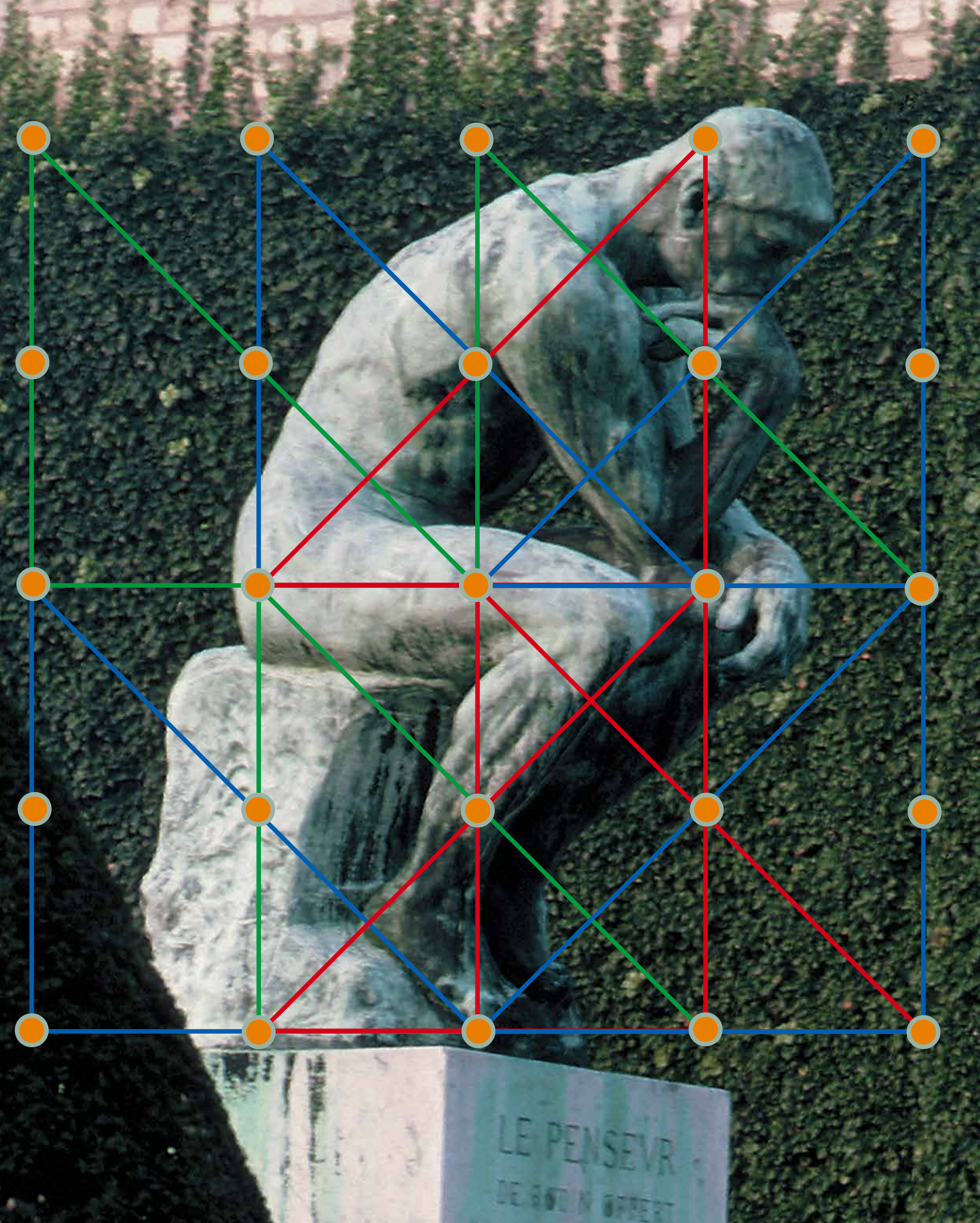
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Taking It to the Next Level: Students Using INDUCTIVE REASONING

Use the Inquiry Continuum model and a geoboard problem as guides to design successful inductive reasoning tasks that blend procedural and conceptual development.

