Assessing for

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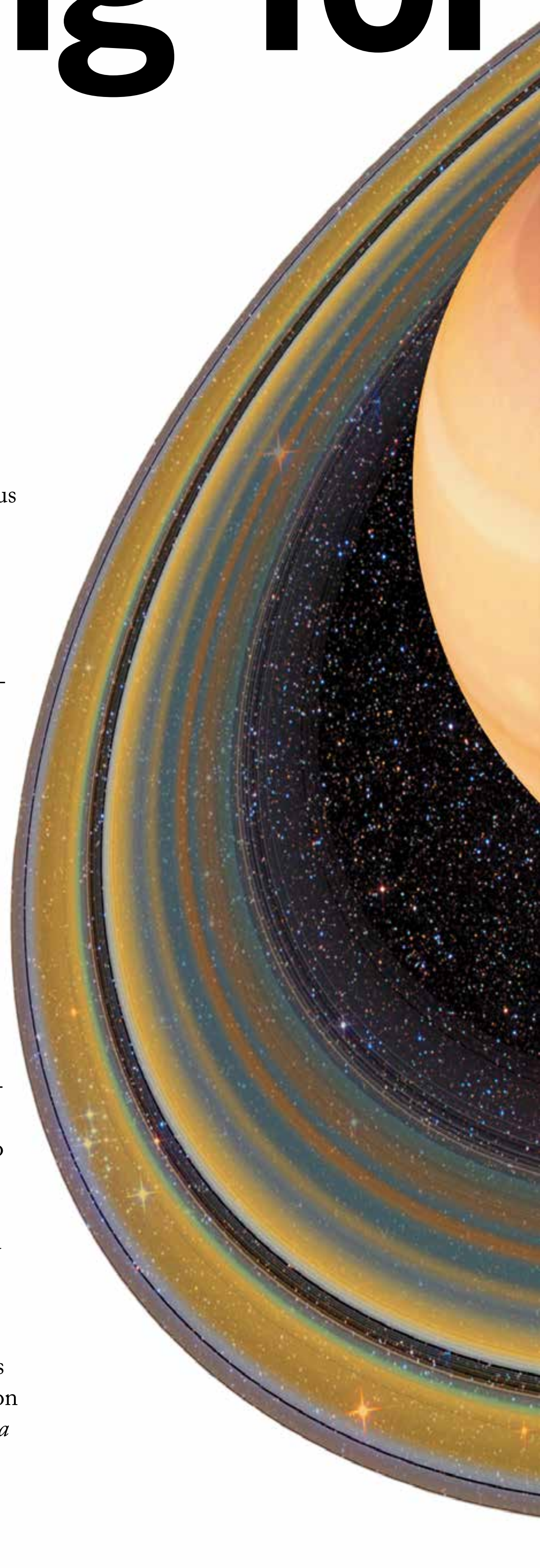
How to design formative and summative assessments that address different types of student cognition.



Continual assessment of student understanding is a crucial aspect of teaching. The adoption of the Common Core State Standards for Mathematics (CCSSM) represents raised expectations for the level and depth of mathematical understanding that is expected of our students. But new standards also mean new tests. What those tests will be like is of concern for many: students, teachers, parents, and school administrators. Although various consortia and groups work on designing better assessments for accountability and reporting at the state and national levels, this article is concerned with improving assessment design and practice in the classroom. How can teachers design test items to meet the goals of CCSSM and assess deep mathematical understanding while also offering opportunities to learn?

We present research-based strategies that use levels of mathematical cognition to design assessment items. These design strategies apply to such formative assessments as warm-ups; informal questioning; class discussions; problem sets incorporating practice, evaluation, and feedback; and summative assessment including unit tests, projects, and quizzes that are used to determine grades. CCSSM provides a common framework of learning goals; in this article, we focus on one standard from seventh grade to illustrate the strategies.

LEARNING LEVELS

To develop meaningful assessment items, it is helpful to organize expectations of student learning into cognitive types or learning levels. **Table 1** lists seven cognitive learning levels, defined by Cangelosi (2002), which will be explored in length: construct a concept, discover a relationship, simple knowledge, comprehension and communication, algorithmic skill, application, and creative thinking.

