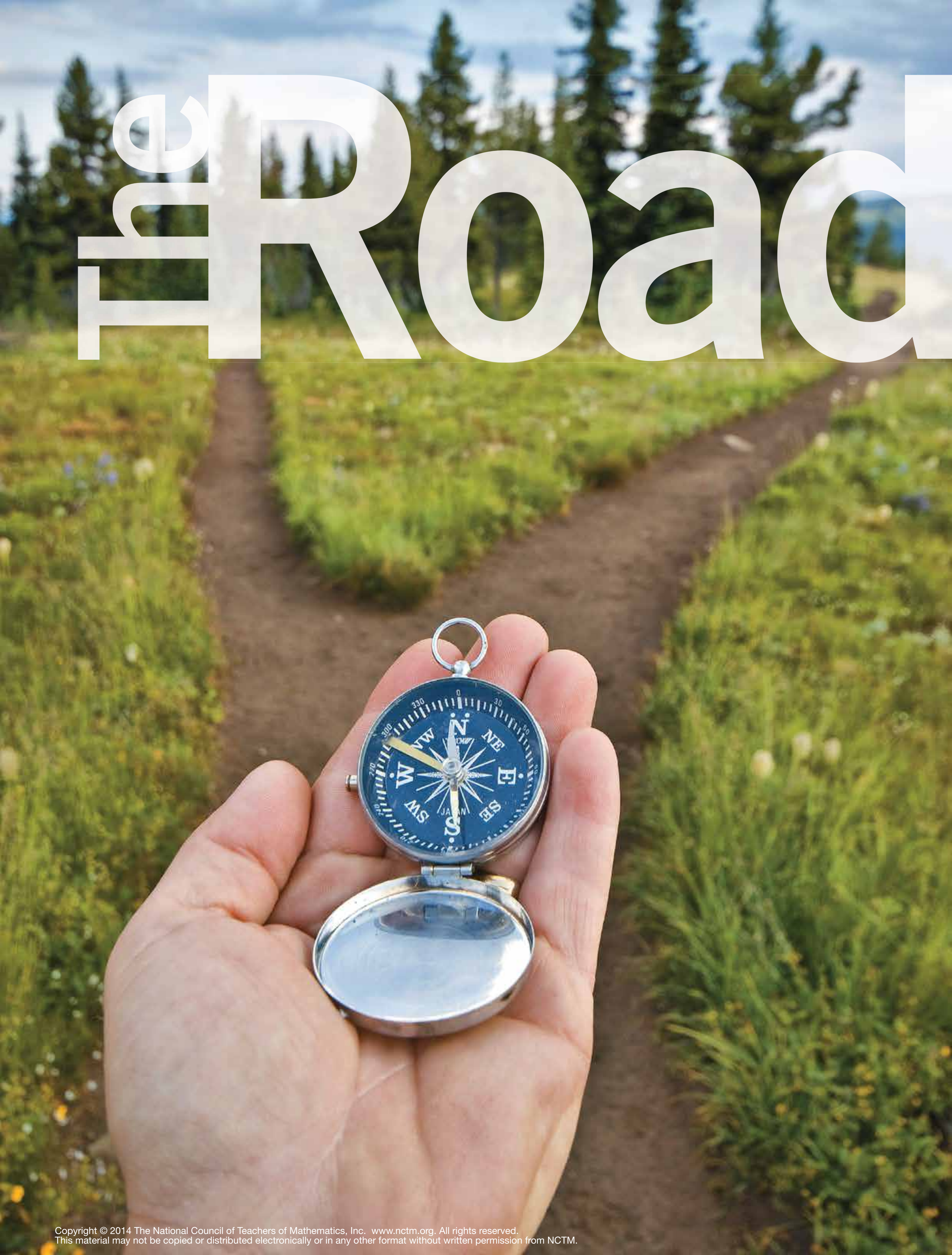
The Road



Less Traveled

As students work on a routine task, look over their shoulders to see how they are using translations to reason and make sense of mathematics.

Stephanie R. Whitney

Two roads diverged in a wood, and I—
I took the one less traveled by,
And that has made all the difference.
—*Robert Frost*

When I became a mathematics teacher, I often found myself wishing that students would take the road *most* traveled, or the most efficient path to a solution. However, I have grown to appreciate and value what happens when students work to make sense of and reason about important content in the mathematical equivalent of "the road less traveled." In doing so, these students actively develop a connected understanding of mathematics, build a sense of self-efficacy, and give others a window into their understanding.

According to NCTM (2009), all students should engage in reasoning and sense making daily in their mathematics classes. It is often through students' divergent thinking as they work on novel mathematical tasks that they make important mathematical connections and that teachers learn about their understandings. Creating opportunities for students to explore allows them to experience mathematics as a connected whole as opposed to viewing the subject as a series of discrete pieces of information that must be committed to memory.

