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“I Don’t Really Know How I Did That!”

By R. Scott Eberle

Geometric tiling’s mathematical aesthetics yield a rich, motivating topic for open-ended mathematical exploration.

I asked fourth grader Kobe if he could tile a floor with triangle pattern blocks. Kobe first sketched with paper and pencil how he could put together little triangles to make larger triangles. He noticed that hexagons and other shapes appeared inside the pattern (see **fig. 1**). When I asked him to make the tiling with pattern blocks, he carefully arranged the first triangles around a common vertex. As he placed the sixth triangle, a perfect hexagon appeared. Kobe paused at the unexpected shape. “I don’t really know how I did that!” he commented.

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FIGURE 1

Kobe's drawing showed how he could put triangles together to make larger and larger triangles to cover the floor. He noticed hexagons and other shapes hidden in these patterns.

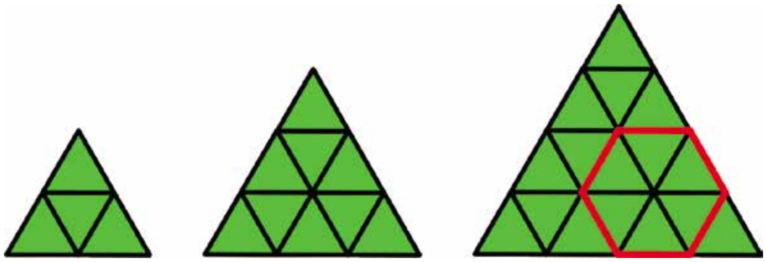


FIGURE 2

Most children are familiar with real-world tilings.

(a) This square pattern is commonly found on indoor floors.

(b) Another typical tiling example is in pavement that uses octagons and squares.

Although Kobe knew that hexagon patterns appear within larger tiling patterns, it had not occurred to him that he could make a hexagon by itself. Such is the excitement of discovery as students are allowed the freedom to explore the infinite possibilities of geometric tilings, or tessellations. This article examines how to organize open-ended tessellation activities in a way that supports the mathematical practices and concepts that we want students to learn for geometry. We will also look at the frequently overlooked importance of paying attention to what children find aesthetic in such activities.

What is a tessellation?

A *tessellation* is a pattern of geometric shapes that fulfills three conditions:

1. No gaps exist between the shapes.
2. The shapes do not overlap.
3. The pattern can go on forever in all directions.

Tilings (see **fig. 2**) are familiar to children and provide a rich resource at all grade levels for learning many different geometric concepts, such as angles and symmetry.

People have been making geometric tessellations since antiquity. Their artistic appeal creates a natural motivation to explore a wealth of geometric ideas as well as patterns that are ever more complex. Surprisingly, the mathematics behind these patterns has been studied for only a little more than a century. Mathematicians are still exploring many questions about tessellations today. New discoveries are still being made, sometimes even by schoolchildren.

The National Council of Teachers of Mathematics (NCTM 2006) recommends that students study tessellations in fourth grade, but the topic is appropriate at all grade levels. Students find the infinite aspect of tessellations challenging because tessellations have two dimensions. It is one thing to make a pattern that continues forever in one direction but quite another to work with two dimensions simultaneously. Children may have to grapple with the concepts of symmetry and transformations. For example, Marie made a tiling from an L-shape tile by randomly placing the tiles (see **fig. 3a**). Rachel created her tessellation by putting the tiles together in pairs to make

a rectangular unit (see **fig. 3b**). This type of unit thinking is important in both arithmetic and geometry (Wheatley and Reynolds 1996) and is included in the Common Core's (CCSSI 2010) Standards for Mathematical Practice (SMP). SMP 3 calls for students to make viable arguments, which includes inductive reasoning (such as Marie used in noting that her tiling seemed to be working so far) and deductive reasoning (such as Rachel used to argue that her tiling worked because her units kept repeating).

For the simplest tessellation activity, show students a geometric shape, such as a pattern block, and ask if it is possible to tile a floor with such a shape. Class discussion will usually conclude that the pattern must have no gaps or overlaps (the first two conditions of the definition). To create a true tessellation, children must also imagine that they are tiling a floor that goes on forever. Some children may insist there are walls *somewhere*—perhaps at the paper's edge—but challenge them to consider how the pattern continues beyond.

