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Opportunities to Develop Place Value through Student Dialogue

The problem 11 + 9 was written on the board at the front of the room. Eleven first graders and nine second graders sat on the carpet, their facial expressions intent as they thought about solutions. I had asked them to try to think of strategies they could use that did not involve counting on their fingers. They did not use paper and pencil because this was what we call "Mental Math" time at our school.

As the students finished their mental calculations, they began to grin and raise their hands. Some began to whisper. I quietly reminded them, "Please wait for the people still working. Be patient. Put your thumb up in your lap to show you are finished, so your arm won't get tired and fall off." The students giggled, but they quieted down until everyone had their thumbs up. Then I asked for answers.

Sharing Computational Strategies

I recorded the two answers the students gave me on the board. Some students thought the answer was 20, others 29. For each answer, a spirited chorus of "Agree" or "Disagree" sounded. I called on Breanna to explain her strategy for solving the problem. She confidently told the class that she started at eleven and counted on nine more to get twenty. I wrote Breanna's name and "Counted on nine more from eleven to twenty" on the board. I always label the counting strategies that students use, such as counting on, counting all, or skip counting, to familiarize students with the vocabulary that describes their actions.

Next, I asked if anyone who got the answer 29 was willing to share his or her thinking. A couple of students raised their hands. I called on Alejandro. He said, "I did ten plus ten equals twenty, plus nine more equals twenty-nine."

Immediately, there was a chorus of disagrees and agrees. Someone called out, "Where did you get your tens?" My students are very active during mathematics. I encourage them to share their ideas, their agreement or disagreement with others' ideas, and their questions. The dialogue that takes place each day during mathematics drives their learning. Over time, they become confident that their ideas have value and that they can learn from their mistakes.

In response to the question about where he got the tens, Alejandro said, "From the ones in the eleven."

I pointed to the one in the tens place and clarified: "So this one is a ten?" He nodded. Pointing to the one in the ones place, I asked, "And this one is a ten?" He nodded again. I turned to the class. "Do you agree with Alejandro?" The students started to nod their heads. "Are you sure?" I asked. Some students said yes and others said no; I saw a lot of puzzled faces.

I pointed again to the one in the tens place and asked, "Is this a one or a ten?" Most of the students said it was a ten. A few of the first graders and one of the struggling second graders, who did not understand place value, said that it was not a ten.

Haley called out, "They are both tens. I agree with Alejandro."

"There are no tens," said Emily, a shy first grader. "They are both ones. There are no zeros."

Aleigha said, "We can add a zero to any one and make a ten."

"You can't just add a zero when you want to,"


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Taylor argued. “It has to be there.”

“There has to be a ten ’cause if you count backwards from eleven, you say ten,” Marcos said. “So it has ten in it.”

Lien said, “Yes, if you take one away from eleven it’s ten.”

Joshua said, “They can’t both be tens. Then it would be twenty. See: Ten plus ten is twenty. That’s too big. The first one is a ten. The other one is a regular one. Ten plus one is eleven.”

Haley came back into the conversation. “Now I agree with Joshua and Lien. It can’t be two tens. I think eleven is ten plus one too.”

Most of the students started to nod their heads to show that they agreed with Haley. I asked Lien to share her strategy. She told the class, “I took the regular one from the eleven and put it with the nine and that’s ten. Then I did ten plus ten is twenty. The answer is twenty.”

I turned to Maria, a second grader with limited English skills. I pointed to the equations I had recorded while Lien was talking and asked, “Is one plus nine ten?” She nodded. I asked, “Is ten plus ten twenty?” She nodded again. “So you agree?”

Maria smiled and softly said, “Agree.” When I asked if Maria had done the same thing as Lien did, she smiled again. Maria was developing place value and had good strategies for solving problems. However, she did not know the English words to explain her thinking to the class. I often told her to write her strategy on the board for her sharing and then asked the other students to try to explain what they thought she did. I also called on her often to give yes or no responses so that she felt she was part of the conversation.

Because the bell rang for recess, we stopped our conversation about the number eleven for the day. I knew we would return to it again because I had discovered that the number eleven can be very confusing to students who are developing their understanding of place value.

The second graders in the group had struggled mathematically as first graders. Most of them did not conserve number when they left kindergarten. They did not become fluent with sums to twenty as first graders and had no understanding of place value at the beginning of second grade.

The Importance of Repeated Discourse

A couple of weeks after our initial conversation about eleven, I asked my students to solve 11 + 11

+ 11. The strategies that I recorded on the board demonstrated that many of them were progressing in their thinking. The students gave four answers: 33, 60, 32, and 31. The answers of 32 and 31 came from miscounts by first graders like Emily. Alejandro stuck with his idea that all the ones were tens, as did a couple of other students, and got 60. Both Haley and Aleigha, however, showed significant progress by using the strategy 10 + 10 + 10 = 30, 1 + 1 + 1 = 3, and 30 + 3 = 33. They both were able to articulate where they got their numbers. Joshua asked me to rewrite the problem as 10 + 1 + 10 + 1 + 10 + 1 = 33 before using the same strategy as Haley and Aleigha. After sharing their strategies, the students engaged in another spirited debate about the meaning of the digits in eleven.

