Norms and Mathematical

How do classroom behavioral expectations support the development of students' mathematical reasoning? A sixth-grade teacher and his students developed this example while discussing a ratio comparison problem.

To challenge students' reasoning and to align with a physical fitness unit that students were studying, Scott Frye and Signe Kastberg adapted a ratio comparison problem from Lamon (1994) to include athletes and doctors who share pizza (see **fig. 1**).

"I know that you found a common number of pizzas in your two mathematical tables. Do you think that you'll ever have the same number of people on your tables if you keep counting up?"

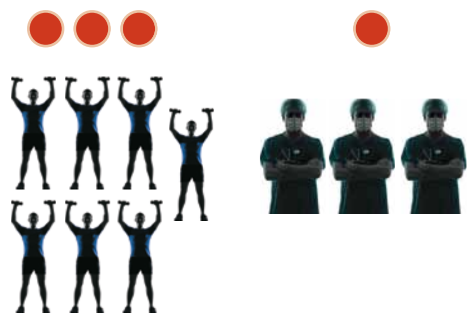
Ashley and her partner considered Stevie's question about their solution (see **fig. 2**) to the Pizza problem but were unsure what Stevie was asking. They asked for clarification.

Stevie's questioning disposition and Ashley's effort to understand the question illustrate norms that Frye had worked all year to establish in his sixth-grade classroom. Social norms develop across all disciplines. In Frye's

FIGURE 1

The authors adapted Lamon's (1994) ratio comparison Pizza problem to include athletes and doctors.

Consider the following drawings of people and pizza. Who gets more pizza, the athletes or the doctors?



OSTILL/VEER; SMIT/VEER

OSTILL/VEER

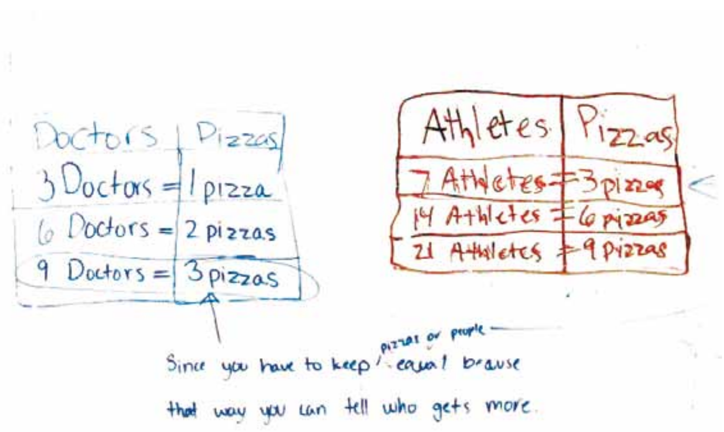
Proficiency

By Signe E. Kastberg
and R. Scott Frye

class, *persisting* and *challenging and questioning* were expected behaviors in all academic discussions. In mathematics class, however, these norms took on new dimensions as "sociomathematical norms" (Yackel and Cobb 1996), or expected ways of engaging in mathematical discussions. These particular ways in turn contributed to students' "mathematical proficiency" (Kilpatrick, Swafford, and Findell 2001). Sociomathematical norms for participation can be fostered when students' attention is focused on contributing to mathematical discussions, understanding one another's ideas, and exploring the merits of those ideas. To illustrate sociomathematical norms that developed, as well as students' mathematical proficiency, the authors draw examples from Frye's students' discussion of the Pizza problem near the end of the academic year.

FIGURE 2

"Do you think that you'll ever have the same number of people on your tables if you keep counting up?"



SIGNE E. KASTBERG

FIGURE 3

Students' inability to use ratios to explain their answers to the Cherry Syrup problem (National Center for Educational Statistics 1996) prompted the authors to revisit the concept with the following problem.

Luis mixed 6 ounces of cherry syrup with 53 ounces of water to make a cherry-flavored drink. Martin mixed 5 ounces of the same cherry syrup with 42 ounces of water. Who made the drink with the stronger cherry flavor?