These learning levels describe the kinds of thinking that are typically required in learning mathematics and are ordered according to a learning progression that is meant to help teachers plan instruction. The ordering in the table reflects the fact that students construct concepts and discover relationships before they are prepared to attach conventional mathematical names and procedures to new ideas and commit those ideas to memory. Hence, lessons focused on *constructing a concept* and *discovering a relationship* should precede lessons

Learning

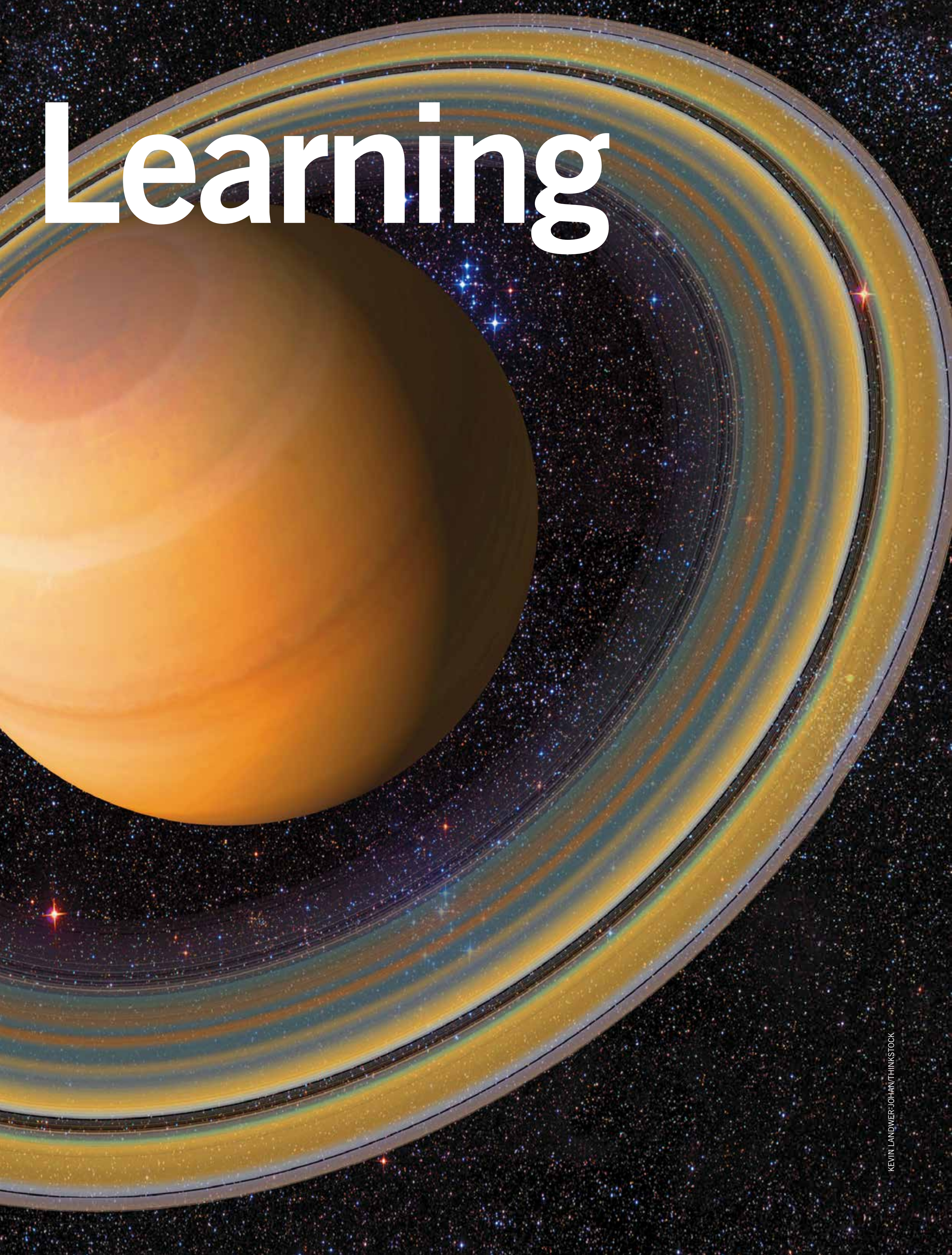


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Table 1 These seven learning levels help organize expectations of student learning.

