CONSTRUCTING MEANING:

STANDARDS FOR MATHEMATICAL PRACTICE

Teachers' insights could inspire further discussion about interpreting the SMPs.

By Sarah K. Bleiler, Wesley A. Baxter, D. Christopher Stephens, and Angela T. Barlow

"How are we going to be able to achieve the goals of Common Core if we cannot agree on the meaning of the Standards for Mathematical Practice?"
—Jonathon, fourth-grade mathematics teacher

This sentiment was shared by a teacher who participated in our recent summer institute focused on implementing the Common Core State Standards for Mathematics (CCSSM). Jonathon's question arose from his engagement in activities earlier that day dealing with the interpretation of the fourth of the Standards for Mathematical Practice (SMP 4): "Model with mathematics" (CCSSI 2010, p. 7). Jonathon was frustrated by different interpretations that were emerging during his small-group discussion around this particular SMP. As Jonathon shared his concerns about SMP 4, other teachers began voicing similar concerns about other SMPs.

Teacher participants in a summer institute unpacked the meaning of the SMPs during intense conversations.

Throughout the summer institute, our teachers had many in-depth discussions about the meanings they attributed to the SMPs. In this article, we share some of the thoughts, concerns, and questions that our fifty-seven participating teachers of grades 3–6 raised. In particular, we share ideas related to those SMPs that provoked the most discussion (i.e., SMP 2, 4, 7, and 8). Our goal is threefold:

1. To describe the interpretation issues that were prominent for our teachers
2. To offer potentially clarifying examples aligned with the constructed meanings of our teachers
3. To catalyze discussions about interpretation of SMPs among mathematics teachers across the United States.

Constructing meaning: SMP 2

Reason abstractly and quantitatively. Mathematically proficient students make sense of quantities and their relationships in problem situations. (CCSSI 2010, p. 6)

Teacher perspectives from the institute

On the seventh day of the institute, we asked teachers, "Which of the SMPs makes the least sense to you?" A large majority of our participants identified SMP 2 as a standard for which they sought clarification. The teachers noted particular words from SMP 2 that they had difficulty interpreting. One was the term *abstract*. Some teachers defined abstract as "using formulas to represent mathematical ideas in general." Others described it in a way that aligned with the common English-language interpretation, using such phrases as *think outside the box* or by referring to abstract artists, such as Picasso.

Teachers most often mentioned two terms related to reasoning abstractly when seeking clarification on SMP 2: *contextualize* and *decontextualize*. The standard reads as follows:

> [Students] bring two complementary abilities to bear on problems involving quantitative relationships: the ability to *decontextualize*—to abstract a given situation and represent it symbolically and manipulate the representing symbols as if they have a life of their own, without necessarily attending to their referents—and the ability to *contextualize*, to pause as needed during the manipulation process in order to probe into the referents for the symbols involved. (CCSSI 2010, p. 6).

To clarify these ideas, we share an example of a task and describe how our teachers engaged in the processes of contextualization and decontextualization through their problem-solving process.

Example: Contextualization and decontextualization

We asked teachers to solve the Aquarium task (see **fig. 1**). To solve this problem, teachers first decontextualized it; that is, they ignored the contextual details in the problem and moved to a more abstract setting. Our teachers accomplished this by first imagining a rectangular prism whose volume was 240 cubic feet and then moving to the process of searching for

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factor triples of 240 in a purely numeric sense. During this phase, the problem solvers did not necessarily attend to the fact that they were suggesting dimensions for an aquarium. In this case, the representing symbols that SMP 2 refers to were the whole-number values that made up the factor triples. For example, our teachers suggested 4, 6, and 10; or 5, 6, and 8. The *referents*—that is, the objects that the whole-number triples referred to—were the dimensions of the aquarium.

In response to the SMP 2 directive to problem solvers to contextualize their solutions, teachers suggested solutions such as a 4 ft. × 6 ft. × 10 ft. aquarium, or a 5 ft. × 6 ft. × 8 ft. aquarium. Jaime, a fifth-grade teacher, found the triple 1, 15, and 16; but in the process of contextualizing, she recognized that this was not a contextually good solution, because such a triple would not make a reasonable shape of aquarium. In this way, Jamie was "attending to the meaning of the quantities" and making sure she had created "a coherent representation of the problem" (p. 6).

Constructing meaning: SMP 4

Model with mathematics.
Mathematically proficient students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace.

