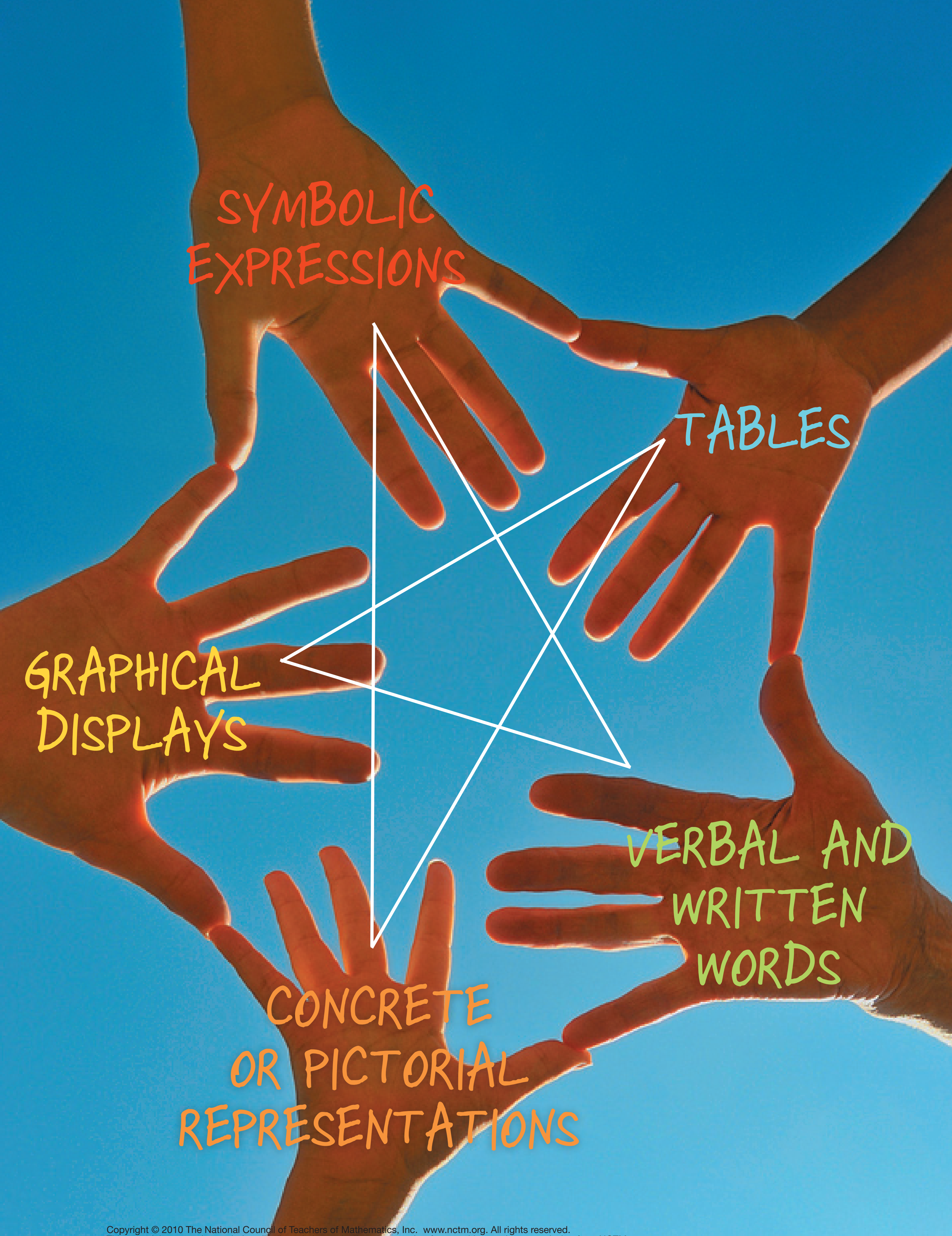

SYMBOLIC EXPRESSIONS
TABLES
GRAPHICAL DISPLAYS
VERBAL AND WRITTEN WORDS
CONCRETE OR PICTORIAL REPRESENTATIONS

Star students make connections

By Anne Marie Marshall, Alison Castro Superfine, and Reality S. Canty

Discover strategies to engage young math students in competently using multiple representations.