Learning Level
Construct a concept: Use inductive reasoning to distinguish examples of a particular concept from nonexamples.
Discover a relationship: Use inductive reasoning to discover that a particular relationship exists or why the relationship exists.
Simple knowledge: Remember a specified response (but not a multistep process) to a specified stimulus.
Comprehension and communication: (1) Extract and interpret meaning from an expression, (2) use the language of mathematics, and (3) communicate with and about mathematics.
Algorithmic skill: Remember and execute a sequence of steps in a specific procedure or multistep process.
Application: Use deductive reasoning to decide how to use, if at all, particular mathematical content to solve problems.
Creative thinking: Use divergent reasoning to view mathematical content in unusual, novel ways.

Fig. 1 This example explores the learning level construct a concept.

Objective: Distinguish examples from nonexamples of the area and circumference of a circle.

Prompt: Write either "area" or "circumference" next to each example according to which label best applies:

a. The quantity of space on a kitchen countertop used to roll out a 9-inch pie crust
b. The hat size for each member of the rodeo team who is ordering a cowboy hat
c. The wire length needed for Evelyn to make a basketball hoop with a coat hanger
d. The amount of pizza in a large pizza costing $14.99 at Pi Pizzeria
e. The chance that a dart hits a certain region of a dartboard

Scoring guide (1 point for each correct response, 5 points total):
a. area, b. circumference,
c. circumference, d. area, e. area

addressing *simple knowledge* and *algorithmic skills*. Throughout the learning process, students must explain their mathematical understanding using more formal notation and vocabulary as they deepen their comprehension. For this reason, it is good practice to integrate *comprehension and communication* learning into many lessons. Finally, students are prepared for deductive thinking required in *applications* and have knowledge to do truly *creative* work with mathematics. In this stage, students bring mathematical understanding into their own practice of creative problem solving. (Note: This ordering is not meant to be rigid. For example, applications can also be effectively used at the beginning of an instructional unit to motivate and stimulate student thinking on the mathematical content that follows in a problem-based learning approach.)

Consider the following standard from CCSSM:

> Know the formulas for the area and circumference of a circle and use them to solve problems; give an informal derivation of the relationship between the circumference and area of a circle. (CCSSI 2010, p. 50, 7.G.4,)

Clearly, asking students to state the formulas (simple knowledge) and compute the area and/or the circumference of a given circle (algorithmic skill) are relevant to this standard. However, this standard calls for more. Students also need a deep enough understanding of these formulas to solve problems (application) and explain derivations (comprehension and communication). The wording in CCSSM calls for high levels of cognition, mathematical reasoning, thinking, and engagement not only in the practice standards but also in the content standards themselves.

What follows is a summary of assessment item design strategies based on the learning levels in **table 1**, illustrated with sample items from a seventh-grade unit of instruction about circles. Each assessment item has an objective related to the chosen standard (7.G.4, CCSSM 2010) and includes either a scoring guide or a detailed rubric. A scoring guide makes the item quick to grade, but it fails to provide much feedback to students. A rubric takes longer for the teacher to design, but it provides an efficient way to communicate detailed feedback about written responses. In our experience, a rubric makes grading easier for the teacher and provides an opportunity for students to learn from the evaluation. Often a version of the scoring guide or rubric can be included in the prompt so students know what is important.

ASSESSMENT DESIGN STRATEGIES AND EXAMPLES

Construct a Concept

To achieve this standard, students must know about circles (a concept they have begun constructing since before kindergarten), and conceptually

Fig. 2 An objective, a prompt, and a rubric explore discover a relationship.

Objective: From measurements and calculated ratios, discover that the length of the circumference of any circle is somewhere between 3.0 and 3.3 times the diameter.

Prompt: In class, we found that for any circle with circumference *C* and diameter *d, C/d* is equal to a constant. Write a paragraph that explains experiments that you and your classmates conducted, which led to the discovery of this relationship. Relate your reasoning that led to your conclusions.