Teacher perspectives from the institute

On the first day of the summer institute, teachers engaged in a task called Arrange Six (Erickson 1995), in which they used six cubes to solve a puzzle. On completion, we asked teachers to reflect on how this task might support students' engagement in one or more of the SMPs. The conversation that ensued suggested that some of the teachers interpreted the use of manipulatives (in general) as engagement in SMP 4. Others in the room perceived SMP 4 differently. The discussion led the group to make a distinction between *modeling mathematics* (i.e., using concrete materials or other visual representations to clarify or give meaning to mathematics) and *modeling with mathematics* (i.e., using mathematics to describe and/or explain a real-world context). We share two examples to illustrate this distinction.

Modeling mathematics

With the introduction of the National Council of Teachers of Mathematics Representation Standard (NCTM 2000), teachers began to frequently support students in *modeling mathematics* through multiple representations and the use of manipulatives, pictures, graphs, and so on. As

FIGURE 1

To solve the task, teachers decided to ignore the contextual details of the Aquarium task and move to an abstract setting, thus engaging in decontextualization.

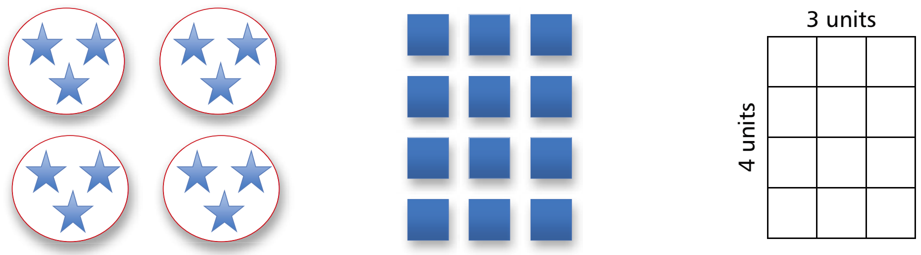
Cari is the lead architect for the city's new aquarium. All the tanks in the aquarium will be rectangular prisms where the side lengths are whole numbers. . . . Cari knows that a certain species of fish needs at least 240 cubic feet of water in [its] tank. Create three separate tanks that hold exactly 240 cubic feet of water.

Source: http://www.illustrativemathematics.org/illustrations/1308

FIGURE 2

By using three different ways of modeling 4 × 3, students are more likely to deepen their understanding of the mathematical concept of multiplication.

Draw a picture that can be represented by 4 × 3.



Equal-size groups model
4 groups of 3 stars

Array model
Four rows with three squares in each row

Area model
Rectangle with dimensions 4 units and 3 units—its area represents the product of the dimensions

an example, suppose you asked your students to draw a picture to represent the multiplication expression 4 × 3. You might receive three different representations (see **fig. 2**). Each of these representations provides a different way of modeling the mathematical concept of multiplication. By modeling mathematical ideas with multiple representations, students are more likely to deepen their understanding of the concepts at hand (NCTM 2000).

Modeling *with* mathematics

Our teachers agreed that the intent of SMP 4 was to describe *modeling* with *mathematics*, which, in contrast to *modeling mathematics*, calls on students to "apply the mathematics they know to solve problems arising in everyday life, society, and the workplace" (CCSSI 2010, p. 7). CCSSM writers indicate that asking elementary school students to represent a problem situation with an equation is an example of modeling with mathematics. However, this statement did not fully support our teachers in understanding the essence of the standard. To this end, we offer the Acrobat task (Burns 1996). We encourage you to examine the task in its entirety, but it can be summarized (see **fig. 3**). One key assumption stated in the full version of the problem is that all acrobats are of equal strength. The same is true for the grandmas.

This task has two particular characteristics that our teachers found relevant for engaging students in the practice of modeling with mathematics. First, the task is not at first glance "mathematical" in nature but can be usefully simplified through mathematical representation. Second, the problem situation requires students to determine which variables are important to keep for the model (e.g., the number of equal-strength grandmas) and which variables are less relevant to the model (e.g., the age of the grandmas). A few weeks before the institute, our teachers observed this task being implemented with fifth-grade students during a demonstration lesson. Our teachers found that students did not immediately think to represent

FIGURE 3

The task below is not overtly "mathematical," and it requires that problem solvers determine which variables are relevant.

The Acrobat task

In round 1 of a tug-of-war, four acrobats tied with five grandmas.

In round 2 of a tug-of-war, one dog (Ivan) tied with two grandmas and an acrobat.