Jaclyn M. Murawska and Alan Zollman

A middle school student asked us, "Why does the pattern . . . 70, 80, 90, 100, go to 110, 120, and not . . . 70, 80, 90, 100, 200, 300?" We understand her thinking, but how do we use her reasoning to deepen her mathematical understanding? We found that inductive reasoning using patterns to be an excellent method of learning mathematical connections. But for students to deepen their understanding and correctly use their inductive reasoning, experiencing cognitive dissonance in these mathematical patterns tasks is important, especially for middle-grades students.

THE IMPORTANCE OF INDUCTIVE REASONING

Although discussions about inductive reasoning can be traced back thousands of years (Fitelson 2011), the implementation of the Standards for Mathematical Practice (SMP) within the Common Core State Standards (CCSSI 2010) is generating renewed attention to how students learn mathematics. The third SMP, "Construct viable arguments and critique the reasoning of others" (CCSSI 2010, p. 6), explicitly calls for mathematically proficient students to be able to reason inductively and be able to

Fig. 1 These four levels of inquiry were originally given to science teachers.

Inquiry Level	Question	Procedure	Solution
1–Confirmation Inquiry Students confirm a principle through an activity when the results are known in advance.	✓	✓	✓
2–Structured Inquiry Students investigate a teacher-presented question through a prescribed procedure.	✓	✓	
3–Guided Inquiry Students investigate a teacher-presented question using student designed and selected procedures.	✓		
4–Open Inquiry Students investigate questions that are student formulated through student designed and selected procedures.			

Source: Banchi and Bell (2008)

judge the validity of their conclusions. NCTM (2000, p. 16) also emphasizes that middle school students should be "proficient in using inductive and deductive reasoning appropriately." The development of these reasoning skills can help promote students' conceptual understanding, linking conceptual knowledge and procedural knowledge (Hiebert and Lefevre 1996; NCTM 2000). Hence, to align with national standards and promote conceptual understanding, middle school mathematics educators should provide ample opportunities for students to engage in inductive reasoning activities that can build to deductive reasoning.

The series of tasks that we propose encourages students to reason inductively and uses problem solving with multiple representations, supported by current mathematics education standards (CCSSI 2010; NCTM 2000). Because the tasks are undergirded by science education inquiry, these tasks support recent STEM literacy initiatives, as well (Zollman 2012).

DIFFERENCES BETWEEN INDUCTIVE AND DEDUCTIVE REASONING

Inductive reasoning, which Baroody, Reid, and Purpura (2012) also refer to as "*empirical induction*, entails examining examples (particulars) and discerning a commonality or pattern (discovering a generality)" (p. 4). Suppose you ask your students to find the parity of the sum of any two odd numbers. An example of inductive reasoning would be if, after students have observed from a few examples that the result is always even, they determine the generalization that the sum will always be even. Inductive reasoning here should not be confused with the mathematical proof by induction that proves a claim by an infinite iterating procedure.

Deductive reasoning "involves reasoning from a premise or premises assumed to be true (a generality or generalities) to logically arrive at a conclusion about a particular case" (Baroody, Reid, and Purpura 2012, p. 4). A common application in the middle school mathematics classroom is when the teacher presents a rule or generalization, such as the formula for the area of a circle, and then students use this formula to find the area of various objects that have circular parts.

THE PROBLEM AND A POSSIBLE SOLUTION: THE INQUIRY CONTINUUM

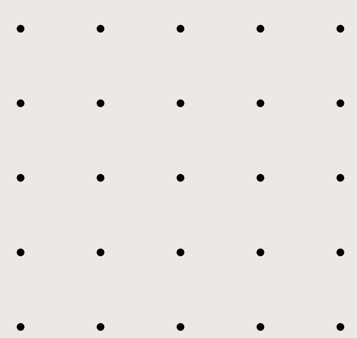
For some of our middle school students, generalizing through inductive reasoning is unattainable. Why? According to cognitive load theory (Kirschner, Sweller, and Clark 2006), some students' working memories may become overburdened. These students may find it difficult to generalize because they have not yet acquired a sufficient knowledge base of relevant mathematical content. For the teacher, this means providing more guidance up front and then fading the guidance during problem solving. To accomplish this, Banchi and Bell's (2008) Inquiry Continuum (see **fig. 1**) for science educators can offer direction to mathematics educators as they design tasks that will provide students with sufficient prior knowledge before they reason inductively.

