



# Changed My Teaching

**Fourth graders' creative thinking concerning a long-standing research problem stimulated changes in instructional strategies.**

**“W**e’ve been trying to tell you, you’re wrong!” With this reprimand, a fourth grader, hand on hip, expressed her classmates’ exasperation with the slow-witted math professor who failed to understand the obvious. These students provided a striking lesson that changed my teaching approach. The message I received? How often do we inadvertently ignore a powerful classroom tool—the hidden, inherent creativity of our students?

Diane Briars had invited me to discuss mathematics with students as part of the Pittsburgh Public Schools’ observance of National Mathematics Awareness Month. I found this easy to do with older students, but trepidation set in when I was informed that a fourth-grade class was on my schedule. What does one say to *fourth graders*, particularly when I find them to be short enough to inspire caution while I am walking about? Perhaps a counting problem would work; my choice involved fifteen fictitious

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## Approach

children who would be permitted to watch one television show. I selected three shows popular with that age group (I replaced the names of the shows here with *Ann*, *Barb*, and *Connie*). Within this hypothetical group—

- six children preferred Ann > Connie > Barb (Ann was their first choice, Connie was their second, and Barb was their last choice);
- five children preferred Barb > Connie > Ann; and
- four children preferred Connie > Barb > Ann.

Which show should they watch? The obvious answer is Ann; in the Ann > Barb > Connie outcome, she is ranked at the top with a 6:5:4 tally.

I presented the problem to the class. Imagine my discomfort upon hearing the unified class response, “Connie.” Were these children too

young to comprehend the relationship between counting and voting? Were they expressing personal favorites? How would I survive this dreadfully *long* 40 minutes?

Calling on tactics I have developed by teaching college students, I challenged the fourth graders’ answer. Their instant reaction included comments like these two: “Well, you see, to choose the best show, you have to count the number of times each show is in first place and how many times the kids like it next best.”

“If you look, some kids like Connie best, and all other kids like Connie next best; but all other shows, many kids like the worst.”

Maybe they had some understanding after all. They treated as silly my suggestion to use our standard plurality rule. Counting how many people liked each show the best, they agreed, would yield the Ann > Barb > Connie ranking.

Then I announced, “Ann is canceled tonight; now what should these kids watch?”

The obvious answer is the second-place show, Barb, but their reply was, again in unison, “Connie!” After I questioned this response, more children joined the debate:

“You can’t just count who likes what show the best; you have to see what they like next best, too.”

“Your counting way—that makes Ann best and Barb next best—is silly.”

“Count! If you count, you’ll see more kids like Connie than Barb.”

Indeed, last-place Connie defeats second-place Barb with a 10:5 landslide. Before I had the chance to announce my punch lines—“Last-place Connie beats even first-place Ann,” and “Second-place Barb beats first-place Ann,”—many of the children had completed the analysis and excitedly called out, “Look; count! See, they like Connie best, Barb next, and Ann last.”

“We tried to tell you, these kids like Connie best.”

I was astonished by their insight and quickness. Then that young girl reprimanded me, and more chastisement followed.

### Plurality vote deficiencies

This example illustrates that our standard tool of democracy, the plurality vote, suffers serious deficiencies: *The winner can be the voters’ inferior choice*. When I have used the same example to introduce this subtle issue to professional economists, political scientists, and mathematicians (replacing *television shows* with *adult beverages*), they have been puzzled—but not these fourth graders. (For more about voting, see Saari 2001; 2008.) These problems are so difficult that asking even professionals to suggest an appropriate voting rule is totally unreasonable. So I asked the fourth-grade class.

Cara answered quietly:

Well, I think we should give three points to the show we like the very best and two points to the show we like next and only one point to the show we like the last. This way, we can also tell what other shows the kids like other than their best one.

Cara’s response nicely described an approach that the French mathematician Borda introduced in 1770. (A few years ago, I proved that Borda’s rule most accurately represents voter preferences [Saari 2008].)