In round 3 of a tug-of-war, if three grandmas and the dog pull against four acrobats, who will win?

this problem with equations or inequalities and that they often used such reasoning as “the grandmas are old” as a means of determining who would win.

Upon analysis of the student work of three fifth-grade pairs, our teachers recognized the various tools for modeling with mathematics that students used in solving this problem. In the first solution (see **fig. 4**), students used diagrams to map relationships from the problem situation in the form of an equation. In the second (see **fig. 5**), students represented relevant relationships through the use of an inequality, using coefficients and variables to describe the problem situation. In the final solution (see **fig. 6**), students used variables in a way that highlighted their mathematical justification. They linked their representation of a given piece of information (i.e., that five grandmas were equal in strength to four acrobats) and the representation of the final round of tug-of-war to draw conclusions about why their narrative makes sense.

Our teachers noticed that in all three sample solutions, students engaged in modeling with mathematics, whether through the use of equations and variables (see **figs**. **2** and **3**) or through the use of diagrams (see **fig. 1**). Moreover, teachers recognized in the sample work how students were able to “identify important quantities” and “simplify a complicated situation” (CCSSI 2010, p. 7), two important

FIGURE 4

In this solution for the Acrobat task, students used diagrams to map relationships from the problem situation in the form of an equation.

We think the left side won because two grandmas and a acrobat equals 1 dog in strength. Left side ~~would~~ be 5 grandmas and 1 acrobat in strength. And in the 1st round it only took 5 grandmas to match 4 acrobats. So if you add more to it the left side would win. Because the dog vs. the two grandmas and the acrobat had a tie in the 2nd round. Left side would win.

FIGURE 5

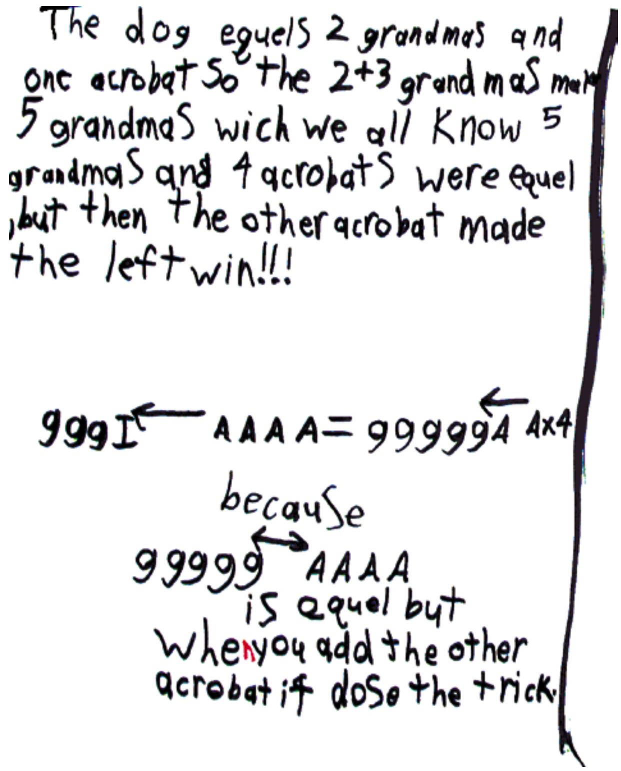
In the second solution for the Acrobat task, students represented relevant relationships with an inequality, coefficients, and variables.

The 3 grandmas and Ivan will win because in the first round five grandmas and four acrobats were equal in strength ~~and~~ But in round 2 Ivan was equal to 2 grandmas and one acrobat. So in the final round Ivan which is equal to two grandmas and 1 acrobat and 3 grandmas has greater strength than the for acrobats.

Ivan = 2 G 1 A + 3 G > ~~for~~ four acrobats

FIGURE 6

Student solution 3 for the Acrobat task shows how students used variables to highlight their mathematical justification, linking their representation of a given piece of information and the representation of the final round of tug-of-war to draw conclusions.



processes that are characteristic of modeling with mathematics.

Constructing meaning

SMP 7: Look for and make use of structure. Mathematically proficient students look closely to discern a pattern or structure. SMP 8: Look for and express regularity in repeated reasoning. Mathematically proficient students notice if calculations are repeated, and look both for general methods and for shortcuts.

Teacher perspectives from the institute

During teachers' discussions, one question that emerged on multiple days of the institute was how to make a distinction between SMP 7 and SMP 8. Thomas, a sixth-grade teacher, summarized the issue: "SMP 7 and 8 cross over, and I am not completely sure about the dividing point between the two." Teachers viewed both of these standards as focused on the idea of identifying structure and patterns when engaging in mathematics, and they sought to pinpoint the distinction between them.

