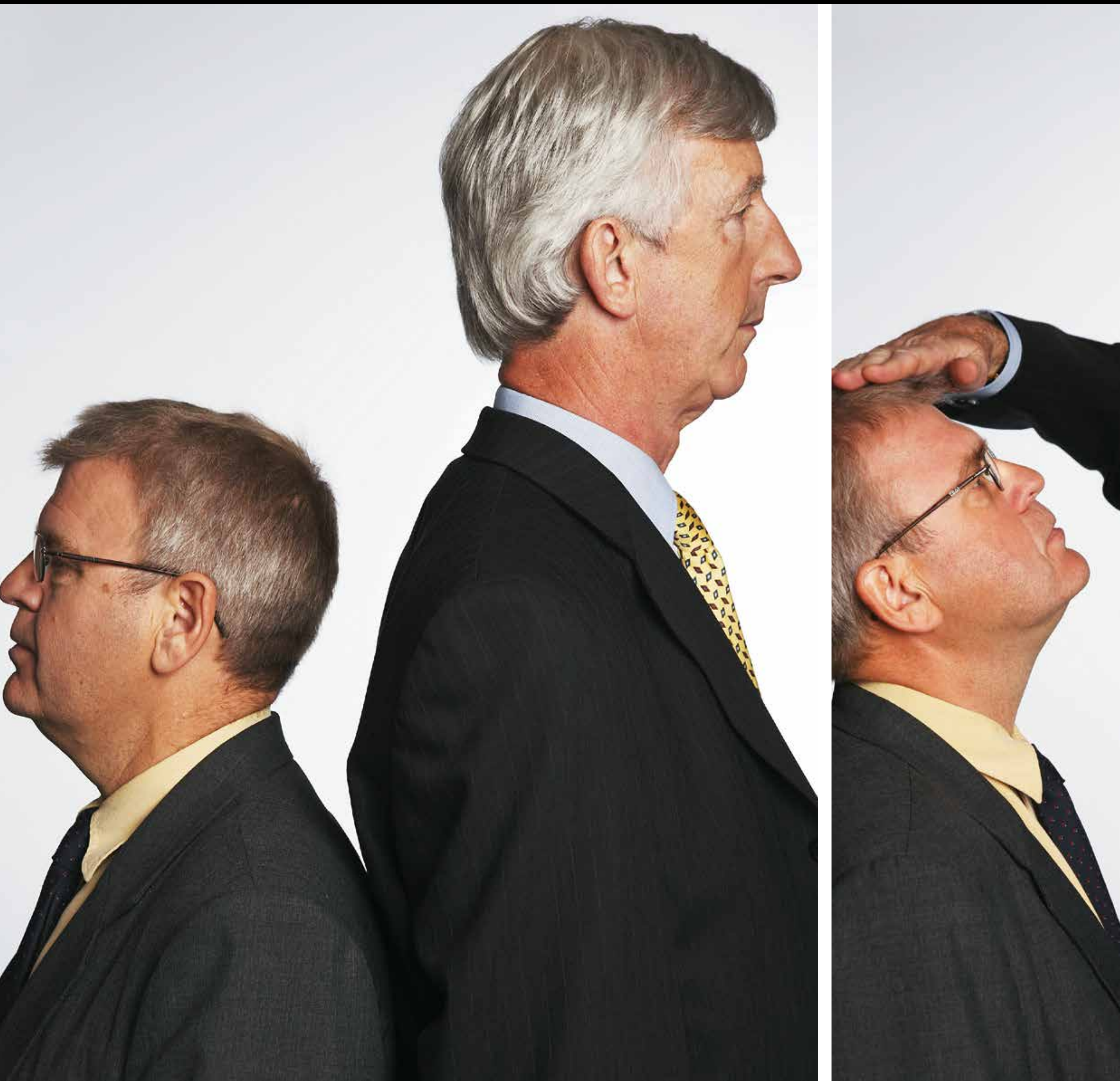
REVISITING Mr. Tall and Mr.



Short

While students are solving a proportion problem, their work in a measure space will enable teachers to take the measure of their thinking.

Suzanne M. Riehl and Olof Bjorg Steinthorsdottir

Ratio, rate, and proportion are central ideas in the Common Core State Standards (CCSS) for middle-grades mathematics (CCSSI 2010). These ideas closely connect to themes in earlier grades (pattern building, multiplicative reasoning, rational number concepts) and are the foundation for understanding linear functions as well as many high school mathematics and science topics. Students develop proportional reasoning slowly, and they need many experiences in diverse contexts to build their conceptual understanding (Lamon 1995). As students journey toward mature proportional reasoning, teachers can gain insight into their thinking by carefully analyzing their solution strategies on a single problem.

