Variations in Both-Addends-Unknown Problems

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Early elementary school students are expected

A math researcher and two teachers pose a structure for thinking about

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Many years ago, an interesting problem that had been implemented in a primary-grade classroom sparked a conversation among the three authors of this article. The problem was a classic both-addends-unknown story problem, one of twelve distinct types of word problems that kindergarten, first-grade, and second-grade students are expected to solve, according to the Common Core State Standards for Mathematics (CCSSM) (CCSSI 2010) Operations and Algebra domain (see **fig. 1**).

Student thinking processes and the relative problem difficulty of the other eleven types

to solve twelve distinct types of word problems.

one problem type that has not been studied as closely as the other eleven.

of problems have been studied extensively. However, both-addends-unknown problems have not been historically included in these studies (see, for example, Carpenter, Hiebert, and Moser 1981; Carpenter and Moser 1984; Nesher, Greeno, and Riley 1982; Fuson 1992; Sarama and Clements 2009). Unlike the other eleven problems in the problem taxonomy (see **table 1**), both-addends-unknown problems can be mathematized by writing a single equation with two unknown variables rather than a single equation with one unknown variable. This situation results in a set of solutions of pairs of numbers rather than a solution of a unique number. Mathematically, this makes both-addends-unknown problems very different from the other eleven problem types in the taxonomy.

During the past four years, we have worked with hundreds of students in primary-grade classrooms as they interact with and solve both-addends-unknown problems. In this article, we share some of our discoveries with regard to the variety of both-addends-unknown problems and offer suggestions for classroom practice when creating and implementing these types of problems.

FIGURE 1

The Common Core State Standards for Mathematics (CCSSM) Operations and Algebra domain for students in kindergarten through grade 2 involve both-addends-unknown problems (CCSSI 2010).

K.OA.2—Solve addition and subtraction word problems, and add and subtract within 10, e.g., by using objects or drawings to represent the problem.

1.OA.1—Use addition and subtraction within 20 to solve word problems involving situations of adding to, taking from, putting together, taking apart, and comparing, with unknowns in all positions, e.g., by using objects, drawings, and equations with a symbol for the unknown number to represent the problem.

2.OA.1—Use addition and subtraction within 100 to solve one- and two-step word problems involving situations of adding to, taking from, putting together, taking apart, and comparing with unknowns in all positions, e.g., by using drawings and equations with a symbol for the unknown number to represent the problem.

TABLE 1

A both-addends-unknown problem is mathematically quite different from the other eleven problems in the underlying framework for the taxonomy of addition and subtraction word problems published in the Common Core State Standards for Mathematics (CCSSM) (CCSSI 2010).

CCSSM addition and subtraction word problems

Major type	Subtype A	Subtype B	Subtype C
Add to	Result unknown	Change unknown	Start unknown
Take from	Result unknown	Change unknown	Start unknown
Put together/ take apart	Total unknown	Addend unknown	Both addends unknown
Compare	Difference unknown	Bigger unknown	Smaller unknown

The following variation of a both-addends-unknown problem was posed to students in a first-grade classroom:

> A classroom has a hamster cage with two parts to the cage, a big part and a little part. The hamsters can move between the two parts at any time, but each hamster must be in one of the parts at all times. If the class has five hamsters, what are all the ways that the hamsters can be in the two parts of the cage? (Adapted from Carpenter, Franke, and Levi 2003)

One student solved this both-addends-unknown problem in the systematic way that the teacher had hoped all his students would discover. This student began with zero hamsters in the big part and five in the little part, recorded his work on a T-chart, and then moved one hamster at a time from the little to the big part, recording as he worked (see **fig. 2**). The following conversation then ensued between the teacher and the student.

Teacher: I noticed that your chart increases here [*pointing to the big cage column*] and decreases here [*pointing to the little cage column*]. Why do you think that happened?
Student: Hmmm. I think it is because the hamster is moving from the big cage to the little cage.
Teacher: How do you know if you have all the ways the hamsters could be in the two cages?
Student: This cage [*pointing to the little-cage column*] is all the way down to zero, and this [*pointing to the big-cage column*] already has all the numbers to five, too.

Examples of both-addends-unknown problems

Each of the five both-addends-unknown word problems (see **fig. 3**) requires that students find combinations of two whole numbers that sum to five. At first glance, the problems may seem essentially the same.

Before reading on, consider the differences among the problems.