Consider the algebra task in **figure 1**. On the surface, it appears simple and straightforward. If students read closely, they will realize that there is not any mathematics that *has* to be done beyond making sense of the problem. The student groups start with different amounts of money and earn the same amount of money each week. This makes the question of when they will have the same amount of money a simple one: never. If students do not immediately stop to reason about the situation, a teacher has two options: (1) Stop them in their work and guide them down the efficient road or (2) step back and get a glimpse into their thinking as they take the road less traveled. This article will focus on the reasoning of four students as they make sense of a mathematical task using multiple representations.

My reflections on taking the road less traveled are based on my

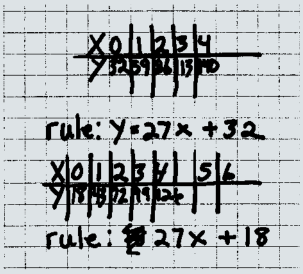
Fig. 1 This School Fundraiser problem was presented to students to solve.

The students at Palmer High School are raising money to paint a community mural on the side of their building. Two groups of students start raising money for the mural. The first group begins with \$32 and earns \$27 each week by working at the school store. The second group of students starts with \$18 and earns \$27 per week by tutoring elementary students.

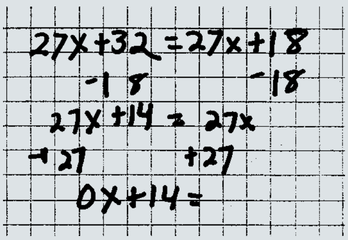
1. Display information on the earnings of each student group in a representation (table, graph, or equation) of your choice.
2. Determine when the two groups will have the same amount of money.

Fig. 2 Katrina chose these representations and explained why.

Katrina answered questions 1 and 2 in this way.



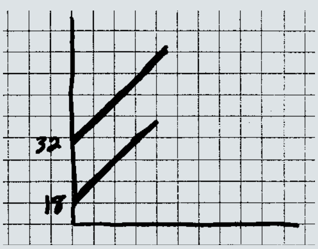
When asked when the groups would have the same amount of money, she said, “I could just keep [building the table], or I could do it like . . . using the equation.”



She paused a bit to think after writing 0x + 14 = , then said, “Oh, so they don’t cross. . . . They just keep going on, they won’t . . . intersect.”

When asked if it makes sense, she said yes and explained: “Because one’s going up by more, so it’s gonna keep going faster, and this one wouldn’t be able to catch up. . . .” After pausing and reflecting, she restated her understanding, “Um . . . this one [referring to the group working at the school store] started off . . . ahead of this one [referring to the tutoring group]. . . . They’re going up by the same rate, but this one’s still further than this. . . .”

She was then asked to sketch what she was visualizing.



When asked how she could tell that the groups were earning money at the same rate, she replied, “[Because] they’re parallel to each other.”

one-on-one experiences with Katrina, Becca, and Harmony and Miguel (the latter two students worked together). They were studying linear equations in classrooms in which teachers were committed to developing student thinking, discussion, and engagement by focusing on understanding tables, graphs, and equations, and the *connections* among these representations. While I was working with the four students outside of their class, I had the opportunity to hear their thinking, watch them produce their written work, observe their chosen mathematical “routes,” and hear how they responded when they encountered mathematics they did not understand. Vignettes of their work on this task are in **figures 2–4**. As you read and view their choices of representations, consider the intellectual activity they engaged in as they reasoned and made sense of mathematics.

While solving this problem, all four students expressed at some point that reading closely and making sense of the problem alone would have been enough to determine that the answer was “never.” Miguel even said to Harmony, “Why didn’t we see that?” So the question that struck me was this: Would it have been better to stop them early in their solution discussion and guide them to read carefully or let them proceed? To explore this question, I began to actively look at the students’ work and explanations for evidence of their understanding and use of connections among representations.

UNDERSTANDING CONNECTIONS AMONG REPRESENTATIONS

One way for students to understand the connectedness of representations is to translate among pairs of mathematical representations (tables, graphs, equations, context) of a concept, as described by Lesh et al. (2000):

> A translation requires a reinterpretation of an idea from one mode of representation to another. This movement and its associated intellectual activity reflect a dynamic view of instruction and learning. (Lesh et al. 2000, p. 450)

In **figures 2–4**, students engaged in the intellectual activity of translating among representations of context, tables, equations, and graphs while finding a viable answer to the question.