Proportional reasoning and the Pizza problem

By May, Frye's students had developed number sense (Sowder 1992) and precision in numerical computation. They often used estimation and quickly solved the types of problems usually found on standardized tests. Frye asked Kastberg, his former university mathematics instructor, to observe his class. During the course of several months, the two teachers noticed that Frye's students did not describe units of measure (quantities) in problem-solving situations. For example, in comparing mixtures of water and cherry syrup (see **fig. 3**), students divided 6 by 53. When asked what the resulting quotient meant, students were unsure. They could not yet explain their answer in terms of a ratio (Lobato and Ellis 2010) of syrup to water, which made comparing the two quotients representing concentrations difficult for them. This example was just one piece of evidence that prompted the teachers to further explore the concept of ratios and proportions with the children.

Lamon (1994) noted that "associated sets" problems, like the one in **figure 1**, "had the effect of eliciting more of the language of ratio" (p. 51) than students typically use in comparison problems. Lamon also found that students tend to solve this type of problem using ratios as units rather than computing and comparing numbers. For example, in the Cherry Syrup problem (see **fig. 3**), the units of water and syrup were the same, which encouraged the computation of a number that was no longer seen as connected to the units of syrup or water. In the Pizza problem (see **fig. 1**), the context discouraged students from dividing numbers and encouraged reasoning with ratios. If three people share one pizza, then students generate a variety of ratios, including 1 pizza: 3 people or 1/3 pizza per person. Frye and Kastberg felt that some students would reason with ratios, whereas others would apply the computational approach used in the Cherry Syrup problem. With both approaches available, potential existed for a rich discussion of ratios and the units involved.

Frye valued working with Kastberg and providing an opportunity for his students to engage with her. Kastberg was motivated to join Frye and his class because the work of teachers and children gave her the chance to build understanding of the teaching and learning of math in the elementary grades. The students in Frye's sixth-grade class were considered to be of mixed academic ability. All students were expected to contribute to mathematics discussions; Frye and the students treated each contribution as an opportunity to learn.

The norm of persisting

Frye and his class defined *persistence* as "working to identify solution paths; finding solutions; working past these solutions to explore other methods of solving the problem; and extending the problem beyond the stated question." For example, when students are presented with a question and discover their perceived solution, they should never believe they are finished. In Frye's class, students ask two types of questions aimed at persistence when they finish any academic work. In the context of mathematics, these questions focus on reflecting on the processes they used to solve the problem and on extending their findings. The first type of question relates to the students' problem-solving process. Frye uses this analogy: A motorsports engineer is looking at a newly constructed Formula One race car. The engineer revisits the design and considers how to make the car faster and safer. So, too, students reflect on their work with such key questions as the following:

- What other mathematical concepts could we have used to solve this problem?
- Which strategy is the most efficient?
- What mistakes did I make that others could learn from?

The second type of question relates to how a problem could be extended or a solution process generalized for use on other problems:

- What other problems might I be able to solve using this method?
- What other questions might I pose and answer about the problem situation?

Jessica's presentation in Frye's class illustrates how the norm of persistence in the context of mathematics enables her team and the class to focus on the unit ratio, just as Frye and Kastberg had hoped for in their planning. Jessica selected one element of her solution to the Pizza problem as the focus of her presentation. She emphasized what she viewed as the most important point of her investigation: when she changed her fractional answer to a decimal (see **fig. 4**) and was unsure what the decimal represented. "I had to track my work back to the point where I knew what the numbers represented," she said. Jessica went on to describe the quantities involved in her answer. She converted the ratios of three pizzas to seven athletes to 3p/7a and one pizza to three doctors to 1p/3d and then to 0.43p/1a and 0.33p/1d. Comparing these results generated a difference of approximately one-tenth pizza per person (see **fig. 4**), with each athlete receiving the larger portion of a pizza. "So I feel that we would rather be athletes," Jessica explained, "because we would receive one-tenth more pizza than each of the doctors. This is made clearer by changing the fraction back to a decimal one-tenth (0.1)," she continued. "Each athlete would receive one-tenth more pizza than each of the doctors."