Because the problems are presented in context, each requires students to think about the context and to interact with the mathematics somewhat differently. As students solve these problems, they must consider various aspects of the problem, such as the inclusion of zero, elements of modeling, the type and quantity of manipulatives needed (if any), and whether to consider the items as individuals or identical copies of each other.

In our recent work, we have come to think that these five problems are considerably different from one another in the way students think about them. In the spirit of the problem types and solution-strategy nomenclature used in Cognitively Guided Instruction (Carpenter et al. 1999), we think that a taxonomic structure and a shared vernacular is necessary for educators, researchers, and practitioners. This common way to discuss these types of problems would enable the efficient use of these problems in assessment and instruction as well as allow for meaningful discussion of student thinking.

Types of both-addends-unknown problems

We propose the following three main types as a working structure for thinking about both-addends-unknown problems.

1. Placement-based problems

Structured such that n objects are each assigned to one of two distinct groups, placement-based problems assume that the objects are indistinguishable from one another. Of interest is the number of objects assigned or placed in each group. The *placement* of each object, not a fixed attribute inherent to the object, determines its status. For placement-based problems, we ask students to find different ways to place a given number of identical objects in two locations (see **fig.** 3). The Game of Tag problem as well as the Hamsters in a Cage problem exemplify placement-based problems, because to solve them, the game players and the hamsters are placed in different areas.

2. Attribute-based problems

For attribute-based problems, students compose a single group of n items that are similar, with the exception of a single attribute. These problems do not necessarily have different mathematical structures, but we set them apart from placement-based problems according to

FIGURE 2

The authors had hoped that all students would solve the Hamsters in Two Cages problem (a both-addends-unknown problem) in the systematic way that this student did.

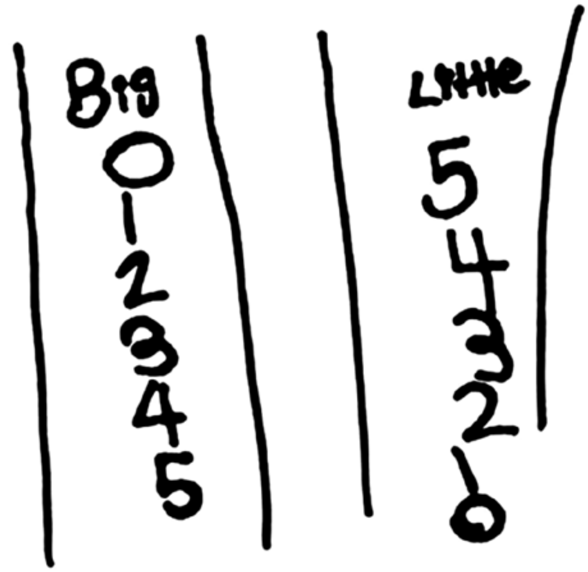

FIGURE 3

At first glance, the five problems below—both-addends-unknown problems involving sums of five—may seem essentially the same. Authors Champagne, Schoen, and Riddell have come to realize that students think differently about each of them.

Problem A
A classroom has a hamster cage with two parts to the cage, a big side and a little side. The hamsters can move between the two cages at any time. If the class has five hamsters, what are the ways that the hamsters can be in the two cages? (Adapted from Carpenter, Franke, and Levi 2003)

Problem B
Five students want to play a game of tag. During this game of tag, when you touch someone on the other team, he or she joins your team. List the ways that students could be on the two different teams during the game.

Problem C
Apples come in packages of five. The packages can contain both red and green apples. What different combinations of five red or green apples can be in a package? (Adapted from Fosnot and Dolk 2001)

Problem D
Crayons come in packages of five and can contain only red and blue crayons. Each package must have both red and blue crayons. What are the ways that the packages could have both red and blue crayons?

Problem E
Five students are asked to vote on their favorite flavor of ice cream—chocolate or vanilla. What are the possible combinations that show which flavor the five students can choose?

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By observing students as they solved the problems, the authors noticed patterns emerging.

the way we see students solve them. Problems C and D in **figure 3** are attribute-based problems. With the crayon and apple examples, the items are grouped on the basis of similar attributes. For example, one could solve problem C by placing three green apples and two red apples in the package; and next, one may find another solution by having four green apples and one red apple. This differs from placement-based problems, because the same item does not move from one place to another; rather, the item changes. In problems C and D, the item changes from a red apple to a green apple or from a red crayon to a blue crayon. However, attribute-based problems are similar to placement-based problems in that objects with the same basic attributes are indistinguishable from one another.