Robert Karplus and his colleagues began using the Mr. Tall and Mr. Short problem (see **fig. 1**) in this way in the late 1960s. Karplus, Karplus, and Wollman (1974), influenced by Piaget, aimed to chart the development of abstract reasoning in young students. The Mr. Tall and Mr. Short task, unlike the original Piagetian tasks, did not require an understanding of physical principles and so was accessible to younger children.

Beyond its use in formal research, this problem is a "classroom challenge" explored in the 2002 NCTM Yearbook, *Making Sense of Fractions, Ratios, and Proportions* (Khoury 2002). Teachers using this challenge are encouraged to assess their students at one of four broad levels of proportional thinking. In this spirit, and as part of a larger project to examine proportional reasoning, we gave the Mr. Tall and Mr. Short problem to over 400 middle school students in a small Midwestern town. Our aim in this article is to share the categories of solution strategies we found and

Fig. 1 The Mr. Tall and Mr. Short problem charted students' development of abstract reasoning.

In the picture, you can see the height of Mr. Short measured with paperclips. Mr. Short has a friend, Mr. Tall. When we measured their heights with matchsticks, Mr. Short's height is 4 matchsticks and Mr. Tall's height is 6 matchsticks. How many paperclips are needed for Mr. Tall?

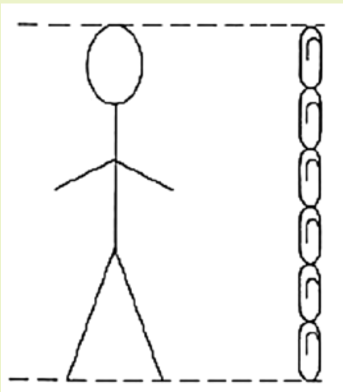
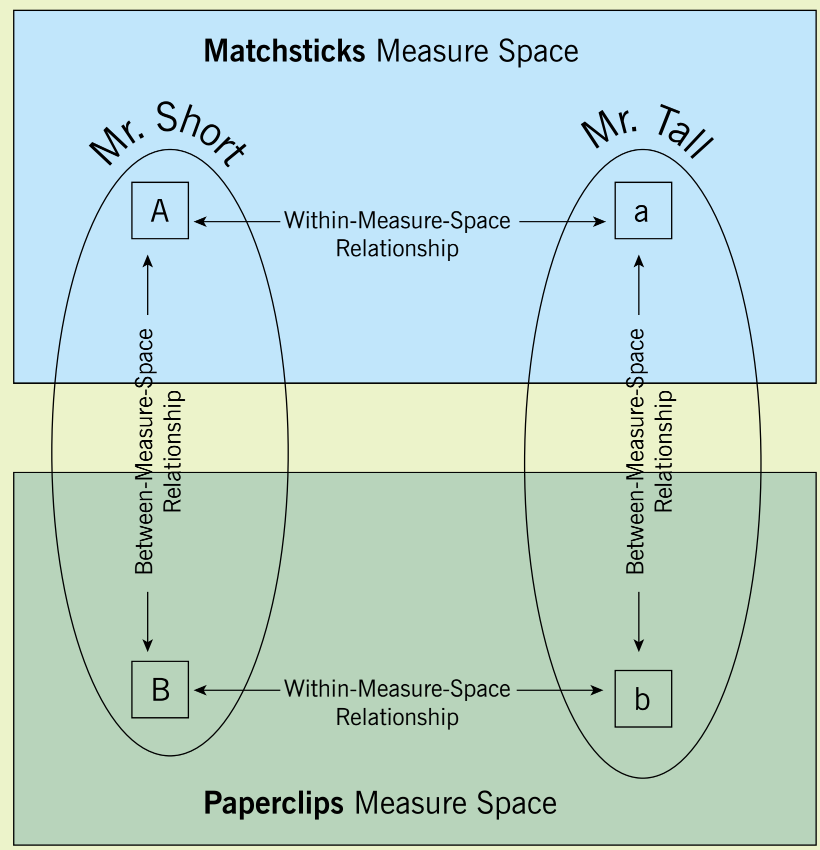


Fig. 2 The number structure can be seen by the fact that the ratio between the matchsticks is the same as the ratio between the paperclips.



to discuss what these strategies reveal about student thinking.

When teachers use the challenge with their students, our categorization may help them determine how best to nurture their students' development. The classic Mr. Tall and Mr. Short problem, unfortunately, has features that blur some levels of understanding. We therefore offer variations of the problem, which give a teacher more information to better gauge a student's reasoning.