Rubric:

	Accomplished: 2	Developing: 1	Below Expectation: 0
Mathematical content	The paragraph not only references measuring circumferences and diameters but also highlights how students discovered the relationship between them.	The paragraph references measuring circumferences and diameters.	The paragraph does not describe any references to circumferences, diameters, and the relationship between them.
Mathematical processes	The paragraph clearly indicates the use of inductive reasoning after examining the values of *C/d* from many examples (10 or more).	The paragraph includes attempts to draw a conclusion about the relationship between *C* and *d*, by examining only 1 or 2 specific cases.	No mention of measurements and reasoning is given. The write-up simply assumes the formula, without any reference to class activities.
Accuracy and relevance	The paragraph is well written, with no errors. The paragraph focuses entirely on the discovery of the relationship between *C* and *d* and contains no erroneous information.	The paragraph is well written, with only one or two minor errors and/or problems in language or presentation. There are no mathematical inaccuracies in the paragraph.	The paragraph contains several errors (i.e., pi is not constant; wrong relationship between *C* and *d*). The paragraph includes a lot of irrelevant details.

understand the difference between area and circumference.

To design an assessment item at the construct a concept learning level, give students prompts that require sorting or categorizing. By sorting examples into categories, students display how they have built a concept in their minds. See the example in **figure 1**.

Prompts like this offer students an opportunity to make connections. For example, with item e, students gain exposure to a connection between probability computations and area models. They also associate attributes of circles with their everyday real-world experience. One critique of this item is that it relies on cultural knowledge about hat sizes and basketball. If the students are unfamiliar with those topics, replace the examples with more personally relevant ideas: Perhaps students know more about trampolines, clock faces, the rings of Saturn, or hula hoops. Teachers customize tests for their own students' individual backgrounds because considering cultural experience is relevant.

Here we have merely provided a quick scoring guide. We recommend that the teacher follow up by conducting a discussion that addresses how or why area or circumference applies in each situation.

Other approaches for testing the construct a concept learning level include asking students to describe concept attributes and create examples or nonexamples of their own. Such activities point out students' understanding of a mathematical concept, which is an object in mathematics worthy of careful consideration and definition, without asking them to state the definition from memory (simple knowledge).

Discover a Relationship

When students have used inductive reasoning to make mathematical discoveries, they may have had a personal and unique experience. Such a scenario can be difficult to test. However, we can look for evidence that students made those discoveries by requiring them to report narratives of the experience of discovery. See **figure 2**, in which students report on what they learned by engaging in a class activity. To avoid an emphasis

Fig. 3 These two examples in (a) and (b) explore the simple knowledge level.

Objective: Accurately state the formulas for the area and circumference of a circle in terms of radius.

Prompt: State the formulas for the area and circumference of a circle.

Scoring guide:
+1 for the response $A = \pi r^2$ or equivalent
+1 for the response $C = 2\pi r$ or equivalent.

(a)

Objective: Name a method used by mathematicians for approximating the value of π.

Prompt: Ancient mathematicians tried to compute π using a specific method. Archimedes used this method in ancient Greece.
In China, Liu Hui used it to calculate π as 3.14159. What was the method these individuals devised?

Scoring guide:
+2 response communicates that the method involved inscribing polygons with a larger and larger number of sides to approximate a circle.
+1 response demonstrates some familiarity with the method by mentioning polygons but fails to communicate the limiting argument.
+ 0 no answer, or the described method does not mention polygons.

(b)

on simply recalling facts, remind students of the relationship that was found. Such a prompt sends the message to students that thoughtfully doing the discovery activity is valued, not just knowing the resulting relationship.

Simple Knowledge

For this level's assessment, think stimulus and response and ask students to respond based on what they remember. **Figure 3a** and **3b** explore the simple knowledge level. Although **figure 3a** is explicitly solicited in the standard of focus, **figure 3b** implies that this unit also includes some history of mathematics that is valued by the teacher. This history may be a motivator for students as they encounter more geometry of circles in future grades or as part of the derivation sought in the second part of the standard. **Figure 3b** can easily be modified to become a comprehension and communication level item, by prompting students to explain the method instead of merely recalling or naming it.

Comprehension and Communication

Assessment at this level requires a prompt that induces students to formulate explanations involving literal and/or interpretive understanding of a technical mathematical expression or a mathematical message. Such items are relevant for assessing students' understanding of mathematics and fluency with their use of the relevant mathematical language. It is important that students should not be asked to merely repeat memorized information (derivations, relationships, or definitions). The prompt instead needs to be novel to the students. Otherwise, they may just try to recall an answer (simple knowledge) instead of focusing their attention on explaining their understanding. **Figure 4** shows an assessment item at the comprehension and communication learning level. Note that the second half of standard 7.G.4 directly calls for students to achieve this kind of learning and explain mathematical arguments.