3. Model-dependent problems

Model-dependent problems fall into the category of either attribute-based or placement-based problems, depending on how students model them. For example, in problem E (see **fig. 3**), if a student were to use five cubes to represent the five students and were to place each of the five students into either the chocolate group or the vanilla group, then the modeling strategy would match that of a placement-based problem. If the student were to model this problem by getting five cubes for chocolate and five cubes for vanilla and then proceeding to make combinations of five with those cubes, the student would be modeling an attribute-based problem.

To date, we have seen students generally solving attribute-based problems and placement-based problems using modeling strategies that are consistent with the problem types we describe here. All these problems could possibly be modeled as a placement-based problem or as an attribute-based problem. However, we are seeing students model some of the problems more consistently as an attribute-based problem and some more consistently as a placement-based problem. Other problems are more ambiguous. To study patterns in student thinking and how they relate to the semantics of the problems, we suspect that systematic research is needed.

Students' solution methods

When examining the student work shown in **figure 2**, one may notice that the student began with zero hamsters in the big cage and five hamsters in the little cage and systematically moved one hamster at a time from the little to the big cage. Initially, we viewed the solution method used by this student as the only systematic method one could use to "prove" that all the ways had been found. However, as we worked more with these problems and watched children solving the problem (rather than simply looking at the artifacts created through the process), we noticed an important pattern emerging.

The most common strategy we see is to first split the *n* items into two equal or near-equal amounts in each group (depending on whether *n* is an even or odd number). Working from these equal or near-equal amounts as a starting point, children often increase or decrease the first amount by one and decrease or increase by one, respectively, the other amount. Next, they typically reverse the order of the two numbers, taking advantage of symmetry and the commutative property of addition. The subsequent steps are to decrease the amount by one again, reverse the order again, and repeat these two steps until the extreme case of zero and *n* is reached (see **figs. 4a** and **4b**).

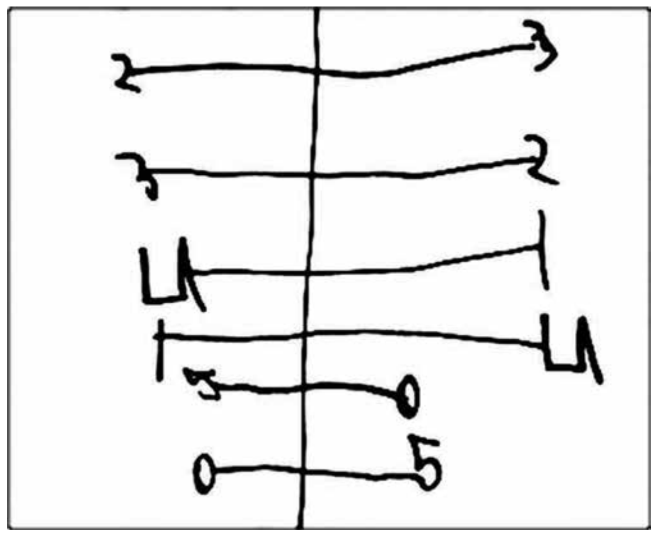
We have seen approximately half the students who successfully solve the problem use this algorithm. That said, this strategy is used more frequently for placement-based problems than for attribute-based problems. Although we did not initially consider the work presented in **figures 4a** and **4b** as a sufficient "proof" that all the ways to place the hamsters have been identified and listed, we have changed our thinking after observing and understanding the processes that students use to create records of their thinking. Additionally, students who used this strategy could consistently articulate that they knew they had found all the ways. In the hamster example, one student responded, "I know I have all the ways because I have each number of hamsters in both the big and the small cage," while checking that each number 0–5 was included on both sides of the cage.

FIGURE 4

After watching students record their thinking, the authors altered their opinion about work like that below being insufficient proof that all the ways to place the hamsters have been identified and listed.

(a) Students who used this strategy expressed consistently that they knew they had found all the ways.

(b) This student's solution for a both-addends-unknown problem involves an even number of hamsters.



Implementing both-addends-unknown problems

Aside from the framework that we hypothesized above, we must consider matters of mathematics, student thinking, and practicality when choosing, creating, or administering to students a both-addends-unknown problem.