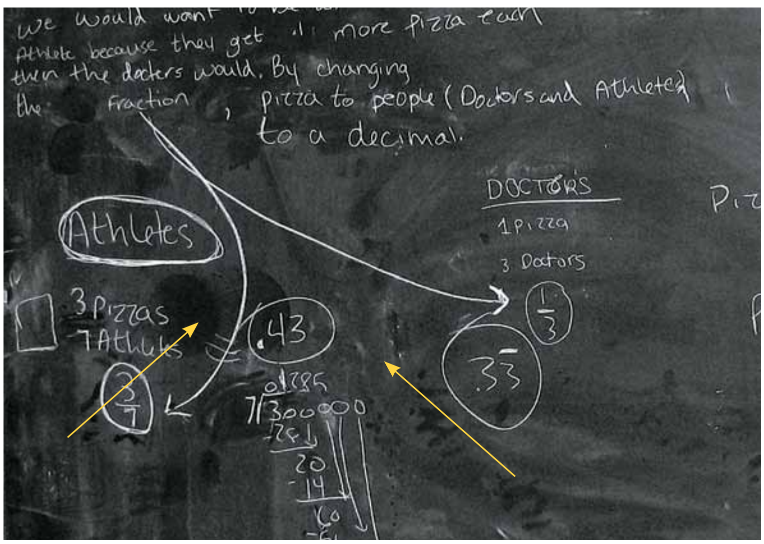
Because Jessica did not stop at number computation, she developed a deeper understanding of the unit rate. She reflected on her solution and was able to identify the meanings of the quantities and the difference in portion size. Jessica calculated an answer to the problem within minutes. Yet her persistence in the form of looking back over her process and looking for ways to extend the problem resulted in questions about the numbers she had computed, her understanding of ratios, and further exploration of the problem in terms of differences in portion sizes. Her approach of moving past the original problem to further explore and confirm her findings exemplifies the persistence expected of everyone in Frye's class.

A sociomathematical norm: mathematical difference

Jessica's work illustrates the social norm of persistence, but it also shows how the sociomathematical norm of exploring mathematical difference is enacted during a presentation. Yackel and Cobb, citing their work with Wood (Cobb, Yackel, and Wood 1989; Yackel, Cobb, and Wood 1991), noted that social norms in classrooms include "explanation, justification, and argumentation" (Yackel and Cobb 1996, p. 460), but sociomathematical norms involve the examination of the mathematics in various solution paths. Persistence is one example of a social norm, because it is not unique to the work in communities of mathematicians. Yet in Jessica's work, we see how persistence helps Jessica and her classmates focus on the mathematics involved in her solution. Jessica shared with her peers her original process of computing quotients. The results of her computations are then compared with her understanding of the decimals as unit ratios of quantities. The students discussed the difference between Jessica's two approaches. Her comparison was one of many instances during the year when the sociomathematical norm of "mathematical difference" was developed. When the teacher and students discuss conceptual differences between solutions and processes used to generate solutions, they focus on the mathematics.

FIGURE 4

Jessica's representation of her solution illustrates the social norm of persistence and the sociomathematical norm of difference.



SIGNE E. KASTBERG

AJA/VEER; STUART JENNER/VEER

This emphasis moves students beyond simply engaging in productive academic ways toward building understanding of ideas unique to mathematics.

Mathematical proficiency, as described by Kilpatrick, Swafford, and Findell (2001), includes five interwoven skills and dispositions needed for successful mathematics learning. Frye's students consistently demonstrated three of these: (1) strategic competence, (2) adaptive reasoning, and (3) productive disposition. Strategic competence is the ability to "formulate, represent, and solve mathematical problems" (2001, p. 116). To solve problems and learn mathematics, students who use adaptive reasoning will think logically, will reflect, will explain, and will justify (p. 116). Productive disposition, one that "see(s) mathematics as sensible, useful, and worthwhile," combined with a belief in the productive potential in one's own diligence (p. 116), are often difficult to foster.

Frye's focus on persistence in mathematics promotes productive dispositions and adaptive reasoning. Students like Jessica come to view mathematics as a sense-making discipline. In her presentation, Jessica described an initial confusion about the meaning of the quotients she had computed. To eliminate her confusion, she carefully reviewed each unit in her computation to make sense of her results as ratios. She assumed that the numbers in her work should make sense.

Abbey, Jessica's peer, commented on the general attitude toward persistence in this class:

> All of us recognize how important it is to use every minute of problem-solving time we have to make us better mathematicians. This is not about following steps or getting done first; it should be deeper than that. We try to really understand why we are doing what we are doing. When we solved this problem, you can tell we looked at it like a real-world problem. If we were hungry for pizza, would we rather be a doctor or an athlete? That is a relevant, applicable application. … I like pizza.