Exploring tessellations

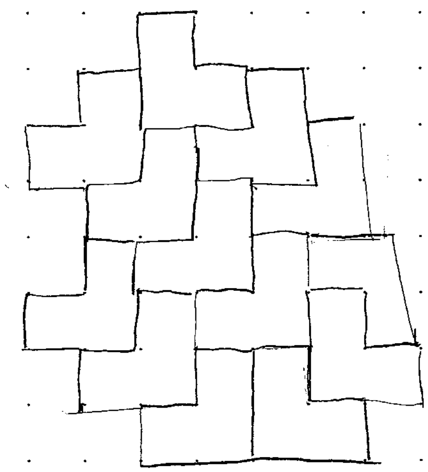
Investigating tiling patterns in a variety of ways is an important task for children. Here are some suggestions for prompts to guide students:

- Start by asking students to sketch or verbally describe—without using manipulatives—how they imagine a tiling will work. Later, when students use physical blocks, the blocks will naturally force mathematical structure even without students understanding the structure, as in the case of Kobe's hexagon. Understanding structure is important (SMP 7). Students' successful tilings with physical blocks or computer software may give the impression that students understand more about the tiling's structure than they actually do. Drawing serves as better formative assessment. For example, before third grade, most students do not understand how squares align in a tiling. But even young children can make a square array with plastic tiles because the tiles naturally line up in rows and columns by themselves (Outhred and Mitchelmore 2000). Kelsey had no trouble making a square array with pattern blocks, but her

FIGURE 3

Two students created tilings from an L-shape tile by placing the tiles at random.

(a) Marie's was a valid tessellation, but she had difficulty justifying how she knew it would cover the plane without leaving any holes.



(b) To cover the plane with a repeating pattern, Rachel found a way to make simple rectangular units from the same L-shape.

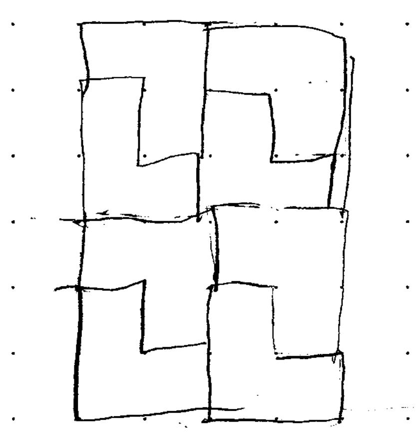


FIGURE 4

Kelsey believed that a rectangle could be tiled with squares like this, an indication that she did not yet grasp the row-and-column structure of a square array.

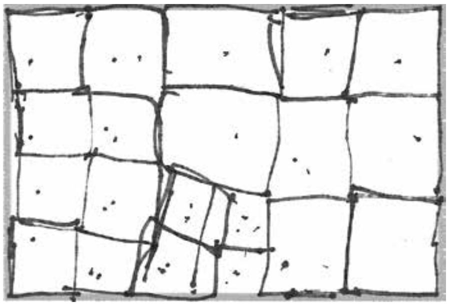
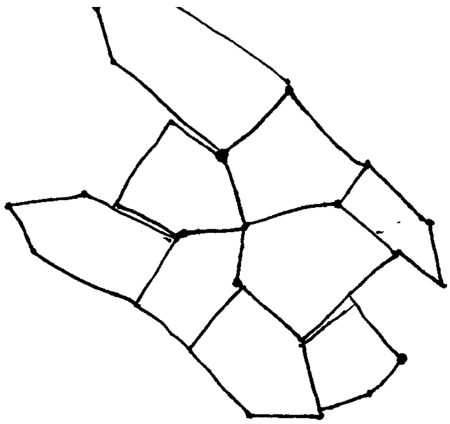


FIGURE 5

Certain that any tile could make a tessellation, Kobe sketched how he thought regular pentagons would fit together. When he tried with plastic pentagons, he discovered that they did not fit together the way he had anticipated.



initial drawing (see **fig. 4**) revealed a lack of understanding of the row-and-column structure. Drawing also prepares students for surprises when the tiling works out differently. After making a tiling with actual tiles, students can record it by carefully sketching again—a good way for them to see the importance of mathematical precision (SMP 6). Students also need experience to learn what conclusions can and cannot be made from a rough sketch (see **fig. 5**).

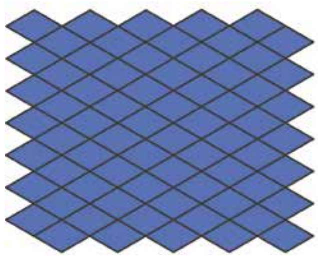
- Use a variety of media. Each tool that students use elicits different types of thinking. What students fail to notice with one type of tool, they may pay attention to when using another. Possible tools for creating tessellations include blank paper, rulers, protractors, dot paper, pattern blocks, computer programs, and construction paper.