The numbers twenty-one, thirty-one, and so on pose the same problems for students who overgeneralize the idea that if there is a one in a two-digit number, it must be a ten. One year, I had a student named Mai who told the class that the one in thirty-one is a ten. Timmy told her, "No way! It's just a one and a thirty." Seven students agreed with Timmy, but twelve agreed with Mai. Sara said it had to be thirty and one because she knew that thirty plus one equals thirty-one and ten plus three is only thirteen. Some students said a student teacher had told them that when one digit in a two-digit number is a one, the one is always a ten. I have no idea what the student teacher actually told them. A lot of dialogue and many experiences are necessary to clear up such misconceptions.

I wrote the number twenty-one on the board and asked Mai to count out twenty-one cubes. I asked her, "Could you make some piles of ten and match your cubes to the numbers on the board?"

Mai carefully counted out twenty-one cubes. She made two stacks of ten and had a single loose cube. Initially, she took one of the groups of ten and put it by the numeral one. She put the other group by the two and said, "This one is left over," referring to the single remaining cube. I did not say anything. Then Mai moved both groups of ten by the one, but that did not satisfy her either. Suddenly, she looked at me and said, "I know." She moved both groups of ten by the two and the single cube by the one. She smiled and said, "I get it."

Indeed, Mai did understand, and she continued to understand from then on. Other students may not solidify their understanding so quickly. Developing place value can take a long time. Students who seem able to articulate clearly their understanding of our number system one moment may seem confused the next. At our school, we say these students are "transitional" in their understanding of place value. We make sure that we hold many conversations during mathematics time that will help these students solidify their understanding. Even though we may do many other mathematics activities, conversations about tens recur throughout the year. We do not simply do one addition unit or place-value unit and move on.

Laying a Foundation for Conversations about Place Value

Long before we discussed the mysteries of the number eleven, my students were working on building their understanding of place value. We started our work on place value at the beginning of the year. The students spent a lot of time playing games that helped them become fluent with all the sums, first to ten and later to twenty. Playing board games was one popular activity. (A file folder with a path marked by colorful stickers makes an easily stored game board; Unifix cubes are game markers.) The children rolled a die and then moved the number of spaces that represented the difference between the number on the die and either ten or twenty. For example, if students roll a three, they move seven spaces. Another way to play is to use two or more dice and roll them, find the sum, and move that number of spaces. Teachers can make the game more complicated by using number cubes instead of dot dice (children may become too reliant on counting the dots on the dice) or polyhedra dice with higher numbers.

During our Mental Math time, we spent a lot of time working with multiple addends that contained sums of ten. One such problem, 5 + 6 + 4 + 5, produced a variety of strategies. Guillermo recognized the 5 + 5 as ten, then counted on six more, then four more to get twenty. Emily counted all four numbers. Several students used the strategy of 5 + 5 = 10, 6 + 4 = 10, 10 + 10 = 20. Aleigha shared her strategy of 5 + 6 = 11, 5 + 4 = 9; she counted to find both sums. Then she was faced with 11 + 9 and counted on from eleven. When I asked her if this had been her original strategy, Aleigha admitted that she had originally counted as Emily had, but she liked this method better. Students often create new strategies during sharing time so they have a chance to share.

Rafael confused the class with his strategy. He started with 6 + 6 = 12. Students asked, "Where did the other six come from?" and noted, "There aren't two sixes." Rafael explained that he got his second six by adding the five and a one taken from the four. He finished by adding the second five and the remaining three from the four to get eight. Then he put eight and twelve together by counting on to twenty. Even though the strategies were not very efficient, I was pleased that the students were beginning to play with numbers more often and use combinations that they knew well.

When I gave the students a problem such as 12 + 4, many of them were comfortable adding two to twelve to get fourteen and then adding the remaining two. Just to put a new idea in their heads, I offered an alternative: "One of my students once said that you could put the two from the twelve with the four, like this [I wrote 2 + 4 = 6], and get six. Then they told me to add the ten from the twelve to the six and you get sixteen. Do you think this person was right or wrong?" I got very little response from the students and left the discussion unfinished.

A few days later, however, Dylan used a ten to solve 12 + 6 + 2. He told the class, "2 + 2 = 4, 6 + 4 = 10, ten plus the ten from the twelve is twenty."