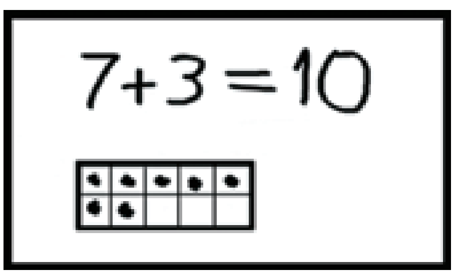
Ms. Beyer's first graders have been working for several weeks on solving problems that encourage the use of such multiple representations as ten frames and number lines. The class is using Math Trailblazers (TIMS Project 2008), a National Science Foundation–supported elementary school math curriculum developed to reflect recent reform efforts in mathematics education (NCTM 2000, 2006).

During one lesson, students gather around Beyer at the front of the room. She asks them to focus on the whiteboard, where she has written a number sentence as well as the corresponding ten frame (see **fig. 1**). Beyer engages students in a discussion that focuses on connecting the two different representations: "How does this number sentence right here, how does that fit in with our number, with our ten frame up here? How can we look at this ten frame right here and see this number sentence? Is there a connection?"

Translating and moving flexibly between representations is a key aspect of students' mathematical understanding. Presenting students with opportunities to make connections between multiple representations makes math meaningful and can help students see the subject as a web of connected ideas as opposed to a collection of arbitrary, disconnected rules and procedures. Creating a learning environment in which students are encouraged to make connections among different representations has been a central feature of mathematics reform efforts for at least the past decade. *Principles and Standards for School Mathematics*, in particular, suggests that instructional programs should provide opportunities for students to use representations to communicate mathematical ideas; select, apply, and translate among mathematical representations to solve problems; and use representations to model and interpret mathematical phenomena (NCTM 2000). Such learning opportunities may support the development of what we posit is students' representational

FIGURE 1

Beyer engaged students in a discussion focused on connecting the corresponding ten frame to the number sentence 7 + 3 = 10.



competence (Novick 2004), which is knowing how and when to use particular mathematical representations. This suggests that students should select representations as well as translate among them with an understanding of how and when representations are most useful. An important aspect of developing a robust understanding of mathematics means not only knowing how to use a representation during problem-solving situations but also being able to make connections between representations.

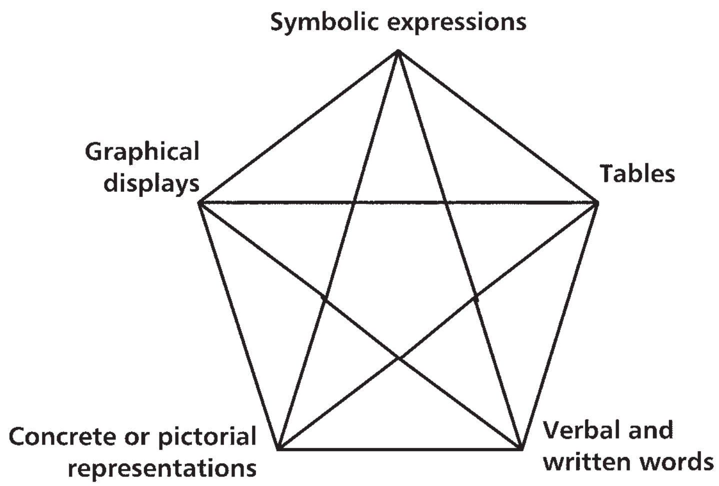
Three specific instructional strategies create opportunities that may support students' development of representational competence:

- **Engaging** in dialogue about the explicit connections between representations
- **Alternating** directionality of the connections made among representations
- **Encouraging** purposeful selection of representations

The identification of these strategies stems from the authors' analysis of Beyer's first-grade classroom. Beyer was selected as the focus of this case study on the basis of findings from a larger study in which students in her classroom were more accurate and flexible problem solvers than students in other classrooms observed in the study.

FIGURE 2

Many educators argue that mathematical representations—each characterized by its own rules of use and relationships—are integral to teaching for understanding (figure adapted from Friel, Rachlin, and Doyle 2001).



Mathematical representations in elementary school classrooms

The use of multiple representations in math classrooms has been the subject of considerable discussion among the mathematics education community in recent years. *Mathematical representations* are words, pictures, drawings, diagrams, graphical displays, and symbolic expressions (see **fig. 2**). Each category is characterized by its own rules of use and relationships. Many educators and researchers argue that multiple representations are important for fostering understanding (e.g., Janvier 1987; Kilpatrick, Swafford, and Findell 2001; NCTM 2000) and should be an integral part of instruction. Teachers should help students understand that representations are tools to model and interpret mathematical phenomena, represent aspects of situations in mathematical terms, and emphasize the importance of representing mathematics ideas in a variety of ways (NCTM 2000). As some Standards-based elementary school curricula creators increase their efforts to design lessons that emphasize the use of and connections among representations, teachers must employ instructional strategies that help students learn how and when to make these connections to support their mathematical thinking.