Abe offered, “I want to give one point to our best show and zero points to our next show and negative one point to our last show.”

Imagine a fourth grader proposing a voting rule involving a negative number of points. Susan suggested, “How about giving two points to our best show and one point to the next show and no points for the last show.”

Other children closed the discussion by arguing that these rules were the same. They

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accurately recognized that rather than the number of points assigned to a candidate, what matters is the point spread among them. The differential for each proposed rule is a single point, so all the methods these children proposed yield the same Borda election ranking: Connie > Barb > Ann. Whereas this outcome agrees with paired comparison majority votes, it reverses the election outcome of our widely used plurality vote. The choice of a voting rule matters.

I introduced an example discovered in 1785 by Condorcet (another French mathematician). After I started writing on the board that five children preferred Ann > Barb > Connie and five preferred Barb > Connie > Ann, a girl quickly volunteered, “And you are going to say next that five like Connie > Ann > Barb.” She was correct. A version of Condorcet’s example has five sets of children each preferring—

- Ann > Barb > Connie
- Barb > Connie > Ann
- Connie > Ann > Barb

Who do they want? By landslide 10:5 majorities, they prefer Ann > Barb, and Barb > Connie, so presumably they prefer Ann > Connie, making Ann their top choice. Yet, the class argued, “Nobody is best; they are all the same.” I loved it when a student slowly and accurately, in a patronizing manner, explained, “See, each [person] is the same number of times in top place and in second place and in last place. That is why there are no favorites; they’re all the same.”

Sticking to my guns, I demonstrated Ann’s 10:5 advantage over Barb, which triggered an avalanche of outbursts: “Yes, and Barb will be better than Connie, and Connie will be better than Ann by the same numbers.” They were correct; Condorcet’s example illustrates that cycles, even with landslide proportions, can accompany paired comparison majority votes. One small boy took pity on my inability to comprehend the obvious:

Let me explain. Nobody is better; they are all the same. It’s like the rock and the scissors and the paper. The rock can dull the scissors, and the scissors can cut the paper, and the paper can cover the rock, so nothing is better than the others.

These difficult mathematical issues continue to puzzle experts from several disciplines. Yet, with an awakened curiosity, with insight not yet stifled by years of accepting our standard but defective election rule, these fourth graders cut through the conceptual difficulties to achieve critical understanding. But then I wondered whether this group was unique; I knew that several of them participated in Briars’s program to enrich the standard mathematics curriculum. To check, I experimented with schools around Northwestern University (my academic home at that time). One class became so upset after discovering that election outcomes can more accurately reflect the voting rule than voter preferences that the students proposed sending a class letter explaining to members of Congress what they were doing wrong. The only fourth-grade failure occurred when a teacher reprimanded a student; this act unintentionally established the teacher as the authority, so subsequent answers were selected to meet her approval rather than to address the problem.

This entire experience crafted my teaching



approach, which is to find ways for students to assume ownership of ideas. To avoid being viewed as the authority, I often appear to be confused and seeking help. Students recognize that this is a game, but it emboldens them to advance creative ideas that then become “theirs.” Transferring ownership of ideas may require that teachers temporarily ignore convention. One fourth-grade teacher, trying to duplicate my experience, experimented with her inequality lesson. She judged it a failure because, rather

than discovering the greater-than symbol for  $5 > 4$ , her students proposed an upward-pointing arrow, which I personally find to be a more intuitive choice. Another teacher proceeded with what her students suggested. By using their symbol, she avoided taking precious time to introduce notation; instead, students could immediately explore consequences, ensuring a successful session. After making the most of her students’ intuition for meaning, she then made the translation to the standard choice.