I asked the class if there were tens in twenty. Mark said, "Yes, the two is like ten plus ten is twenty; that's two tens." After my alternative suggestion, the students' talk about tens came naturally. I always made sure to question the meaning of what the students were saying and to ask more than one student to put things in their own words. "Kid language" seemed to make more sense to the students than did adult explanations.

The strategies that I recorded on the board demonstrated that many students were progressing in their thinking

As we progressed in our work, I gradually increased the numbers. As the numbers moved into the double digits, some students continued to use their counting strategies, but others began to talk more confidently about place value. For the problem 12 + 12, Emily actually experimented by using 2 + 2 = 4, 1 + 1 = 2, and 4 + 2 = 6. Most of the students disagreed with her, so I asked her to pay close attention to their comments. Aleigha said, "No, those ones are really tens. So it should be 10 + 10 = 20, 20 + 4 = 24."

Chris commented, "Yeah, tens because you've passed it. If you count to twelve, you have to pass ten." Meanwhile, Emily tried to count on her fingers and reluctantly agreed that 24 was the correct answer.

Chris became a leader in the tens talk. His explanations and those of other strong students influenced the development of the group. Other students started to think about tens and agreed that there were tens in the greater numbers. These children did not use tens to solve problems, however. Each student eventually reached his or her own breakthrough point.

One day, T'ni described her strategy for the problem 10 + 13 + 7: "I got this from Chris. I did 10 + 10 = 20, 7 + 3 = 10, 10 + 20 = 30."

At about this time in the group's development, I introduced the problems 11 + 9 and 11 + 11 + 11. Once most of the students had resolved their conflicts about the values of the digits in the number eleven, they started to make great progress.

Using Multiple Problems to Differentiate Instruction

Some of the students were having a more difficult time with place value than the others were. I began to use two problems, one easier than the other, during Mental Math, and I let the students choose to do one or both problems. For example, one day I put up the problems 26 + 13 + 4 and 44 + 26 + 36. I called on Emily first, because she had been developing a tendency to simply repeat someone else's strategy as her own. I wanted to encourage her to think for herself. I let her know in advance that I would call on her first. She was ready. I recorded her strategy for 26 + 13 + 4 as she shared it: 20 + 4 = 24, 24 + 6 = 30, 30 + 10 = 40, 40 + 3 = 43. I was thrilled that she had come a long way in her understanding of place value and her ability to effectively solve problems.

T'ni, another student who had been struggling, chose the same problem and showed how much she was progressing when she shared her strategy: 20 + 10 = 30, 30 + 6 = 36, 36 + 4 = 40, 40 + 3 = 43. Both girls were confident in their explanations and proud that their classmates agreed with their thinking. I was impressed by the variety of strategies that the group was producing.

Students had a range of successful strategies for the second problem of 44 + 26 + 36 as well. Aleigha told the class, "I know 40 + 20 + 30 = 90. 90 + 6 = 96, 96 + 6 = 102, and 102 + 4 = 106."

I probed for a more detailed explanation of a couple of Aleigha's steps. "How did you know that 40 + 20 + 30 = 90?"

She replied, "I know 40 + 20 is 60, plus 30 more is 90."

Then I asked, "What did you do when you got here?" I pointed to the step 96 + 6 = 102.

She said, "I counted on from 96 to 102."

I asked the students if they could think of a way to add 96 and 6 without counting on. After a moment of reflection, Joshua raised his hand. "Take the six from the ninety-six and put it with the other six. That's twelve. Take the ten from the twelve and put it with the ninety. That's one hundred. Now put the two from the twelve with the one hundred. That's one hundred two."

Conclusion

Principles and Standards for School Mathematics (NCTM 2000) recommends that all K–12 students should be able to "organize and consolidate their mathematical thinking through communication" and "analyze and evaluate the mathematical thinking and strategies of others" (p. 60). Our Mental Math sessions are very stimulating because the students focus their explanations on how they can decompose numbers to facilitate finding solutions. They explore with numbers and become very flexible in their thinking. They build strong mathematical vocabulary to describe their thinking.

Too often, teachers underestimate the importance of these recommendations. Perhaps they realize that they are not easily achieved. The first conversations that children have are not immediately productive or clear. Building a classroom environment in which children have the confidence to share their developing ideas about our number system takes time. The payoff, however, comes when a child such as Alejandro, who once thought that both the ones in eleven were tens, shares a strategy for the problem 34 + 48 + 21: 20 + 30 = 50, 50 + 40 = 90, 4 + 1 = 5, 5 + 8 = 13, 90 + 10 = 100, 100 + 3 = 103. Alejandro confidently communicates his competency with number and operations in a manner that *Principles and Standards* envisions.

Reference

National Council of Teachers of Mathematics (NCTM). *Principles and Standards for School Mathematics.* Reston, Va.: NCTM, 2000. ▲