Algorithmic Skill

To demonstrate algorithmic skill learning, students will be required to recall a sequence of steps and execute the specified mathematical procedure. When designing the prompt, emphasize the process, not the outcome. One might even state the answer up front and ask students to show the computational steps that led to that result, thereby providing an opportunity for students to check their work as they practice the procedure (as they frequently do with answers in the back of the book). Stating an answer up front is not required.

Figure 5a, an algorithmic skill level prompt, may appear in a textbook as an "application" because there is a context and it is phrased as a word problem. A more typical algorithmic skill level item might consist of a drawing of a circle with the radius or diameter length labelled. However, in **figure 5a**, students are explicitly told what to compute and to work with given information only. Hence, they exercise no further cognitive

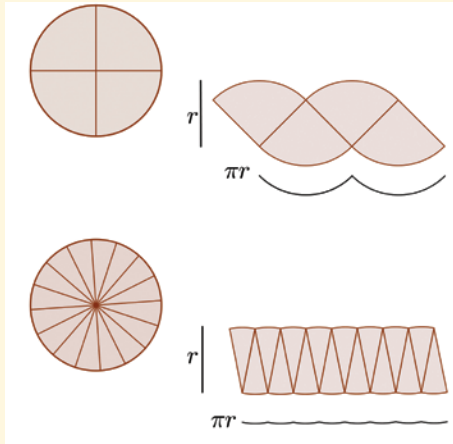
If the students are unfamiliar with the chosen topics, replace the examples with more culturally relevant ideas; perhaps they know more about trampolines, clock faces, the rings of Saturn, or hula hoops.

Fig. 4 An objective and a rubric explore the comprehension and communication learning level.

Objective: Explain a method for finding the area of a circle with a given circumference and articulate the mathematical flow of reasoning in the method.

Prompt: In class we developed a formula for the area of a circle with radius r and circumference C by imagining cutting the circle into equal segments and rearranging the pieces. We decided based on our diagrams (at right) that the area of circle $A = (1/2)Cr$.

Write a letter to convince a new student that $A = (1/2)Cr$. Also explain why our formula is consistent with another formula for the area of a circle ($A = \pi r^2$).



Rubric:

	Accomplished: 2	Developing: 1	Below Expectation: 0
Main idea	The letter clearly references the limiting process, taking more and more wedges and rearranging them. Student argues that the rearranged shape will look more and more like a parallelogram.	The letter alludes to the limiting process, but the ideas are incomplete and given without argument.	The letter contains only figures given in the prompt and includes no allusions to the limiting argument.
Accuracy	Student correctly identifies both the base and height of the parallelogram in relation to the circle's circumference and radius. This information was used correctly to reach the expression for area.	Student identifies the relationship between the sides of the parallelogram and circle, but the derivation is incomplete or incorrect.	Student attempts to identify the necessary sides, but the conclusions are faulty.
Communication	The paragraph is well written with no errors. The paragraph focuses entirely on finding area of a circle and contains no erroneous information.	The paragraph is well written with only one or two minor errors or problems in language or presentation.	The paragraph contains several problems in language and presentation. The paragraph includes a lot of irrelevant details.
Novel mathematics	The letter explains that $\pi r = (1/2)C$ so that the formulas are consistent.	The student attempts the explanation, and notes that $2\pi r = C$, but fails to complete it.	The student does not attempt the explanation.

work than algorithmic skill to plug the numbers into the appropriate procedure to respond to this prompt.

Another algorithmic skill relevant for this standard is the practice of drawing circles of specified dimensions using a compass and a ruler. Developing these skills provides a good way for students to practice making computations of radius, diameter, area, and circumference, and to gain comfort using tools important for further geometric exploration in subsequent grades. Note that this prompt also reverses the typical given (radius) and find (area), requiring students to more thoughtfully apply equation-solving techniques they learned previously. (See **fig. 5b**.)

Application

Students show achievement of application level learning if they demonstrate proficiency deciding how to solve problems. They show that they can decide if a particular situation is an example of a broader mathematical principle or not, exhibiting deductive reasoning. Hence,

Fig. 5 Two algorithmic skill level prompts in (a) and (b) are explored next.

