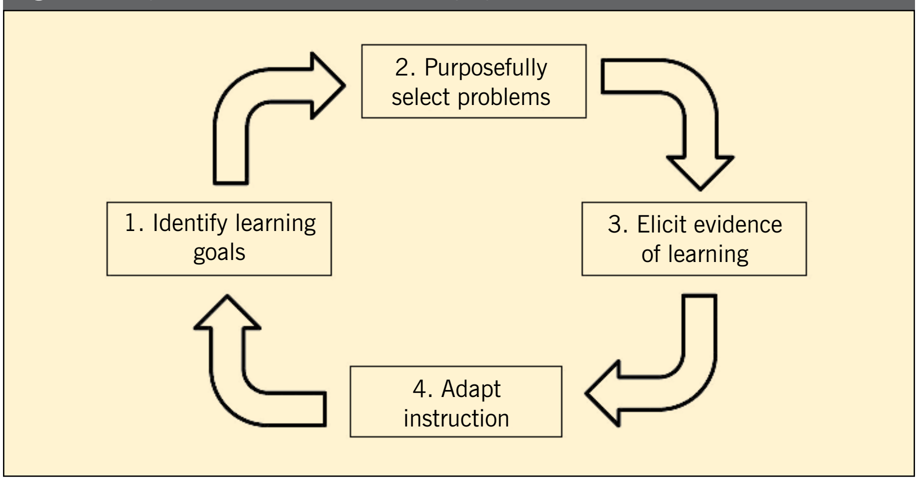

**Fig. 5** In this conversation about slope, the teacher was able to guide students to recognize their misconceptions.

*Teacher*: So what would this graph look like?

*Carry*: Straight up, starting at zero.

*Teacher*: OK, so the line starts right here [beginning at the origin], and he runs so fast that, boom, he goes like this [tracing the *y*-axis].

*Aaron*: But he cannot.

*Teacher*: Why not? You said that it had to start at zero.

*Carry*: Because he can't stay that fast.

*Teacher*: Well, isn't he running away with super speed, though?

*Megan*: But the time; it can't stay at zero.

*Carry:* Actually, it would be impossible because the time is not changing. This can't happen.

*Aaron*: It is weird. He would have to be in different spots at the same time.

*Teacher*: So how would you describe this slope?

*Aaron*: Zero slope.

*Carry*: But I thought zero slope went like this [putting her arm out horizontally].

*Megan*: I think that since this is impossible, it can't have a slope.

*Aaron*: So if the time can change but his [location does not], then it is zero slope. But if his [location] changes but not his time, then it is impossible. Is that right?

*Megan*: Yeah.

*Teacher*: So is it zero slope?

*Carry*: It's like what Aaron said, it can't be.

students' understanding to adjust instruction or learning experiences to meet the learning objectives.

Although discourse was frequently used to elicit evidence of understanding, I also used students' work, such as calculations, drawings, or other representations. Regardless of the type of evidence collected, my role was to identify the nature of their learning and misconceptions so that I could adapt my instruction for individuals, small groups, or the class as a whole.

### 4. *Adapt Instruction*

While students were in their groups working on their problems, I was able to adapt instruction and address questions based on the needs at the time. I often began with the Yellow Tape groups to ensure that misunderstandings did not inhibit their initial progress. When students are unsure of what to do, how to begin, or if what they are doing is correct, they will often stop working until they get the help they need (Vatterott 2009).

During the focused small-group instruction, I often heard similar questions asked between groups or from section to section. Sometimes I asked all Yellow Tape members to address a misunderstanding or clarify a concept. At other times, I brought the sections back together as a whole class and led discussion on an issue before students continued with their respective explorations. At yet other times, students presented solutions to the whole class. Because all students were working on similar themed mathematics, like the Pythagorean theorem, they seemed to understand the presentation, as evident in the questions and comments they posed, despite working in a different tier.

## REFLECTING ON CSMI

Creating a classroom structure that allows the teacher to promote students' mathematical reasoning, formatively assess learning, and adapt instruction to support all students is critical, regardless of students' current understandings or readiness. CSMI created a structure within my classroom by which I could accurately gauge student learning and address misconceptions in a timely manner while helping students develop relational understanding. Furthermore, students believed that they could be successful in mathematics.

In a reflection on CSMI, one student wrote, "I think it's really helpful because it's a choice we can make for where we want to be and what we can do." With respect to the differentiated aspects of CSMI, another student commented that it was "helpful to have a place to know where you are [easy, medium, hard] instead of everyone gets the same thing and others might not get it." Furthermore, CSMI helped students *believe* that they could be successful in mathematics. Not all students are ready for the same mathematics at the same time, but they can explore, discuss, and develop reasoning skills on the same concepts. We just need to create structures that provide access to this kind of thinking, one investigation at a time.

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**Cory A. Bennett**, benncor3@isu.edu, is an assistant professor of mathematics education at Idaho State University in Pocatello and a National Board Certified Teacher in Early Adolescence—Mathematics. He is interested in mathematical discourse; problem solving to advance teachers' pedagogical, content, and process knowledge in mathematics; and middle level education.