MULTIPLICATIVE RELATIONSHIPS

Inherent to a proportion are the multiplicative relationships between the numbers involved. To distinguish among these relationships, we use the concept of a *measure space* (Vergnaud 1983). Two quantities are considered to be in the same measure space when their units are the same (see **fig. 2**). In the Mr. Tall and Mr. Short problem, for example, there is a matchsticks measure space and a paperclips measure space. We call the multiplicative relationship of Mr. Tall's height measured in matchsticks to Mr. Short's height measured in matchsticks a *within-measure-space ratio*. The multiplicative relationship of Mr. Short's height in paperclips to Mr. Short's height in matchsticks is a *between-measure-space ratio*. When we refer to the *number structure* of a proportion, we mean both the *within*- and *between*-measure-space ratios.

In a proportion, the two *within*-measure-space ratios are equal and the two *between*-measure-space ratios are equal. That is, the matchstick-to-matchstick relationship is the same as the paperclip-to-paperclip relationship (A:a = B:b), and the matchstick-to-paperclip relationship for Mr. Short is the same as the matchstick-to-paperclip relationship for Mr. Tall (A:B = a:b). (See **fig. 2**.)

Students succeed in solving proportion problems when they have at least a rudimentary understanding of one of these multiplicative relationships. When they understand both, they will flexibly choose to use whichever relationship permits an efficient solution. This more robust understanding allows students to apply and extend the power of proportional reasoning in diverse settings. As their experience with proportional situations increases, students will also be able to decontextualize and think of proportions algebraically or as equivalent fractions. Only at this point, we maintain, is an algorithmic approach of solving the resulting equation a

desirable strategy (Steinthorsdottir and Sriraman 2009).

HOW DO STUDENTS SOLVE THE PROBLEM?

In our study, the Mr. Tall and Mr. Short problem was part of a pencil-and-paper instrument in which students were instructed to explain their thinking. We coded student explanations by using two categories for erroneous strategies, illogical and additive, and two categories for correct reasoning, build-up and multiplicative. We then refined the multiplicative category to capture how students used the multiplicative structure of the proportion. **Table 1** defines these categories; **table 2** provides a breakdown of strategy by grade. In the following paragraphs, we illustrate each category with student work and discuss possible interpretations.

The illogical category is a collection of error strategies for work that

Table 1 Student explanations are coded using categories for erroneous and correct reasoning.

Types of Student Reasoning		
Illogical	No explanation is given; guesses and random computations are used.	
Additive	The difference between two of the quantities (either in the same or different measure spaces) is computed and applied to the third quantity. The comparisons are absolute rather than relative.	
Build-Up	The given ratio of first-measure-space quantity to second-measure-space quantity is repeated and combined using addition or multiplication as a form of repeated addition.	
Multiplicative	A multiplicative relationship is explicitly used.	
	Between-Measure Space	The given between-measure-space ratio is maintained in the target ratio.
	Within-Measure Space	The scale factor is determined and applied within each measure space, a reduced rate is scaled, or a unit rate is scaled.
Ambiguous	It is impossible to distinguish whether the student is building up (using addition) or scaling down and then scaling up (using multiplication).	
Multiplicative-Ambiguous	It is impossible to distinguish whether the student is using the within- or between-measure-space ratio.	

Table 2 This table enumerates student explanations and their frequency as a percentage.

Category		Grade 5 (n = 62)	Grade 6 (n = 88)	Grade 7 (n = 107)	Grade 8 (n = 155)
Illogical (error)		48	23	23	22
Additive (error)		39	60	50	33
Build-up (correct)		6	10	11	8
Ambiguous (correct)		5	3	3	8
Multiplicative (correct)	Between	0	0	1	3
	Within	0	0	6	16
	Ambiguous	2	3	6	10

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Fig. 3 These samples of student work demonstrate illogical strategies.