Objective: Set up linear equations, substitute values, and solve by direct computation or by applying inverse operations.

Prompt: A circular banquet table has a diameter of 6 feet.
a. Find the area of the table.
b. Determine how many people can fit around the table if each person needs at least 2 feet of the circumference to sit comfortably.

Scoring guide:
a. Up to 6 total points for employing a correct procedure for calculating the area, showing work, and reporting the solution:
 +1 for determining the radius
 +1 for squaring the radius
 +1 for multiplying the square of the radius by π
 +1 for using a reasonable estimate of π (3.14, or 22/7)
 +1 for no errors in the computation
 +1 for including the units of ft.^{2} in the area

b. Up to 3 total points for determining the circumference of the table using a valid procedure:
 +1 for multiplying the diameter by π (or for multiplying radius by 2 and by π)
 +1 for using a reasonable estimate of π (3.14, or 22/7)
 +1 for no errors in the computation

c. 3 additional points for solving the problem:
 +1 for dividing the circumference by 2 ft./person
 +1 for truncating the solution to the nearest whole person (and answering with units of people)
 +1 for no errors in the computation

(a)

Objective: Use a compass and ruler to construct circles of a specified size.

Prompt: Construct a circle with an area of 85 cm^2.

Scoring guide:
+2 for determining the correct radius with a valid procedure (5.2 cm)
+1 for fixing the radius on the compass to the found radius using the ruler
+1 for placing the compass point properly at a fixed point and sketching the circle

(b)

unlike the algorithmic skill prompts, assessment items at this level should be crafted to avoid clue words that tell students what they need to do. In these problems, students may encounter extraneous data or they may need to find missing data to complete the problem. Some applications incorporate knowledge about nonmathematical topics.

Figure 6 is an extended performance task rather than an item that can be used easily on a test because of the time and tools required to answer it. We would use it as a project type of assessment due on the unit test day. The item induces students to demonstrate their achievement of the first part of the standard, "Know the formulas for the area and circumference of a circle and use them to solve problems," at a high level of cognitive activity, rather than simple recall. The results of this assessment would be nice to display in the classroom.

Assessment design at the application level can also be done using shorter-format problems. One might mix examples and nonexamples using a prompt similar to **figure 1**, asking students to recognize and perhaps set up equations for situations requiring area or circumference calculations. Including circle-related prompts that do not require the area or circumference formulas make the conclusions less obvious and should be used to ensure that deductive reasoning is employed.

Creative Thinking

Some teachers integrate art into their teaching of mathematics, which is commendable. However, requiring students to employ technical artistic skills should not be confused with creativity in mathematics. For example, at some point in this unit, it would be great to lead students through a procedure for creating a six-petal flower with a compass (an algorithmic skill). Even if the teacher encourages artistic originality by asking students to make an elaborate picture or design including the flower, assessing their skill of replicating the design should not be considered mathematical creativity.

To design items that test your students at the creative thinking level, we recommend using *synectics*, the juxtaposition of seemingly unrelated ideas. It helps to be a little spontaneous and playful in your interactions with students to spark curiosity and creativity. Open-ended application prompts also may be used to assess creativity. The marking should ensure that different and unusual mathematical thinking

Fig. 6 An application task is described.

Objective: Identify situations that require area or circumference computations, and decide what information is needed to carry them out.

Prompt: Curtis is hosting a banquet for approximately 50 people and is going to have circular tables at the banquet. Create a blueprint design of the room layout that includes the size of the tables needed and the size of the room. Make your drawing to scale and include a mathematical justification for the dimensions you choose.

Write two paragraphs explaining why your design provides adequate space for the banquet attendees and your considerations in the design process.

Extension problem: How much material would Curtis need to purchase to make tablecloths for each table?