In one discussion, third-grade teacher Linda shared her developing understanding of the distinction, suggesting that SMP 7 refers to the exploitation of structure *within* a given problem and that SMP 8 refers to the exploitation of structure *across* different problems. Our teachers found this distinction helpful for their own constructed meaning of these two standards. To clarify what our teachers meant by exploitation of structure *within* a given problem versus exploitation of structure *across* different problems, we offer the following example that one of our fourth-grade teachers used in the classroom after the summer institute.

SMP 7, exploitation of structure within a problem

Consider the Squares task (see **fig. 7**). When presented with this problem, students initially counted the small 1×1 squares and concluded that the solution is nine total squares. Other students, though, began to notice other squares. For instance, the outer boundary of the figure is a square, and perceptive students also noticed that the figure has 2×2 squares.

How, then, should the squares be counted? Given the fact that the various squares may overlap, counting by simply looking is no longer easy. Having some way to keep track of exactly *which* squares have been counted is necessary.

Students began to "discern a pattern or structure" (CCSSI 2010, p. 8) within the context of this problem. They grouped the squares into

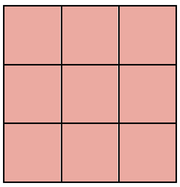
Participants questioned the nuanced differences between SMP 7 and SMP 8.

FIGURE 7

Working on the Squares task, perceptive students noticed more squares than the original 1 × 1 squares.

The Squares task

How many total squares can you find in the drawing below?



three types: the 1 × 1 squares, the 2 × 2 squares, and the 3 × 3 square. In this way, they concluded that the total number of squares is 9 + 4 + 1, or 14. This is an example of SMP 7: “Look for and make use of structure” (p. 8).

SMP 8, exploitation of structure across problems

After students had been given the opportunity to “look for and make use of structure” (SMP 7) within the original Squares task, their teacher wanted them to use their observations from that task to “look for and express regularity in reasoning” across other similar tasks. The teacher gave students a 4 × 4 square and a 5 × 5 square (see **fig. 8**).

Students began with the 4 × 4 picture. Because of their experience with the previous task, students were quick to notice the existence of 2 × 2 and 3 × 3 squares in the 4 × 4 picture. Also because of their prior experience, they were able to categorize the squares by size and count the total number of squares. They found that the 4 × 4 picture had 16 + 9 + 4 + 1 squares. Their solution process, therefore, was more efficient because they looked for and noticed “repeated reasoning” as in SMP 8 (CCSSI 2010, p. 8).

As the discussion proceeded, students continued to rely on structure, patterns, and repeated reasoning. Some students noticed that 9 + 4 + 1 appeared in the solution to the 3 × 3 problem as well as the solution to the 4 × 4 problem. Other students recognized that 16, 9, 4, and 1 are all perfect squares. The combination of these observations led students to conjecture that the 5 × 5 square would give them 25 + 16 + 9 + 4 + 1 total squares. This conjecture was readily verified, as by now students had already twice applied their plan of “group the squares by size.”

We see that students exploited structure across problems in two ways. First, they used their earlier plan of grouping squares by size to quickly count the total squares in the 4 × 4 picture. Second, they noticed a numerical pattern in 1 + 4 + 9 + 16, which allowed them to conjecture an answer for the 5 × 5 square without even looking at the figure. These are examples of “look[ing] for general methods and for short-cuts” and “express[ing] regularity in repeated reasoning” (CCSSI 2010, p. 8).

Final thoughts

Throughout our summer institute, teachers recognized the complexity of the SMPs and that a cursory reading of these standards was inadequate. They saw this complexity as relating to the idea that many of the practices outlined in the SMPs seem to overlap, and therefore, clear-cut distinctions between them might not always exist. They recognized the value of discussion and contemplation for constructing meaning of the SMPs.

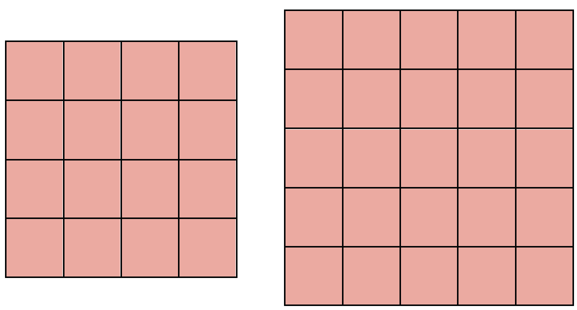
We encourage readers to engage in their own discussions with colleagues about some of the

FIGURE 8

Students were then to use their observations from the Squares task to “look for and express regularity in reasoning” across other similar tasks.