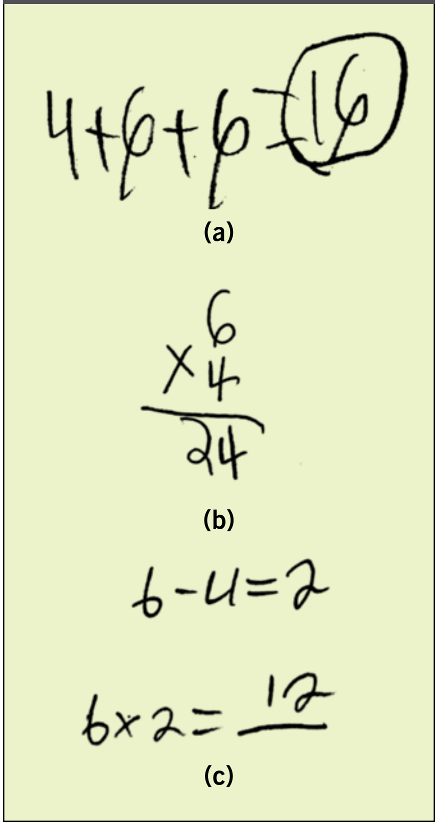
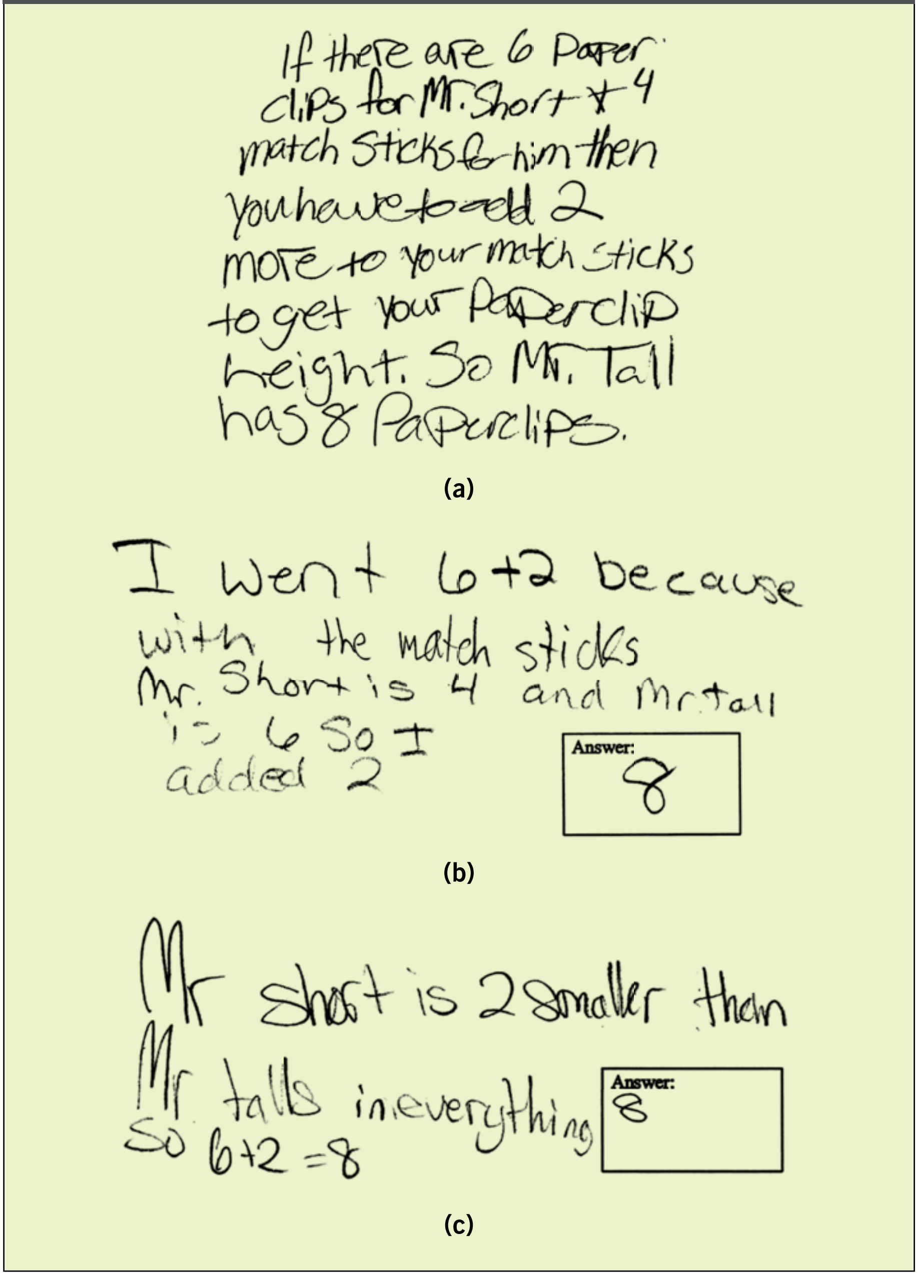


Fig. 4 With these additive strategies, students were using absolute comparisons rather than relative comparisons.



shows little understanding of proportions. Numbers given in the problem may be combined haphazardly, as in **figure 3a**, in which the three numbers are summed, or in **figure 3b**, in which two numbers are multiplied. Also classified as illogical is the work in **figure 3c**; in this example, the student seems to compare two quantities additively and then applies the result multiplicatively.