Much of the research related to mathematical representations has focused on translation, the processes involved in moving from one representation to another (Janvier 1987; Knuth 2000; Lesh, Post, and Behr 1987). Translating from one representation to another, however, is a complex skill that requires knowledge of the representations involved (Castro Superfine, Canty, and Marshall 2009). Specifically, translation requires "knowledge about the relations between different types of diagrams that enable problem solvers to translate information from one representation to another, such that the new representation preserves the structural information conveyed by the original representation" (Novick 2004, p. 210). For example, in the aforementioned classroom episode, translation between the number sentence and the ten frame requires knowledge of the meaning of

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First graders in Beyer's classroom were more accurate, flexible problems solvers than students in other classrooms because they had practiced translating between representations.

number sentences and ways of using ten frames in order that 7 + 3 = 10 is represented according to the conventions of the ten frame. This aspect of representational competence is necessary for students to deepen their understanding of mathematics as they make connections between concepts (NCTM 2000). The knowledge constructed while translating may serve as the basis of students' mathematical understanding (Lesh, Landau, and Hamilton 1983).

Three representations

The case study presented here is part of a larger study (Castro Superfine, Canty, and Marshall 2009) that examined elementary school students' whole-number reasoning with representations. To provide a picture of how teachers used representations in different classrooms, the larger study analyzed multiple data sources, including teacher surveys, teacher interviews, classroom observations, and student interviews. Teachers were surveyed about their use of the curriculum and were interviewed about their views of representations in their instruction and how they use them, particularly to support students' learning.

Thirteen students were interviewed three times between February and June of the 2006–2007 school year. The interviews were designed to yield information about students' whole-number reasoning with representations. First graders were selected from classes in two public elementary schools in a large Midwestern metropolitan area. They were using the field test materials of the next edition of the Math Trailblazers elementary school curriculum. The interviews lasted approximately 20–30 minutes and focused on one of three different mathematical representations from the curriculum:

- **Box diagrams** (interview 1)
- **Ten frames** (interview 2)
- **Number lines** (interview 3)

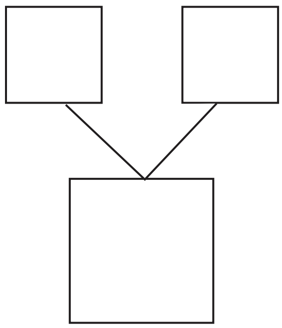
Box diagrams

Also referred to as *part-part-whole diagrams*, box diagrams (see **fig. 3a**) are visual models showing that two partitions of the whole can be

FIGURE 3

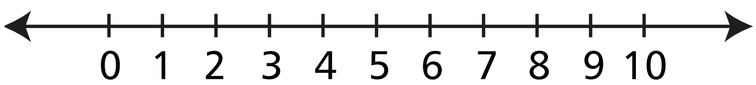
Thirteen first-grade Midwesterners were interviewed for 20–30 minutes about one of three different mathematical representations from their curriculum:

(a) Box diagrams, also known as part-part-whole diagrams (interview 1),



(b) Ten frames (interview 2), or

(c) Number lines (interview 3).



represented by various number sentences. A box diagram is intended to illustrate the part-part-whole relationship of the numbers involved and aims to support students as they compose and decompose numbers. Specifically, two parts are inserted into two small boxes, and the whole is inserted into the larger bottom box to allow students to determine the corresponding addition and subtraction statements and record their partitions of a given number.

Ten frames

A ten frame (see **fig. 3b**) is a two-by-five rectangle divided into ten squares. Ten frames foster mental imagery that can help students build understanding of basic number concepts, especially around the important benchmarks of five and ten. They are frequently recommended for primary grade mathematics instruction (Thompson 1990; Thornton 1990). As part of the representation system associated with the ten frame, students are expected to place an object or a symbol in each square to represent numbers less than or equal to ten. Two (or more) ten frames can be used to represent numbers in the teens, twenties, and beyond.

Number lines

In the curriculum materials in Beyer's first-grade classroom, students use number lines (see **fig. 3c**) as tools for counting and computation, particularly addition and subtraction. As part of the representation system associated with number lines, students are often taught to make "hops" or "jumps" on a number line, moving from one number point to another, and then to count the number of hops or jumps in order to indicate a given number.