Rubric:

	Accomplished: 2	Developing: 1	Below Expectation: 0
Thoughtful design	Student's explanation indicates thoughtful consideration of all the restraints (e.g., size of tables and room, number of people, walking space, quantity of cloth, width the cloth is available in, and so on).	Student's explanation indicates consideration of majority of the items (at most two are missing).	Student's response lacks thoughtfulness and several items are missing.
Representation	The blueprint is drawn with a consistent and appropriate scale and using appropriate tools.	The blueprint has minor inconsistencies, or the drawing is inaccurate.	The blueprint appears to be drawn freehand, and no apparent attention to scale is present.
Mathematical accuracy	The text includes clear writing with expressions and equations appropriately stated and incorporated into the sentences. All computations are accurate.	There are minor errors in computations, or the computations seem somewhat disconnected from the written portion of the assignment.	There are major computational errors, and the work is not incorporated into the written portion of the assignment.

receives higher scores. Note that the scoring guide in **figure 7** gives higher scores for originality.

Typically in schools, students learn to increasingly exhibit conventional and convergent thinking and are not rewarded for divergent thinking. An item like **figure 7** is rarely found on tests, and students may even feel intimidated by such prompts. The older the students, the more intimidation they might feel, as they have grown accustomed to certain types of thinking in schools. It might be more natural to use this example as a warm-up exercise and follow up

Fig. 7 This example explores a creative thinking prompt, not often found in math class, but one that may require spontaneous and possible playful interactions.

Objective: Express novel ideas about circles, irrational numbers, and geometric symmetry.

Prompt: How is pi like a snowflake? Write a paragraph with illustrations to explain.

Scoring guide:
Sort the responses into categories based on the ideas expressed. Make one category for nonmathematical thoughts. The nonmathematical responses each get 1 point for the attempt. For the remaining responses, if n students have similar results, each gets $1 + 2\pi/n$ points. Hence, a maximum score of $1 + 2\pi$ points are possible.

with a class discussion about some of the mathematical ideas expressed in the responses. Perhaps some students have thought about the uniqueness of individual snowflakes and the unique patterns that can be found at random places in the never-ending decimal representation of pi. Others may have focused on the roundness of a flake of snow and noted that pi emerges in the geometry of round things. Maybe only one student describes the geometric similarity of all circles and compares that with the self-similarity exhibited in snowflakes, also noting that circles have infinitely many lines of symmetry and the decimal representation of pi continues indefinitely. Resist the temptation to overexplain students' most novel ideas and take over their thinking. After students get more accustomed to creativity in mathematics, you can start including creative thinking prompts on summative tests.

CANGELOSI'S LEARNING LEVELS

We have shared a structure of categorizing mathematical cognition that we find very useful for assessment item design, Cangelosi's learning levels, and we have demonstrated various design principles in the examples for the chosen standard. We believe that by employing these strategies in test design, students will actually learn from taking the test. We have shared assessment items for each level of cognition in this structure merely to illustrate them and also to share some strategies we use to instruct and assess seventh-grade level content particular to circles. However, these methods of designing assessments around the levels of cognition expected of students can be applied to any topic in our mathematics courses.

We further suggest using this cognitive structure to evaluate test items or exercises in textbooks. Once you decide, for example, that comprehension and communication level cognition is of particular importance for a given standard, design or select items that address that learning level. Imagine that the district pops a "common formative assessment" on your doorstep and asks you to use it in place of your regular teaching activities. Look over the items. Are they primarily at the simple knowledge level or algorithmic skill level, perpetuating the notion that mathematics is all about memorizing a slew of facts and procedures? Or do the district-provided assessments help assess students' progress on the high levels of cognition that CCSSM demands?

The reason for concern about tests is also clear: High-stakes standardized tests have been misused to evaluate teachers and schools, encouraging poor instruction focused on test preparation and superficially "covering" topics, thus consuming large amounts of time in the school year. Associations with rampant cheating have also occurred (Gojak 2013; Picciotto 2014). Teachers need to be outspoken about poor assessment practice that is drowning out high-quality instruction. The strategies in this article can be helpful for improving assessment design and for evaluating assessments that other test writers provide.

We feel encouraged by the fact that a wider range of assessment types are called for by CCSSM and are pleased that teachers are becoming more engaged in test writing within schools and districts (Gojak 2013). Teacher-designed assessments are valuable and relevant because they cater to the students as individuals and

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are part of a central and meaningful human relationship in education: that between the student and her teacher. This view of assessment is in line with the results of a recent survey in which students, teachers, and administrators declare the classroom tests to be far more valuable than end-of-year testing (NEA 2014).

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Any thoughts on this article? Send an email to ***mtms@nctm.org.***—*Ed.*

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