APPLICATION FOR THE MATHEMATICS CLASSROOM

Through the lens of this continuum, mathematics educators can design tasks at differentiated levels of inquiry so that students receive sufficient guidance in the beginning before being asked to develop a valid generalization. As an illustration, we present a suggested sequence of four related tasks to correlate with the four levels of inquiry using an adaptation of Speer's (2011) Geoboard Segment problem (see **fig. 2**). This problem includes applications of the Pythagorean theorem as listed in the eighth-grade Common Core State Standards for geometry (CCSSI 2010).

Fig. 2 The Geoboard Segment problem opened the reasoning lesson.

Question 1: How many segments of length one unit can you find on this geoboard?



Level 1: Confirmation Inquiry

Question: How many segments of length one unit can you find on the geoboard?

The role of the teacher in confirmation inquiry is to provide students with a question and procedure, where the solution is known in advance. This question helps students develop their mathematical knowledge base for the subsequent related questions. (The solution to the question is 40 segments of length one.)

In our experiences administering the level questions, many students preferred to draw every segment and then count the segments, as shown by the student work in **figure 3**. This question not only confirms a correct answer but also supplies a basis of justification for the following tasks.

Level 2: Structured Inquiry

Question: How many segments of length two, three, four, and five units can you find on the geoboard?

The role of the teacher in structured inquiry is to provide the question and the procedure, but students generated their solutions on the basis of empirical evidence. In the level 2 question, students may first use deductive reasoning because the number of segments of length one can be represented as $4 \times 5 \times 2$ (4 in each row $\times$ 5 rows $\times$ 2 for both horizontal and verticals segments). Some students saw only 4×5; however, after discussions with others, these students saw both the horizontal and the vertical that changed their results to $4 \times 5 \times 2$. It follows logically that the number of segments of length two is $3 \times 5 \times 2$, and the number of segments of length three is $2 \times 5 \times 2$, and so on.

Fig. 3 A student work sample from the level 1 question showed careful counting.

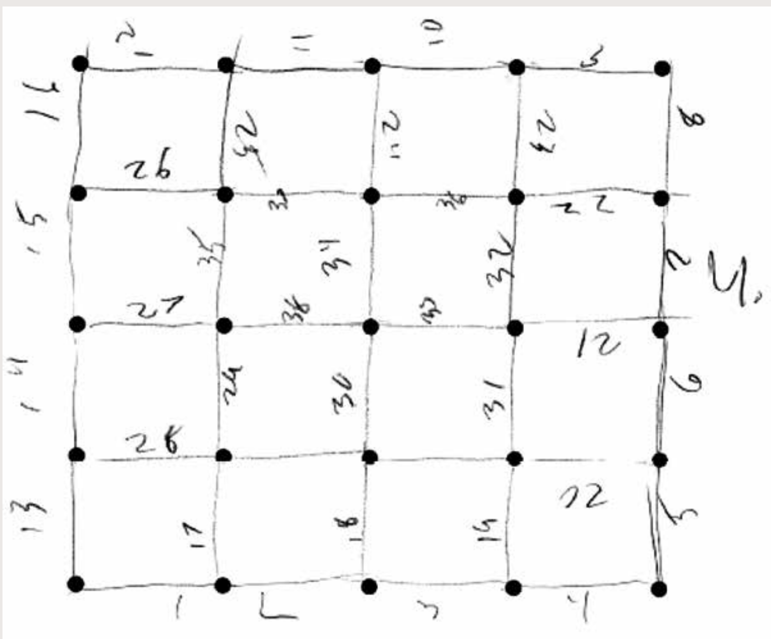
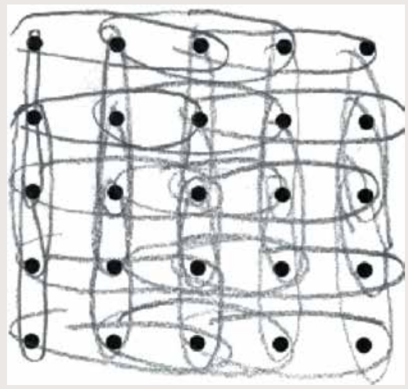