A case study

Students in Beyer's classroom who participated in the larger study outperformed students in other classrooms on the interview problem situations involving multiple representations. Those findings prompted an in-depth case study analysis of Beyer's enactment of three lessons within the Math Trailblazers curriculum, each of which explicitly promoted the use of representations and translations among them. The enactment of these lessons occurred with the entire class of twenty-six students. The analysis examined Beyer's instructional moves and decisions during a series of lessons aimed at providing experiences for students to make and use connections among the box diagram, the ten frame, and the number line. Researchers carefully reviewed and coded videotaped lessons, paying particular attention to the ways that Beyer engaged students with representations.

Beyer's thoughts about her teaching

An analysis of Beyer's first-grade classroom indicated that, in general, she closely follows the lessons within the curriculum. In particular, she consistently follows suggestions for representation use and raises opportunities for students to translate between them. Asked about using representations with her students, Beyer replied,

Using multiple representations helps the students see that there are different ways to solve a problem. Many children get frustrated because they do not "understand" what to do; but, in reality, they just understand differently. So having more than one way to represent a solution gives other children a way to see the problem differently.

Beyer also recognizes that using such representations as ten frames and number lines does not imply that students will automatically make sense of these representations and the mathematical ideas that they represent. In talking about a subtraction lesson that uses the box diagram, Beyer stated, "I'm not sure if they are making the connection about subtraction or if they are just kind of plugging numbers in."

Throughout Beyer's reflections on the three lessons that the researchers observed, she indicated the importance of using multiple representations in her teaching. Beyer admitted she was concerned that students understand *when* to use representations (e.g., when a problem is difficult, try a different representation), as well as understand *how* to use them for computation (e.g., "just kind of plugging numbers in") in order to enhance conceptual understanding as opposed to rote memorization.

Instructional strategies

The case study analysis revealed that Beyer uses the three instructional strategies previously mentioned (also see **sidebar**, **p. 46**) to prompt her students. The authors' research suggests that using these strategies in math instruction may enhance students' knowledge of the relationships among multiple representations and perhaps promote representational competence.

Engage in dialogue

Students who are able to translate among multiple representations of the same problems or of the same mathematical concepts may have not only flexible tools for solving problems but also "a deeper appreciation of the consistency and beauty of mathematics" (NCTM 1989, p. 146). Connecting mathematical representations and translating between them can occur in multiple ways during instruction. For example, specific structural features in two or more representations can be identified and compared (e.g., identifying the number 8 on the number line and then in a ten frame). Corresponding parts of different representations can also be identified and compared. For example, parts in a box diagram are represented by two small boxes, whereas parts in a ten frame are designated by two different object features (e.g., Xs and Os or markings versus no markings).

In the following classroom exchange, students gather around Beyer and engage in a whole-group discussion about one problem situation in the lesson. She focuses students' attention on a small whiteboard at the front of the room. Two number sentences (7 + 3 = 10 and 10 – 3 = 7) and a corresponding ten frame are posted on the whiteboard, and students discuss the relationship between the two representations, the number sentences and the ten frame. The following transcript highlights Beyer's use of dialogue to illuminate the connections between representations in the lesson:

Beyer: How does this number sentence right here [*pointing to* 7 + 3 = 10 (*see* **fig. 4**)], how does that fit in with our number, with our ten frame up here [*pointing to ten frame*]? How can we look at this ten frame right here and see this number sentence? Is there a connection there?

Lindsay: It's backwards. Seven plus, and the first time it has a take away, and on the bottom, seven plus isn't a take away.

Beyer: OK. So these two problems are different. One is take away, and one is

FIGURE 4

Teachers can use dialogue to explicitly illuminate the connections for students between two number sentences and a corresponding ten frame.

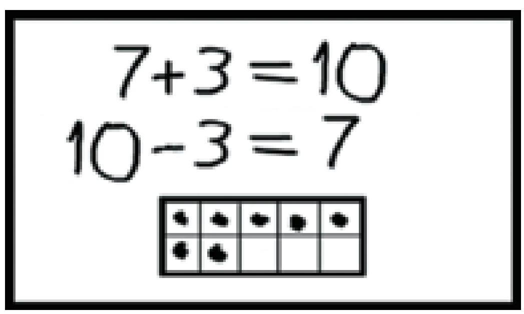
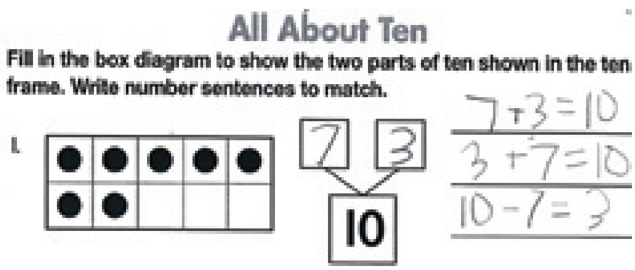