In the additive category, students use absolute comparisons rather than relative comparisons. **Figure 4a** shows how a student computes the *between*-measure-space difference (Mr. Short is 6 paperclips or 4 matchsticks tall, which are 2 apart) and maintains that difference for Mr. Tall. In **figure 4b**, the student instead computes the *within*-measure-space difference (6 matchsticks for Mr. Tall compared with 4 for Mr. Short) and answers that Mr. Tall's height in paperclips is 2 more than Mr. Short's. **Figure 4c** illustrates a strategy in which a student declares that there is a difference of two "in everything," and, therefore, Mr. Tall is 8 paperclips in height.

As **table 2** shows, although the majority of students reason incorrectly, the distribution of errors changes with grade level. Between the fifth grade and the sixth grade, answers falling into the illogical category decreased from 48 percent to 23 percent, whereas those falling into the additive category increased from 39 percent to 60 percent. For seventh grade and eighth grade, the percentage of illogical answers is essentially the same as that for the sixth grade, whereas the percentage of additive answers declines to 50 percent and 33 percent, respectively. This may indicate that more mature students

recognize the need to compare quantities, although many students choose the incorrect type of comparison. In the next section, we conjecture that this specific problem has features that lure students into additive thinking when they might not use it to solve a different problem.

In using a build-up strategy, students recognize that the ratio of Mr. Short's 4 matchsticks to 6 paperclips forms a unit that needs to be coordinated. This unit (the *between*-measure-space ratio) can then be joined repeatedly to the same ratio, joined to an equivalent ratio, or partitioned (Lobato et al. 2010). In each case, the result will be an equivalent ratio. Students use this knowledge to generate equivalent ratios of 4:6 until a desired ratio is found.

Figures 5a and **5b** are examples that show how the total number of paperclips is determined by computing the number of extras needed by Mr. Tall to account for his 2 extra matchsticks. In **figure 5a**, the rate of 1.5 paperclips/matchstick is added to the original ratio 6:4 twice. In **figure 5b**, the unit rate is computed (using the figure), and then the equivalent ratio 2:3 is added to the original 4:6 to reach the ratio 6:9. In both cases, students join equivalent ratios to create a newly composed ratio of 6 matchsticks to 9 paperclips. In **figure 5c**, the student uses 6 iterations of 1.5:1, one for each matchstick, to create a new composed ratio of 6:9.

Multiplicative strategies explicitly use one of the proportion's multiplicative relationships (either the *between*-measure or *within*-measure space). The *between*-measures strategy (comparing paperclips with matchsticks) is applied when students explicitly use the fact that Mr. Short's measurement is 1 1/2 times greater in paperclips than in matchsticks (or 2/3 as much in matchsticks as in paperclips), and hence so is Mr. Tall's. In **figure 6a**, the

Fig. 5 The number of paperclips is determined by employing the build-up strategy.