Fig. 4 This sample of student work for the level 2 question showed all segments of length three.



(a) Identifying every segment

(b) Identifying half the segments, before rotating.

However, our students approached their search for these segment lengths differently. Some students again found comfort in identifying every segment, whereas others stopped counting after determining the number of horizontal segments of length three, then mentally rotated the geoboard a one-fourth turn to obtain an equal number of segments vertically (see **fig. 4**). Providing a separate geoboard recording

Fig. 5 The solution to the level 2 question required students to explore right triangles.

Segment Length	Number of Segments
1	40
2	30
3	20
4	10
5	8*

* As hypotenuses of 3-4-5 right triangles

sheet seemed to help students test and verify their conjectures.

At this point in the task, students can begin to use inductive reasoning because they will often see a pattern as the lengths of each segment increase (see **fig. 5** for the solution to the level 2 question). Using the empirical evidence given by the examples thus far, they may come to a conclusion on the number of segments of length five. Some students reasoned that the number of segments of length five must be zero, which also was incorrect. After the teacher suggested the Pythagorean theorem, students were able to generate all the segments with length five, but only after discussion among the students of 3-4-5 right triangles on the geoboard.

The pattern that surfaced in this question provides a "desirable difficulty" (Speer 2011, p. 3) in which the end result for the number of segments of length five is not anticipated. This part of the task creates disequilibrium in students' minds. It leads to a rich discussion of how the procedure changes to use the Pythagorean theorem to find all segments of length five. In fact, as Allen (2013) had found with the Geoboard Triangles problem, finding all the segments of length five brings up "issues of orientation, reflection, and rotation" (p. 115).

Level 3: Guided Inquiry

Question: How many segments of length $\sqrt{2}$ units can you find on the geoboard?

The role of the teacher in guided inquiry is to provide the question, and then students determine both the procedure and a justification for a solution. Because students have had prior experience identifying segments of length five using right triangles, we expected that they would have sufficient mathematical tools to find the number of segments of length $\sqrt{2}$ without their working memories being overburdened in the level 3 question. Our students seemed confident with their correct solution: 32 segments of length $\sqrt{2}$ from the hypotenuse of $1-1-\sqrt{2}$ right triangles.

This investigation can lead to a powerful extension, such as a discussion of how irrational numbers (e.g., $\sqrt{2}$) are represented as finite lengths on the geoboard. Thus students can problem solve using multiple representations: hypotenuse lengths of right triangles on a geoboard and the symbolic representations of the lengths as irrational numbers.

Level 4: Open Inquiry

Questions: Formulated by students

The role of the teacher in open inquiry is to let the students design the questions, procedures, and solutions. Banchi and Bell (2008) suggest having students write some "I wonder" questions that they could investigate. Here are some possible student-formulated questions.

I wonder—

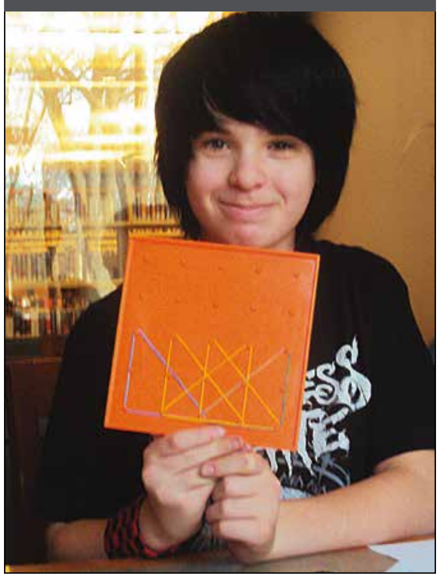
- how many segments of length $\sqrt{5}$ are there?
- how many segments of length $\sqrt{8}$ are there?
- how many different (i.e., unique) length segments are there?
- what happens if we use a smaller or larger geoboard?
- what is the relationship between the number of segments of a given length and the number of pegs on the geoboard?