FIGURE 5

During instruction, students focus on specific parts of one representation and think about the correspondence with parts in another. The teacher asks questions that require students to translate between parts. Alternating directionality supports students' thinking with various representations.



addition. OK. But what I'm asking you to look at right now is not this problem but this ten frame right here. Do you see these numbers up here on this ten frame? Chris, do you see the number seven on the ten frame? Where do you see the number seven?

Chris: Right there [*gesturing at the ten frame on the whiteboard*].

Beyer: Right here where the chips are right [*pointing to the ten frame*]? OK. Where do we get the number three? Where can we get the number three from on the ten frame? Look at the ten frame. Do you see three up there anywhere? Julie?

Julie: In the empty boxes.

Beyer: In the empty boxes. There are seven filled boxes and three empty boxes. How many boxes are there altogether? Mark?

Mark: Ten.

Beyer: Ten. So we can use the ten frame to help us do take away and help us do addition.

Beyer makes explicit connections between the corresponding parts of the ten frames and the number sentences. Making explicit connections between these two representations helps preserve the structural information conveyed in the original number sentence, a key feature of representational competence.

Alternate directionality

The directionality of the connections made between the representations and the problem situation is another important feature of representational competence. For example, translating from a number line to a ten frame, and vice versa, promotes the use of different mathematical thought processes. According to Knuth (2000) an important aspect of developing a robust understanding of mathematics means not only knowing how to use a representation during problem-solving situations but also being able to move flexibly between different representations, making connections from one representation to the other, and vice versa.

During instruction, Beyer often asks students to focus on specific parts of one representation and to think about their correspondence with parts in another representation. For instance, during one lesson, Beyer asked students to identify the 3 in the box diagram. Next she asked them to identify where that 3 was represented in the ten frame (see **fig. 5**). She then changed the directionality of the connections by asking students to locate a part represented in a ten frame and then identify the corresponding part in a box diagram.

By alternating directionality, Beyer may be helping her students form an understanding that will enable them to reason about whole numbers with ten frames, box diagrams, and number sentences, and thus promote their flexible thinking about representations.

In this lesson, Beyer asks questions that require students to translate between corresponding parts of the representations, moving first from a ten frame to a box diagram, and then vice versa. Alternating directionality in this way can support students' thinking with various representations—another key feature of representational competence.

Encourage purposeful selection

Different representations often highlight different mathematical concepts or features of a concept. Therefore, it is important for students to understand the purpose for using particular representations (Kilpatrick, Swafford, and Findell 2001). This may be achieved by creating opportunities for students to reflect on the use of representations (NCTM 2000). More specifically, the use of multiple representations in instruc-

First, students brainstormed as a class about strategies and tools. Then, they worked individually on the problem. Finally, they were asked to defend to their classmates their choice of a representation.

tion may help students form an understanding of the relative costs and benefits associated with a representation for a given range of situations (Dufour-Janvier, Berdnarz, and Belanger 1987).

In one lesson, Beyer asked students to solve the following problem:

> One evening, a family had some dinner guests. The father wanted to set the table. He said, "There are four of us in the family, and we have sixteen guests. How many plates do we need?"

After the class had brainstormed multiple strategies and tools for investigating the problem, students were allowed to work individually to solve the problem. As they worked, Beyer circulated among students to observe their problem-solving strategies and to question their thinking. After several minutes, Beyer pulled the class together to discuss their reasoning. In the following exchange, Beyer asks students to purposefully select a representation on the basis of its efficiency in the given problem situation.

Beyer: What I would like for us to do right now is I would like for us to pick one of these strategies, either the ten frame or the cubes [*referring to interlocking or connecting cubes*] or the, um, what else did we do? Drawing a picture or the number line. Those were the four strategies that most people used. You drew a picture, used your number line, used your ten frame, or used the cubes. What I'd like you to do right now is I would like you to think for a second about which of these strategies do you think is going to give us, you know, the quickest answer? Which strategy do you think maybe we should use? The strategy that we think is the most efficient. Which strategy do you think was the most efficient? Julie?

Julie: Number line.