Abbey, like Jessica, shares a view of math as "sensible, useful, and worthwhile" (Kilpatrick, Swafford, and Findell 2001), a productive disposition and one facet of mathematical proficiency.

The norm of challenging and questioning

In Frye's class, every student is responsible for developing understanding and contributing to and supporting the understanding of others. Frye encourages students to be curious about others' ideas and reasoning in all academic disciplines. In mathematics, this curiosity takes the form of questioning. Students are encouraged to demonstrate the value of peer solutions by asking thoughtful questions about the problem-solving process and solutions. Students eventually come to compare processes and results and challenge peers to build strategies that are more efficient and mathematical justifications that are more convincing.

In another solution to the Pizza problem, Ashley and her partner created a table (see **fig. 2**) and found a common number of pizzas using a "build-up strategy" (Lamon 1994). This strategy allowed students to generate proportions based on the original ratio: If three doctors can share one pizza, then six doctors can share two pizzas, and so on.

Ashley explained their findings:

> Seven athletes would share three pizzas, and nine doctors would share three pizzas. Therefore, I would rather be an athlete, because we would be sharing with a fewer number of people.

Stevie, who was introduced at the beginning of this article, had used several different strategies to solve the problem, including Ashley's. Stevie found that twenty-one athletes would share nine pizzas and twenty-one doctors would share seven pizzas. At this point, Stevie challenged Ashley. "I know that you found a common number of pizzas on your mathematical table. Do you think that you'll ever have the same number of people on your table if you kept counting up?"

The presenters thought Stevie wanted to know if extending the tables would produce a number of doctors and athletes with the "same" amount of pizza.

Ashley responded, "If you keep adding to the doctors' and athletes' column, the number of pizzas will be getting bigger also. So, they will never be the same."

Stevie reframed his question: "I mean, would just the number of athletes and doctors be the same? The number of pizzas could be different. You know, if you were to continue adding to your column of people?"

Now Ashley understood Stevie's point; she shared her thinking:

> Right here [*pointing to the table*], when you look at three doctors and seven athletes, if you keep going, you'll arrive at twenty-one in both columns. I already have twenty-one athletes, and over here [*adding to the table while counting by threes*], if you keep going, you'll end up with fifteen, eighteen, and twenty-one doctors. So, that's a different way to solve the problem and arrive at the same answer. Cool.

Ashley responded to Stevie's challenge by adapting the strategy she had applied to the athletes and pizzas, to arrive at a 21:7 ratio of doctors to pizzas.

Reflecting on the Pizza problem and his challenge, Stevie pointed out that—

> the goal isn't just for me and my classmates to learn during presentations. It is also important for the presenter to be learning, too. Other classmates have done the same thing for me in the past.

Stevie described his own learning and that of his classmates as the impetus for his challenging and questioning:

> We don't have to use the same approach, but we should understand each other's strategies to determine which we believe is more efficient, and we all know that the more ways we can solve a problem, the deeper we understand it. Everything just becomes clearer.

Stevie's description illustrates that he and his peers value "mathematical differences" in search of efficient solutions. His emphasis on efficiency suggests that a second sociomathematical norm developed in Frye's class: "mathematical sophistication" (Yackel and Cobb 1996, p. 461). In Frye's class, efficient strategies that were generalized and could be applied to new mathematics problem situations were valued as more mathematically sophisticated than those strategies that appeared to apply to only one problem-solving situation. Stevie's reflection and interaction with Ashley illustrate the value he gives to efficient approaches to problems, his work to generalize their approach, and the mathematical sophistication that is valued by his peers and teacher.

The social norm of questioning and challenging that Frye and his students value contributed to the development of mathematical proficiency as illustrated in the discussion of the Pizza problem. Ashley adapted her reasoning as she responded to Stevie's challenge. She made sense of Stevie's question and related it to her own solution. She recognized that Stevie's idea was different from her own. Focusing on the mathematical difference between strategies and on solving other problems that require a comparison between ratios will also encourage students to identify what is common among strategies. Students' strategic competence and understanding of ratios will evolve as they use strategies that they have developed with ratios to solve similar problems.