Depending on the questions they generate, students have the opportunity to use inductive or deductive reasoning to find solutions to their chosen problems. For example, the solution to the question regarding the number of unique segment lengths is 14:

$$1, 2, 3, 4, 5, \sqrt{2}, \sqrt{5}, \sqrt{8}, \sqrt{10}, \sqrt{13}, \sqrt{17}, \sqrt{18}, \sqrt{20}, \text{ and } \sqrt{32}.$$

One student asked, "How many right triangles with 2 sides each being 2 can I make?" **Figure 6** shows the student's solution to the problem. This question led to another rich discussion among the students regarding the correct answer. They ultimately concluded that there were 18 such triangles. This is an example of the inquiry continuum

Fig. 6 Having students display their work often leads to rich discussion.



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transferring to other problems and moving students toward deductive reasoning.

DIFFERENTIATION

The use of the inquiry continuum can not only help mathematics teachers design tasks in which inductive reasoning can be successful but also help teachers adjust their instruction to differentiate to students' needs because they are allowed to create their own questions. Interestingly, one student sought to create what he thought would be an easy question, "How many squares are there [on the geoboard]?" However, his anticipated solution of 16 was incorrect, as he originally considered only the 1×1 squares. Luckily, this provided a challenge that he successfully met; he concluded that this geoboard had 30 total squares on it.

In general, listening to our students' responses as well as encouraging students to listen to their peers' reasoning are important elements to the effectiveness of the lesson as prescribed by the third SMP: "Construct viable arguments and critique the reasoning of others" (CCSSI 2010, p. 6).

SOME LIMITATIONS

Students' False Generalizations

Obtaining a false generalization is a legitimate and serious limitation of inductive reasoning. But teachers can make this limitation clear to their students by choosing tasks that do not always elicit a valid generalization every time (NCTM 2000). This is why including the level 2 questions, in which the number of segments of length five did not follow the anticipated pattern, is an important component of the activity.

Accepting Inductive Reasoning Justifications as Proofs

A second limitation of using inductive reasoning is that finding generalizations often makes students think that this type of argument is a proof (Rips and Asmuth 2007). Even though inductive reasoning may elicit the big ideas, which can lead to a formal proof, empirical evidence alone is not sufficient. To mediate this, we used the example of how the sum of two odd numbers makes an even number. It is important to ask students why this always happens. If a student can only cite the examples, this is not a proof. If the student can articulate an argument based on combining those extra numbers to form another pair, the student has the essence of the key idea of the proof, which is an important step between inductive reasoning and formal proofs.

In fact, what students consider as "proof" is influenced highly by how the teacher responds to student discussion. For example, in the questions for levels 2, 3, and 4, the teacher can prompt students to justify their correct responses, leading students to begin to see that there is more to the mathematics than just identifying the correct patterns. Such an approach aids students to extend their inductive reasoning to a more sophisticated deductive argument leading to formal proof.

Teacher Proficiency in Content Knowledge and Pedagogy

A third limitation of inductive reasoning is one of practicality. To facilitate inductive reasoning, middle school teachers must be proficient in content knowledge, pedagogical knowledge, and pedagogical content knowledge (Shulman 1986). In particular, it is essential that the teacher be skilled in questioning techniques to elicit the students' thinking strategies in mathematics. Further, classroom norms must be developed that provide a comfortable environment conducive to meaningful discourse throughout the inductive reasoning activity.

For students to deepen their understanding and correctly use their inductive reasoning, experiencing cognitive dissonance while working with mathematical pattern tasks is important.

Therefore, a series of tasks that gives students a chance to conjecture and discuss is a great way to help cultivate this type of classroom norm.