Extending the Squares task

How many total squares can you find in the drawings below?



We encourage readers to engage in discussions with colleagues to contruct meaning of the SMPs.

issues that arise when interpreting the SMPs. The examples we have provided can serve as a starting place for such discussions. In addition, work through further examples to deepen your understanding of the intricacies of these SMPs.

Common Core Connections
SMP 2 SMP 7
SMP 4 SMP 8

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An assistant professor of mathematics education in the Department of Mathematical Sciences at Middle Tennessee State University (MTSU), **Sarah K. Bleiler**, **sarah.bleiler@mtsu.edu**, is interested in students' development of proof-related reasoning and in the dynamics of collaboration between mathematicians and mathematics teacher educators. **Wesley A. Baxter**, **wab2n@mtmail.mtsu.edu**, is a Ph.D. candidate in the MTSU Mathematics and Science Education program. He aims to understand more about teachers' interpretations of CCSSM as well as the obstacles teachers face as they attempt to implement these new standards and practices. **D. Christopher Stephens**, **chris.stephens@mtsu.edu**, is an associate professor of mathematics in the MTSU Department of Mathematical Sciences. Along with his research in pure mathematics, he is interested in how community understandings of the nature of mathematics align with community expectations for how mathematics is taught. **Angela T. Barlow**, **angela.barlow@mtsu.edu**, is the director of the Mathematics and Science Education Ph.D. program at MTSU. Her interest lies in developing teacher beliefs that align with standards-based instruction.

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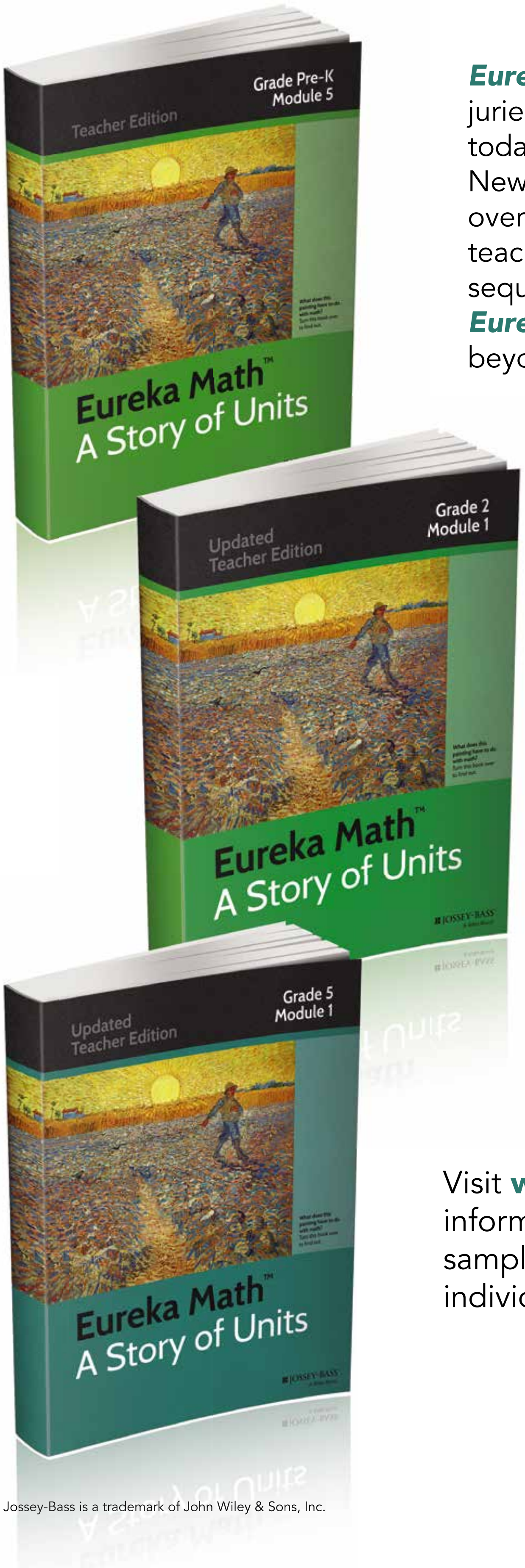

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