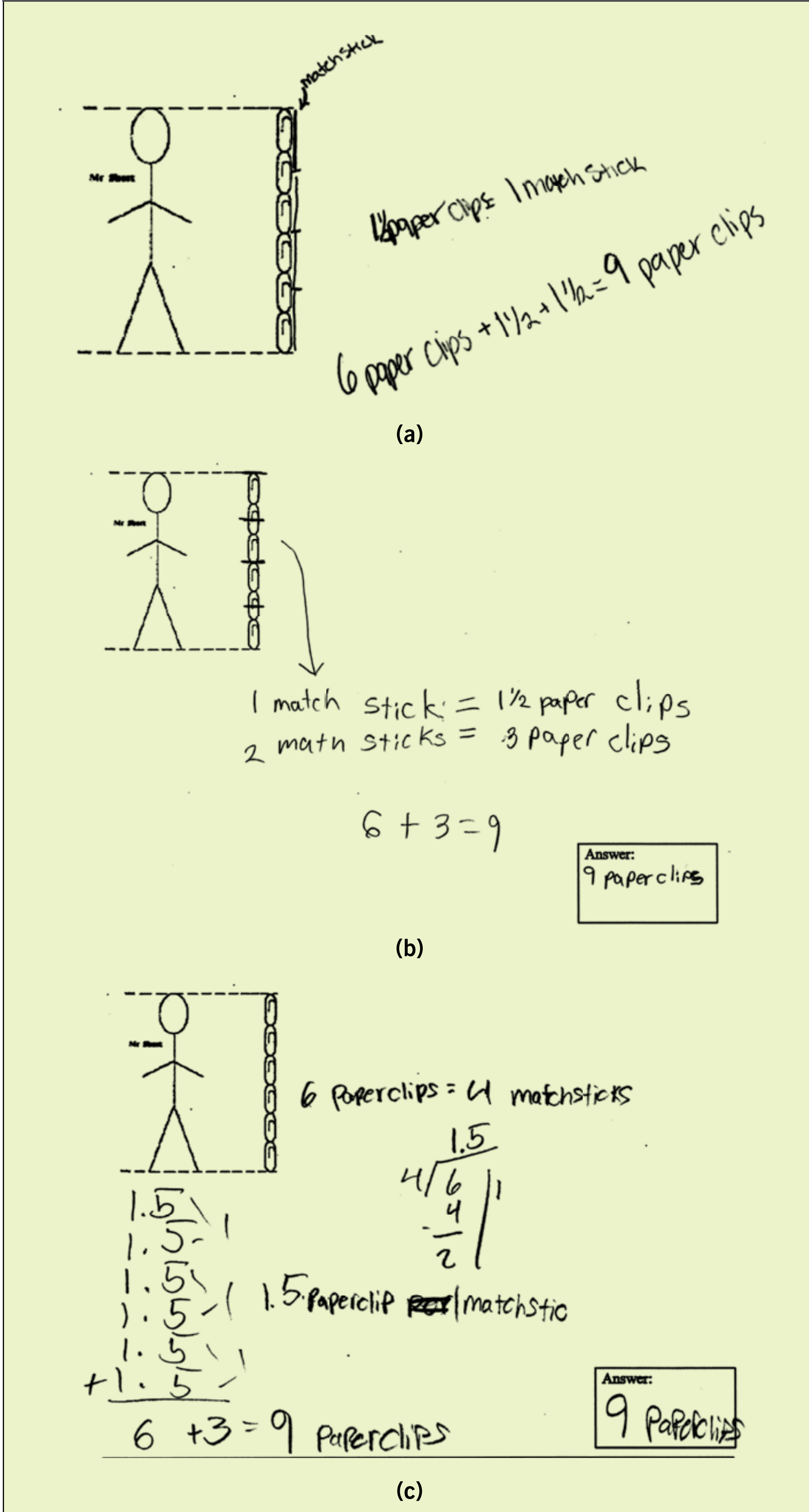
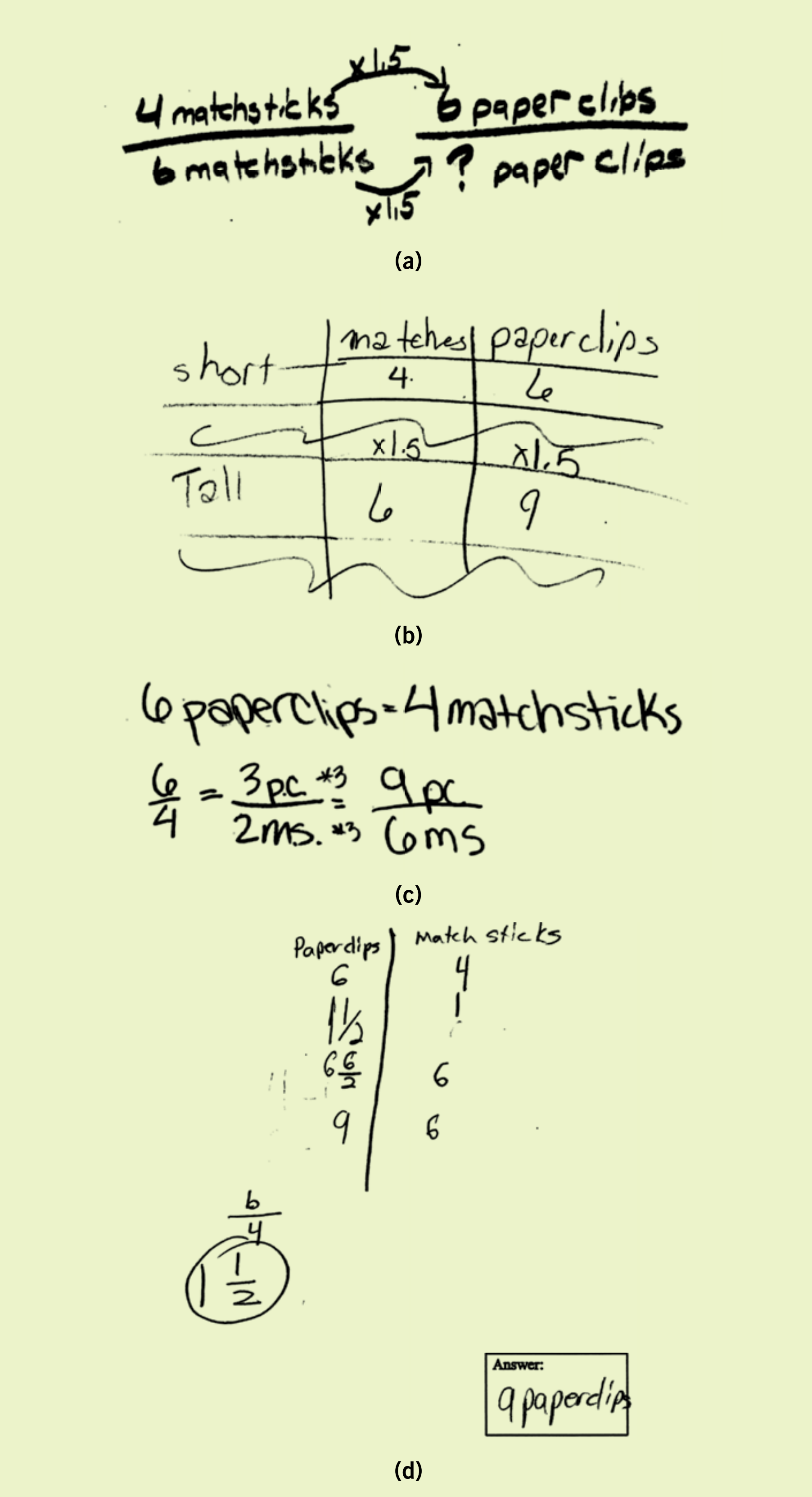