Even considering these limitations, the advantages of promoting inductive reasoning in the classroom outweighs the potential obstacles. Not only has inductive reasoning been used extensively in real life and across disciplines, but it can also promote conceptual understanding and mathematical proficiency, thus aligning with current mathematics education initiatives (CCSSI 2010; NCTM 2000).

In this particular series of related tasks using an Inquiry Continuum, the content aligns to the Common Core eighth-grade standards that call for students to "understand and apply the Pythagorean theorem" (CCSSI 2010, p. 56); "know that there are numbers that are not rational, and approximate them by rational numbers" (CCSSI 2010, p. 54); and "verify experimentally the properties of rotations, reflections, and translations"

(CCSSI 2010, p. 55). An inquiry approach upsets the one-to-one correspondence between standards and daily lesson assumed by some teachers. This is an example not of teaching the particular standard but of teaching the student via connected standards.

INQUIRY CONTINUUM MODEL PROMOTES SUCCESS

At times, inductive reasoning might not be effective if students' working memories are overloaded. However, we have found that using this Inquiry Continuum model from science education is a helpful guide in designing successful inductive reasoning tasks. This continuum is a strong model for middle school teachers to blend procedural and conceptual development to deepen student mathematical understanding.

REFERENCES

Allen, Kasi C. 2013. "The Geoboard Triangle Quest." *Mathematics Teacher* 107 (September): 113–18.

Banchi, Heather, and Randy Bell. 2008. "The Many Levels of Inquiry." *Science and Children* 46 (2): 26–29.

Baroody, Arthur J., Erin E. Reid, and David P. Purpura. "An Example of a Hypothetical Learning Progression: The Successor Principle and Emergence of Informal Mathematical Induction." Paper presented at the International STEM Research Symposium, Savannah, GA, May 30–June 2, 2012.

Common Core Sate Standards Initiative (CCSSI). 2010. Common Core State Standards for Mathematics. Washington, DC: National Governors Association Center for Best Practices and the Council of Chief State School Officers. http://www.corestandards.org/wp-content/uploads/Math_Standards

Fitelson, Branden. 2011. "Inductive Logic." In *Philosophy of Science: An Encyclopedia*, edited by Jessica Pfeifer and Sahotra Sarkar, pp. 384–94. New York: Routledge Press. http://fitelson.org/il.pdf

Hiebert, James, and Patricia Lefevre. 1986. "Conceptual and Procedural Knowledge in Mathematics: An Introductory Analysis." In *Conceptual and Procedural Knowledge: The Case of Mathematics*, edited by James Hiebert, pp. 1–27. Hillsdale, NY: Lawrence Erlbaum Associates.

Kirschner, Paul A., John Sweller, and Richard E. Clark. 2006. "Why Minimal Guidance during Instruction Does Not Work: An Analysis of the Failure of Constructivist, Discovery, Problem-Based, Experiential, and Inquiry-Based Teaching." *Educational Psychologist* 41 (2): 75–86.

National Council of Teachers of Mathematics (NCTM). 2000. *Principles and Standards for School Mathematics*. Reston, VA: NCTM.

Rips, Lance J., and Jennifer Asmuth. 2007. "Mathematical Induction and Induction of Mathematics." In *Inductive Reasoning: Experimental, Developmental, and Computational Approaches*, edited by Aiden Feeney and Evan Heit, pp. 248–68. New York: Cambridge University Press.

Shulman, Lee S. 1986. "Those Who Understand: Knowledge Growth in Teaching." *Educational Researcher* 15 (2): 4–14.

Speer, William R. "Creating Desirable Difficulties to Enhance Mathematics Learning." Paper presented at the 11th International Conference of Mathematics into the 21st Century Project, Grahamstown, South Africa, September 10–16, 2011.

Zollman, Alan. 2012. "Learning for STEM Literacy: STEM Literacy for Learning." *School Science and Mathematics* 112 (1): 12–19.

Any thoughts on this article? Send an email to ***mtms@nctm.org.***—*Ed.*

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