Students often learn how to create representations, but making connections among the representations is something that is not necessarily emphasized (NCTM 2000). They translated from one representation to another when grappling with something puzzling within a representation, selecting from known representations to solve the problem, or checking to see if their solutions made sense. The summaries of students' explanations of their thinking are used to illustrate each situation.

Grappling with Something Puzzling

Throughout the vignettes, when students puzzled over something within a representation, they frequently translated to a new one. They would make a conjecture, translate to a new representation, test the conjecture in the new representation, and interpret the results. For example, when Becca tried to determine the time at which the two groups would have the same amount of money, she wrote the equation $27x + 32 = 27x + 18$ (see **fig. 3**). Before doing any symbolic manipulation, she realized that this was "gonna be weird," indicating that the term $27x$ was on both sides of the equals sign. Clearly, she knew that the common term $27x$ was an important feature of this equation, but she did not know how to interpret this attribute.

Fig. 3 When Becca was unsure about the meaning of 27x, she extended her table in anticipation of graphing the ordered pairs.

Becca answered questions 1 and 2 in this way. When asked why the first rule matches the situation, she said, "Because they start off with 32 . . . and then every week they get 27. . . ."

To find out when the two groups would have the same amount of money, she wrote the following:

$$27x + 32 = 27x + 18$$

She looked at the equation and said, "Hold on, this is gonna be a weird one . . . because . . . this is both the same." When asked what was the same, she said, "The 27*x*." Although she realized that this was important, it seems that she did not quite know what it meant, so she said, "We could graph it, and I could see when they both cross the same thing. Or we could make a table. . . ." She started to extend her table (as shown below).

X	0	1	2	3	4	5	6	7	8
y	32	59	86	113	140	167	194	221	[illegible]
y2	18	45	72	99	126	153	180	207	234

After realizing that she had not found the point at which the lines cross, she said, "Or maybe it's negative, where they cross. . . ."

She was asked to evaluate the difference in money earned between the two groups for various weeks. "Fourteen apart. And then right here they're 14 apart also. . . . They'd be 14 apart each time." When asked the question of when the two groups will have the same amount of money, she stated, "They won't . . . 'cause parallel lines, they don't intersect."

For students in a similar situation without a connected understanding of representations, this could be a point of impasse. However, for Becca, translating among representations was a tool to make sense of an unfamiliar situation.

When she recognized that the equation was a special case, she refocused her attention to the table. She then extended the table she had previously produced to make sense of what was happening. After translating to a tabular representation, she realized that the difference in money between the two groups was constant from week to week. She then concluded that because this situation resulted in the same multiple of x on both sides of the equals sign, the result was that there was no solution. The next time Becca sees an equation in the form $ax + b = ax + c$, it is unclear whether she will be able to conclude that there is no solution, but this experience may have begun to help her build tools to find the answer by translating to other representations.

Selecting a Representation for Solving the Problem

In each of the three vignettes, students communicated that they were making a choice among representations. At various times, I heard the students think aloud and weigh their options for how to proceed and select from among the representations they knew and understood. For example:

- Katrina specified that she could extend her existing table or use equations (see **fig. 2**). She opted to

Fig. 4 Harmony and Miguel worked together and decided to create a new equation from the existing information.

When given questions 1 and 2, Miguel wrote the equations, and Harmony created a table:

graph #1

$y=27x+32$

$y=27x+18$ #2

x	0	1	2	3	4	5
y	32	59	86	113	146	167

x	0	1	2	3	4	5
y	18	45	72	99	126	153

Miguel explained his second equation in this way: “$27 is what they’re making per week and $18 is what they started with. And *y* is what they have each week.”

When asked when the two groups would have the same amount of money, Miguel looked at Harmony’s table and stated, “It hasn’t happened yet. . . . I know an easy way to do it. If we solve for *x*.” After some discussion, they wrote $27x + 32 = 27x + 18$ and they (incorrectly) solved for *x*. The last line read $x + 14 = 0$ (the written work below was amended later in the task):

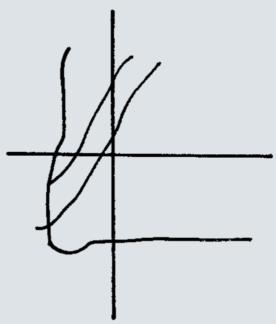
$$27x+32=\cancel{27x}+\cancel{18}$$
$$-27x-18 \quad \cancel{-27x}-\cancel{18}$$
$$\cancel{x}+14=0$$

Initially, they interpreted the last line of $x + 14 = 0$ to mean, “At 14 weeks they will be the same.” Then they extended their tables to verify this result (see below). When their tables did not match their interpretation, they checked and rechecked their math. First, they added 27 to the starting values to check their tables. Then they used their equations and substituted 14 for *x* to see if they got the same result. When this was inconclusive, I guided them back to their system of equations to check their work.