Number choice

When posing both-addends-unknown problems to young children, number choice is an initial and critical decision. The specific learning goal of a lesson or assessment and an individual child's level of understanding of number should dictate number choice. For example, if the intention is for students to learn or practice listing all combinations of a given number, then ten may be the most important choice in the base-ten number system. If ten is beyond the scope of what a given student understands about number, then a lesser number may be more appropriate. We initially thought that choosing an odd or an even number would have an effect on student strategies or student difficulty. After carefully presenting students with problems having both odd and even numbers, our experience has been that the parity of the number (whether a number is defined as even or odd) typically has little effect on student thinking or performance.

Other mathematical considerations

Many of the both-addends-unknown problems we have seen, both those written for textbooks and those written by teachers, do not include zero as an option. For example, having teams with zero players on one team (see **fig. 3**) would not make sense for that context, but having zero hamsters in the small part of the cage is plausible. We have not seen children struggle any more with problems that included zero as a plausible solution than with problems that do not. We currently think problems that include zero are preferable over those that do not, because the former provide opportunities for students to consider and articulate their thinking about zero being the additive identity. Also, the set of possible solutions results in a complete list of every pair of whole numbers that sums to n.

Also carefully consider the symmetry of the problem when posing both-addends-unknown problems. When you choose placement-based problems, make certain that students understand that the two groups that items are placed into (or move between) are clearly different from one another. For example, using a big cage and a small cage makes the two places (big or small) different. A problem with too much symmetry in the two categories may result in a solution involving only three combinations: (5, 0), (4, 1), and (3, 2). For example, if the two parts of the cage are not distinguished by the size, big and small, students tend to see the combination of (3, 2) as the same as (2, 3).

In addition to creating two distinguishable *groups* in placement-based problems, consider that students may also see the *items* as unique. For example, if the five hamsters all had names and were considered to be unique individuals (as they would likely be in a classroom that had actual hamsters), more than six ways exist for the hamsters to occupy the cages. This challenge seems more likely to arise if the problem involves a set of people or animals rather than inanimate objects. Finding every variation in all the ways that are possible to arrange individual hamsters in cages involves not only higher-level thinking but also mathematics that is beyond the scope of most first-grade students. When this issue has arisen in our interactions with students, we generally note that for *this* problem, we are going to pretend that the hamsters (or people) are identical and we will not treat them as individuals. In each case so far, students have been perfectly happy to solve the simpler problem after we direct them to do so.

Finally, there is the matter of whether the problem asks students to determine one way, some of the ways, all the ways, or to answer the question of "How many ways are there?" Of course, finding one way is a much easier problem and does not require students to approach the problem systematically. For that matter, even listing all the ways does not constitute a proof that all the ways have been identified. Producing an answer to "Find all the ways" or "How many ways exist?" requires higher-level reasoning that involves many of the eight Standards of Mathematical Practices defined in CCSSM.

Practical considerations

When implementing attribute-based problems, one practical consideration is that students may choose to use twice the number of manipulatives to model attribute-based problems than they do for placement-based problems. For example, if a student uses red and green cubes to represent red and green apples, he or she would need at least five cubes of each color to physically model each way five apples could be arranged. For placement-based problems, only five cubes are necessary. As teachers implement attribute-based problems, we encourage them to be aware of this possibility when they make manipulatives available for students. Of course, some students may choose to use a drawing to solve these types of problems, and the number of manipulatives becomes irrelevant.

Recommendations

With the implementation of CCSSM in the coming years, many kindergarten, first-grade, and second-grade teachers as well as their students will be interacting with both-addends-unknown problems, and much can be learned from observing and interacting with students as their mathematical thinking is revealed through solving these problems. In our work, we have noticed important differences in the way both-addends-unknown problems can be written, and some of these differences seem to result in differences in levels of difficulty and strategies that students use to solve these problems.

We intend that the analytical structure posed in this article will initiate discussions about important considerations with regard to using both-addends-unknown problems in instruction and assessment. We recommend that teachers, curriculum developers, and assessment-item writers carefully consider the thoughts and experiences we have shared in this article when selecting or writing both-addend-unknown problems to pose to students. In particular, we encourage careful consideration of number choice, the type of problem (attribute, placement, or model dependent), the inclusion of zero, and how manipulatives will be used in administering the problem. Such careful implementation will offer crucial insight into how students interact with the mathematical ideas inherent in these problems.

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