Beyer: Number line. You think the number line was most efficient. Why, Julie?

Julie: Because she just started at four and then counted sixteen.

Beyer: And do you think that she got her answer real quick, or do you think it took her a long time?

Julie: Real quick.

Beyer: Real quick. All right. Raise your hand if you agree with Julie, that you think the number line is the most efficient strategy for that problem. Which one do you think, Andrew?

Andrew: Ten frame.

Beyer: You think the ten frame? Why do you think the ten frame?

Andrew: Because it's more quicker.

Beyer: What? How is it quicker? To make all these Xs and Os, you think, is quick? Do you think that's quicker than making hops on a number line? OK. All right. Lindsay?

Lindsay: I mean, um, ten frame.

Beyer: OK. Why do you think the ten frame is quicker?

Lindsay: Um, because they already got it drawn for you. You don't have to draw, um, like on a picture, um, and you could even do, all you have to do is do, um, Xs and Os and the Xs.

Beyer: What do you have to do once you make those Xs and dots? What do you have to do?

Lindsay: Count them.

Beyer: You have to go back and count them. OK. All right.

Curtis: I actually, um [*inaudible*].

Beyer: Oh, OK. Which one?

Curtis: Ten frame.

Beyer: OK, why do you think the ten frame is the most efficient?

Instructional strategies

Teachers can prompt students' discussions and interactions with representations to enhance their knowledge of the relationships among multiple representations and promote representational competence.

- **Engage** students in discussions that make the connections between representations explicit. Allow them to select from a range of representations when solving problems. Also encourage them to try solving the same problem using multiple representations. Then discuss with them the similarities and differences of the multiple representations within the context of one problem.

  Ask them to show where specific numbers are situated within representations. Ask them to think about how the structure and rules of representations are related. For example, students can discuss how the parts and whole of a number sentence are connected to the small and large boxes of a box diagram. Pose challenges:
  - How is that number represented in the ten frame?
  - Show me the 10 in the ten frame and in the box diagram.
  - Show your partner where the 6 is in each of your representations.
- **Pose** questions that require students to alternate the directionality of the connections they make between representations. Coach students to learn to move flexibly between representations. Alternate the directionality of the connections. When discussing multiple representations, ask focused questions:
  - Where do you see a 3 in a box diagram?
  - Where is that same 3 in the number sentence?
- **Encourage** students to consider the suitability of a representation. Discuss a variety of reasons to use particular representations, including but not limited to the following: efficiency, accuracy, ease of use, appropriateness with respect to the problem context, and student preference. By comparing the use of multiple representations for the same problem, students can more easily see the suitability of one representation over another.

Curtis: Because on the lower one, it's all dots; so you can just count by ten.
Beyer: OK.
Curtis: Instead of counting by ones.

In the excerpt above, Beyer asks students to consider the use of four different representations. Communicating about mathematical ideas involves making choices about which representations to use (Kilpatrick, Swafford, and Findell 2001; Novick 2004; Novick and Hurley 2001). This classroom dialogue suggests that such a choice may be based on the child's capacity to balance the characteristics of each representation in light of the problem situation.

Beyer prompts students to consider the ease and efficiency of the use of a menu of representations. For example, she poses questions that push students to consider whether the number line or the ten frame provides a more efficient means of counting in the problem. Thus, encouraging purposeful selection may help students acquire the competence to consider the way in which information is conveyed in the representation in relation to what they are being asked to do in the problem.

Conclusion

Students should have frequent opportunities to not only learn to use and work with representations in mathematics class but also to translate between and among representations. Sufficient opportunities help them see mathematics as a web of connected ideas. In supplying such opportunities, teachers should engage students in dialogue about representations and the relationships between them in order to help develop students' representational competence, an important aspect of mathematical understanding.

By closely examining Beyer's teaching in a first-grade classroom, the authors identified her use of three instructional strategies that likely support her students' development of representational competence. Other research has shown that instruction incorporating the use of multiple representations strongly influences students' performance in mathematics (e.g., Cramer, Post, and del Mas 2002). Teachers ought to create a learning environment in which students' use of multiple representations is encouraged, supported, and accepted by their classmates (NCTM 2000). This case study provides evidence that employing these three instructional strategies may help students' acquire representational competence and form a more coherent and robust understanding of mathematical ideas.

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Explore interactive student activities that use representations such as five frames, ten frames, box diagrams, and number lines by searching the Activities tab on NCTM's Illuminations Web site: **http://illuminations.nctm.org/**.