From social norms to sociomathematical norms

The value that Frye and his students' place on behaviors identified with the social norms he called *persisting* and *challenging and questioning* in the context of mathematics encouraged the students to focus on mathematical differences. Morgan, Frye's student, described the social value of developing this norm:

> You can see why everyone who goes to the board thinks they are right. They go so deeply into it [the process of solving the problem] so that you understand how they are thinking. In life, if you want to show why you are right and you really care, then you have to go deeply into it [your reasoning]. You have to love each other to change people's mind. In our class, we love each other so you can change each other's mind. You share facts and statistics and reasons. For the other person to know that you love and care about them makes it easier [to challenge them] because they know why you are trying to change their mind.

Morgan's description of her experience in Frye's class highlights the significance of a community of respect and care for others in being able to genuinely confront or question the problem-solving process or findings of a peer. Students in Frye's class, like Morgan, shared their processes and findings so that peers understood how the presenter was thinking. Challenges came after the problem-solving process and findings were understood and differences among ideas were identified. Because the children care about one another and know that they are cared for, they understand that a challenge is an opportunity to learn and that changing your mind because a peer has convincingly justified his or her claim is part of building understanding. Although challenging and questioning would be considered a social norm, the students' discussions that focused on identifying and understanding differences in solutions and processes and evaluating their efficiency were sociomathematical norms.

The development of mathematical proficiency can emerge from teachers' efforts to establish sociomathematical norms. Encouraging and drawing attention to students' comparison and evaluation of mathematical ideas is an important first step. In Frye's class, the discussion and comparison of students' solutions helped build mathematical proficiency. Because students were supported in persisting, representing, and sharing their findings and methods—as well as debating the differences in findings and solution methods—they developed strategic competence, adaptive reasoning, and productive dispositions. Building social norms is native to every teacher. Frye's story illustrates how attention to students' discussion of their findings and processes can help develop sociomathematical norms and mathematical proficiency.

Common Core Connections

6.RP.A.2
SMP 1
SMP 3

REFERENCES

Cobb, Paul, Erna Yackel, and Terry Wood. 1989. "Young Children's Emotional Acts while Doing Mathematical Problem Solving." In *Affect and Mathematical Problem Solving*, edited by Douglas B. McLeod and Verna M. Adams, pp. 117–48. New York: Springer-Verlag.

Kilpatrick, Jeremy, Jane Swafford, and Bradford Findell. 2001. *Adding It Up: Helping Children Learn Mathematics*. Washington, DC: National Academies Press.

Lamon, Susan. 1994. "Ratio and Proportion: Connecting Content and Children's Thinking." *Journal for Research in Mathematics Education* 24 (1): 41–61.

Lobato, Joanne, and Amy Ellis. 2010. *Developing Essential Understanding of Ratios, Proportions & Proportional Reasoning*. Reston, VA: National Council of Teachers of Mathematics.

National Center for Educational Statistics. 1996. NAEP Questions Tool, Block M12, Question 8. Washington, DC: U.S. Department of Education. http://nces.ed.gov/nationsreportcard/itmrlsx/landing.aspx

Sowder, J. 1992. "Estimation and Number Sense." In *Handbook of Research on Mathematics Teaching and Learning*, edited by Douglas Grouws, pp. 371–89. New York: Macmillan.

Yackel, Erna, and Paul Cobb. 1996. "Sociomathematical Norms, Argumentation, and Autonomy in Mathematics." *Journal for Research in Mathematics Education* 27 (4): 458–77.

Yackel, Erna, Paul Cobb, and Terry Wood. 1991. "Small-Group Interactions as a Source of Learning Opportunities in Second-Grade Mathematics." *Journal for Research in Mathematics Education* 22 (5): 390–408.

Kastberg and Frye express many thanks to the following students whose ideas and discussions helped them develop new perspectives on mathematics teaching and learning: Stevie Hoffman, Morgan Matt, Abbey Collester, Ashley Gray, and Jessica Langely.

Signe E. Kastberg, **skastber@purdue.edu**, is a mathematics teacher educator at Purdue University in West Lafayette, Indiana. She is interested in the development of understandings of multiplicative reasoning. **R. Scott Frye**, **sfrye@paramountindy.org**, is the assistant director at Paramount School of Excellence in Indianapolis, Indiana. Frye advocates highly engaging, Standards-based instruction that meets student readiness levels and interests.