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### When students “own” ideas

Insisting on convention before learning hinders comprehension. A telling example comes from young hawkers on the beaches of Brazil who could convert, faster than I, a professional mathematician, any currency and make correct change for purchases. Yet, as I discovered, they could not handle far more elementary classroom problems. The message is that helping students assume ownership of concepts requires relating the concepts to their experiences. For instance, when helping a student who was totally mystified by tenth-grade geometry, I recalled that he helped his father in carpentry, so I asked him to describe alternating angles formed by tossing a board over joists (with parallel supporting timbers for floors). By relating geometry to his carpentry experiences, he nicely mastered several weeks of material in one session.

Approximations provide a rich supply of challenging problems. I borrowed an example from my calculus lectures to try with sixth graders: a student safety campaign to honor the most responsible driver. I reminded students, who were armed with a stopwatch and measuring tape, that they could determine how far a car travels in fifty seconds. Then, by using the usual  $r = d/t$  equation, the car’s speed can be computed in terms of feet per second. To toss in a complication, I described how someone could game the system. Speeding at 100 feet per second in a 50-feet-per-second zone (approximately 34 mph), the driver could quickly slow down—and win the prize. The challenge to the students was to find a way to catch the person. After a discussion, they quickly recognized that they needed to use shorter time intervals; only in this way could they capture how fast the driver was going before putting on the brakes. Using 10 or 20 seconds rather than 50 leads to

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more accurate speed estimates. This notion, which they discovered (with minimal direction), is the essence of the definition of a *derivative* in calculus. I loved hearing one student worry that the best choice would be to use zero seconds, but because this could not be done, she opted for radar. Actually, an answer for her concern is the only step missing in using approximations to define a derivative.

Another approximation challenge starts with the standard width-times-height formula for the area of a rectangle. After finding the area under a curve for several adjacent rectangles of different heights, the offered challenge is to approximate the area under a smooth curve. Rather than drawing a curve on the board, I found it useful to supply students with scissors and a sheet of paper cut in a curved fashion at the top. Within a surprisingly short amount of exploration time, they discovered that cutting the paper into narrow strips (edges parallel to the  $y$ -axis) creates objects that closely resemble rectangles, so the area of each strip can be estimated. Adding the strip areas gives the approximate value for the total area. Students realized that the narrower the strips, the closer they resemble a rectangle, so the sharper the estimate. This is the essence of the definition of an *integral* in calculus.

Experiences with elementary school students have changed how I handle my classes—from freshman calculus to advanced graduate studies. My goal is to have students assume ownership of ideas (Bain 2004); I want them to believe that had they been born before Newton, they could have invented calculus, or whatever the basic concept is for the course I am teaching at the time.

Is there enough time for such an approach as I take? This is a realistic concern. At the university level, my section would already be behind schedule, compared to all others, by the end of the second week. However, combining my approach with pop quizzes allowed the pace to accelerate as soon as students grasped this critical-thinking approach. With their deeper understanding of the concepts, we needed less time for review, so my sections finished well before any others.

At Northwestern, I taught the only large calculus section (about 200 students); all other sections had about 35 students. A successful performance on common departmental finals (written

primarily by the other instructors) would be if my large class—roughly 40 percent of all calculus students—earned, say, 35–40 percent of all A's on the common departmental curve. Instead, my classes always monopolized the A's by garnering more or less 90 percent of them; rarely did any of my students receive a poor grade.

## Telling stories

A key step is to tell stories that capture interest and attention—a story that separates concepts from supporting details. A way to find stories is to convert a word problem from the exercises into a simple but interesting challenge that emphasizes concepts. Indeed, stories have become my standard instructive tool, even at the PhD dissertation level. To help a former student who was experiencing a mental block on his thesis research, for example, I had him explain the difficulty to me as if I were a high school junior. After several failures to get me, an eleventh grader, to understand him, on the fifth day he announced that *even I* would follow his story. Partway into it, he suddenly overcame his mental block, dashed out of my office, and completed the project. (I never heard the rest of his story.) Training students to tell stories helps them separate concepts from supporting details and leads them to assume ownership.

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