Fig. 6 These student-work samples demonstrate various multiplicative strategies, either using the *within* or *between* ratios.



student uses the factor of 1.5 with the unwritten units *paperclips/matchsticks*, and so is employing the *between*-measure-space ratio.

On the other hand, the *within*-measures strategy compares matchsticks with matchsticks and paperclips with paperclips. Students determine a scale factor, perhaps in one step (6 matchsticks ÷ 4 matchsticks; see **fig. 6b**) or after reducing the original ratio of matchsticks to paperclips to 2:3 or 1:1.5 (see **figs. 6c** and **6d**). The scale factor is then applied to Mr. Short's paperclip measurement to find Mr. Tall's measurement. Note that the extra step of reducing the original ratio results in a simpler integer scale factor, and thus, finding the unit rate does not necessarily imply the use of the *between*-measure-space relationship.

The number structure of this problem makes it difficult to determine which relationship is used unless the students label quantities. In **figure 7a**, the student uses a multiplicative strategy, but it is not clear whether the 6 represents matchsticks or paperclips. This type of work is coded as multiplicative-ambiguous in **table 2**.

For other responses, we cannot determine whether the students used a build-up strategy or a multiplicative strategy. The strategy in **figure 7b** is an example. It is clear that the student reduced the given ratio 4:6 to 2:3, but it is unclear whether the student then found the equivalent ratio of 6:9 by adding the ratios 4:6 and 2:3 or if the student applied the scale factor of 3 to the 2:3 ratio. This type of work is coded as ambiguous in **table 2**.

The percentage of students using a correct strategy increased from 13 percent in fifth grade to 16 percent in sixth grade, to 27 percent in seventh grade, and to 45 percent in eighth grade. As expected, the use of multiplicative strategies increased by grade level, from 2 percent in fifth

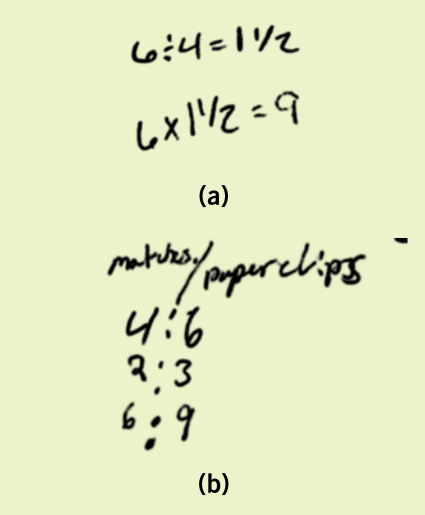


Fig. 7 Multiplicative strategies are used by these students, but it is unclear into which subcategory they fall.

grade to 29 percent in eighth grade (see **table 2**). When students annotated their work, the *within*-measures multiplicative strategy was preferred.

We believe that many students begin with a build-up strategy, in which the tendency is to compute how to transform the given number of matchsticks into the target number and then duplicate the operation for the paperclip quantity. The transition from repeated addition to a scaling operation (a *within*-measures strategy) is natural. Prior research, although not conclusive, supports this idea (Steinthorsdottir and Sriraman 2009). Problem context, number structure, and other factors also affect student choices, and we perhaps define the *between*-measure-space strategy more narrowly than do other researchers.

WHY IS THE SUCCESS RATE LOW?