X	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
y	32	59	86	113	146	167	194	221	248	275	302	329	356	383	410

X	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
y	18	45	72	99	126	153	180	207	234	261	288	313	342	369	396

When they arrived at the last line and realized that it should have read $14 = 0$, I questioned what it meant, “Are you telling me that $14 = 0$?” Harmony said, “That doesn’t make sense.” Miguel says, “They’re *never* going to be the same. . . . Group 1 started with 32 [dollars] and group 2 started with 18 [dollars], and they’re going up at the same rate.” I asked what this would look like, and Harmony produced this sketch:



set the two equations equal to each other and solve for *x*.

- When Becca realized that the term $27x$ on both sides of the equation was “weird,” she indicated that she could either graph the equations or make a table. She chose a table.
- Miguel considered the information in Harmony’s table and then introduced an “easy” way to solve the problem by creating a new equation (see **fig. 4**).

These instances indicate that the students were aware that there were multiple ways to use representations to answer the question. They were choosing a representation that they found effective. It was clear that the students were familiar with representations as tools to use in solving the problem, and at various points they made informed choices regarding which representation to pursue.

Checking the Reasonableness of a Solution

Within each vignette, students were able to verify the reasonableness of their solutions by using a different representation. Consider when Harmony and Miguel made an algebraic error that resulted in the equation $x + 14 = 0$ and misinterpreted the solution to be $x = 14$. Without prompting, they extended their table to 14 and realized that the results in their table did not confirm that at 14 weeks the two groups would have the same amount of money. Initially, they checked and rechecked their table. When prompted to look at their equation, they revisited their symbolic work and quickly found the error.

Similarly, when Katrina reached a solution, she checked the reasonableness against the context. At first, she said that one group had more money to start and then earned more. However, after looking at her equation, she revised her statement to reflect that they were earning money at the same rate.

These students were so focused on reasoning through the task that they continued their stance of checking and making sense of the mathematics even when they reached their answers. Rather than consider an answer as an ending point, they treated it as a conjecture for a solution, then worked to prove or refute the validity of the conjecture.

In all three cases, the students did not draw graphs to find a solution. These four students primarily translated among the representations of context, tables, and equations to make sense of both the problem and their understanding. However, when prompted, they were able to discuss and sketch a graph that would represent the situation presented in the problem. In these cases, the students were able to use graphs as a tool to retell the story and offer visual evidence to show why their solution made sense.

LETTING STUDENTS PROCEED

To answer this question that was raised earlier, would it have been better to stop students early in the problem and guide them to read carefully or let them proceed? One needs to decide whether the overriding goal of having students do mathematics problems is to reach a solution or to deepen their understanding of mathematics. Because the purpose of engaging students in mathematical problems is, at least in part, to understand mathematics as a connected subject, there are merits in students taking a longer, less traveled, path to a solution. During this task, students translated among representations to make sense of questions that arose as they worked toward a solution and assessed the reasonableness of their solutions.

Skemp (1987) describes intellectual learning as a goal-driven activity in which learners use flexible plans as they test hypotheses and work toward the goal. Katrina, Becca, and Harmony and Miguel engaged in intellectual learning in that they adapted their thinking, used representations to move forward in their ideas, and reflected on their conjectures and potential solutions to the question. Given mathematical tasks and learning environments that foster opportunities for students to translate among representations, students will develop connected understandings and strong foundations for how to read, interpret, and understand the relationships among tables, graphs, equations, and context. As their study of linear equations gives way to nonlinear equations, the connected understanding that students have of linear equations might be a foundation on which they can integrate, synthesize, and make meaning of new families of functions, thus adding to their connected understandings.

In addition to developing their understanding of mathematics, by allowing these students to take the road less traveled, it became clear to me that they were developing their self-efficacy in using mathematics. Katrina, Becca, and Harmony and Miguel were confident in their mathematical ability. When they were presented with questions that made them unsure how to proceed, they explored alternative representations rather than claim defeat.

In the cases presented here, the answer to the problem was not as important as the students' thinking as they pursued a solution. Perhaps we can allow students the room to explore the connections among representations and to take less efficient routes on tasks that offer an efficient solution. In the process, we will gain insight into their understandings.

Are problems assigned so that students find an answer or for students to reason through a process? Both. As with a walk in the woods, solving problems in math class should be as much about the journey as the destination. And that *can* make all the difference.

CCSSM Practices in Action

SMP 1: Make sense of problems and persevere in solving them.

SMP 2: Reason abstractly and quantitatively.

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Any thoughts on this article? Send an e-mail to ***mtms@nctm.org.***—*Ed*.

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