Older students correctly solve this problem more frequently than younger students, but their success rate is surprisingly low. Our numbers are similar to those reported by Karplus, Karplus, and Wollman (1974). For example, 29 percent of our 412 students (grades 5–8) correctly solved the problem, and 44 percent used additive reasoning. Karplus, Karplus, and Wollman (1974) report that 37 percent of 610 students (grades 4–9) used correct reasoning and 32 percent used additive reasoning.

It is important to consider what makes this problem harder than expected. The Mr. Tall and Mr. Short problem uses a scaling context, which Lamon (1993) argues is difficult. Although the numbers involved are small, both the *within*- and *between*-measure-space ratios are 1 1/2. Research is clear in stating that integer relationships are easier than non-integer relationships, and yet one would expect that 1/2 relationships would be the next easiest. This does not appear to be true for the 1 1/2 relationship because, we argue, 1 1/2 is less than 1 full repeat.

Consider, for example, the problem 4:6 = 14:x, in which the scale factor is 3 1/2. Students with a beginning understanding of proportion can build up using whole units: 4:6, then 8:12, and then 12:18. The target is not quite reached. Some students will partition 4:6 to correctly finish the build-up (2:3 joined to 12:18 gives 14:21). Other students will fall back on additive thinking to finish the problem, adding 2 to both quantities and getting the incorrect answer of 14:20. Two levels of understanding are uncovered in this case.

With a scale factor of 1 1/2, students must immediately deal with the fractional part. In this case, students who can build up with integer repeats but fail when faced with “leftovers” are indistinguishable from students who are thinking additively.

The small numbers in this problem also mean that the relative difference ($6 \div 4 = 1.5$) and absolute difference ($6 - 4 = 2$) between the numbers are approximately the same. The incorrect additive answer is thus “in the ballpark” and may not alert students to an error. The fact that 6 appears as a measurement in both ratios may also be a source of confusion.

SHOULD YOU ACCEPT THE MR. TALL AND MR. SHORT CHALLENGE?

No one problem can fully assess proportional reasoning. In fact, mature proportional reasoning is indicated by the successful navigation of a variety of problems from diverse contexts, no matter the complexity of the number structures involved. On the other hand, teachers do learn much from a careful analysis of a single problem and can use the resulting information to make instructional decisions.

We recommend trying one of the following variations of Mr. Tall and Mr. Short to delve more deeply into your students’ thinking. First, consider which broad category of reasoning (illogical, additive, build-up, or multiplicative) you expect most of your students to use. See **table 1** and note that **table 2** suggests that you may see all reasoning categories in your classroom. Then choose a variation that will enable you to verify and refine your assessment. The first two problems are appropriate for novice students and are designed to reveal a range of sophistication in build-up strategies. The last two problems should allow you to assess how robust and flexible your students are in the use of multiplicative strategies. The classic problem (see **fig. 1**) may be most illuminating in assessing experienced students. You may discover that some students fall back into additive thinking, and you will thus be able to address their misconceptions.

- *4 matchsticks:6 paperclips = 20 matchsticks:x paperclips*. This problem uses a whole-number scale factor. This simple case can launch discussions on relative versus absolute comparisons.

Mature proportional reasoning is indicated by the successful navigation of a variety of problems from diverse contexts.

Correct solutions will distinguish between students using build-up and multiplicative strategies.

- *4 matchsticks:6 paperclips = 14 matchsticks:x paperclips*. This problem has a scale factor of 3.5. Because this factor is more than 1 full repeat, you can determine how students deal with "leftovers." Which students have a basic understanding of the build-up strategy, and which use techniques that are more sophisticated?
- *4 matchsticks:6 paperclips = x matchsticks: 9 paperclips*. This problem is similar to the classic problem but removes the potentially distracting second appearance of the 6 and allows you to determine which multiplicative strategy your students are using. Because the scale factor is 1.5, you may see some students lured back into additive thinking.
- *6 matchsticks:12 paperclips = 22 matchsticks:x paperclips*. This problem uses a "messy" scale factor of 3 2/3. Which measure space ratio do your students use? The *between*-measures ratio clearly leads to a more efficient solution, but you may still see students scaling by the *within*-measures ratio. Use this opportunity to compare and discuss solution strategies and the multiplicative relationships in a proportion.

Once you know how your students are thinking about proportions, you can guide them to explore all aspects of the multiplicative structure of proportion. As they expand their repertoire of solution techniques, they strengthen the connections among division, fractions, and rational numbers, and they lay the groundwork for working with slopes and rates of change in functions. You can help your students navigate their routes to reason.

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Any thoughts on this article? Send an e-mail to ***mtms@nctm.org***.—*Ed*.

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