- Ask students if more than one way exists to tile with a given shape. Some shapes, such as the yellow hexagon pattern block, can be used to tile in only one way. But the blue rhombus pattern block can tile in an infinite number of ways (see **fig. 6**). Students must understand that a mathematical task often has many solutions. To make solid arguments in favor of a tiling possibility, SMP 3 will require students who make random tilings to discover repeating tilings. On the other hand, random tilings can show that not all tilings have symmetry. Finding such counterexamples is also an important form

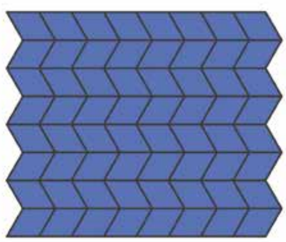
FIGURE 6

Shown below are three—of an endless number—of popular and valid ways for children to tile with a rhombus pattern block.

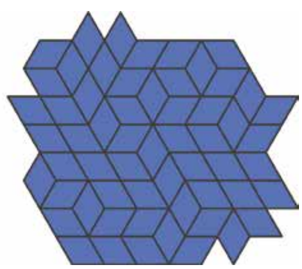
(a) A simple tiling



(b) A tiling with alternating rows



(c) A tiling with randomly turned tiles



of mathematical reasoning. In this case, a random tiling answers the question "Must all tessellations be symmetrical?" in the negative.

- Ask students if they can use different triangles and quadrilaterals to tile a floor. Can they use a scalene triangle or a trapezoid? In fact, any type of triangle or quadrilateral will work. Furthermore, each type of triangle—scalene, right, isosceles, and so on—has several different ways of tiling. Some tilings will highlight the special properties of the triangle. For example, students may tile with a right triangle by putting two together to make a rectangle unit (see **fig. 7**). In doing so, they discover that the other two angles combine to make a right angle—a special case of the fact that the sum of the angles in a triangle is 180 degrees. By measuring or by noticing that a quadrilateral can be decomposed into two triangles, older students can learn that the sum of the angles in a quadrilateral is always 360 degrees. The fact that the angles add up to 180 degrees or 360 degrees is one of the reasons that triangles and quadrilaterals can tile: In any tessellation, the sum of the angles around every vertex must add to a full turn (360 degrees), which can be done by using each of the angles in a triangle twice or by using all four angles of a quadrilateral once around each vertex (see **fig. 8**). This is an example of Common Core content standard 4.MD.7, an important principle that fourth graders will not immediately grasp in this context because they will focus on the holistic shapes rather than the angles. (Note also that just because certain angles of a polygon add to 360 degrees does not guarantee that a tiling is possible. In fact, no general test exists for knowing if a polygon can make a tessellation. All triangles and quadrilaterals will work, but mathematicians have yet to find a rule for knowing which pentagons will work. Students may enjoy knowing that they are exploring territory that mathematicians have not yet conquered.)

- Challenge students to make tilings using several different types of polygons at the

FIGURE 7

One way to tile with right triangles is to combine them in pairs, making rectangular units. From this structure, students can see how the two acute angles, *A* and *B*, add to make a right angle.

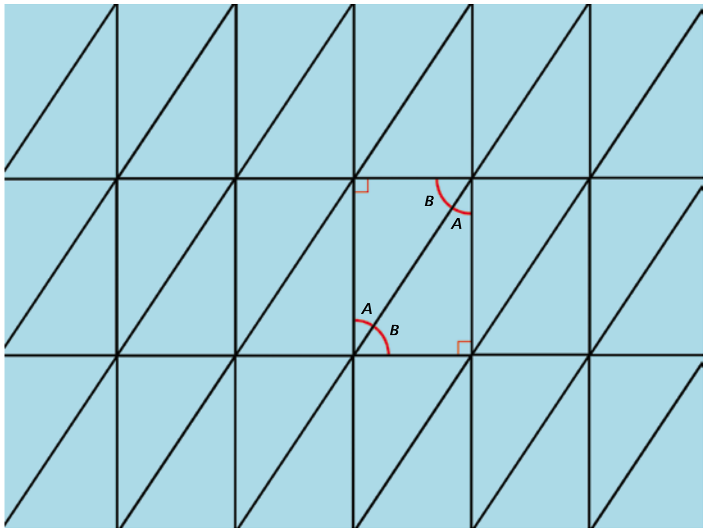


FIGURE 8

Any quadrilateral can be used to create a tiling. The sum of the angles of a quadrilateral is always 360 degrees. All four angles come together at every vertex to make a full 360-degree turn without leaving any gaps.

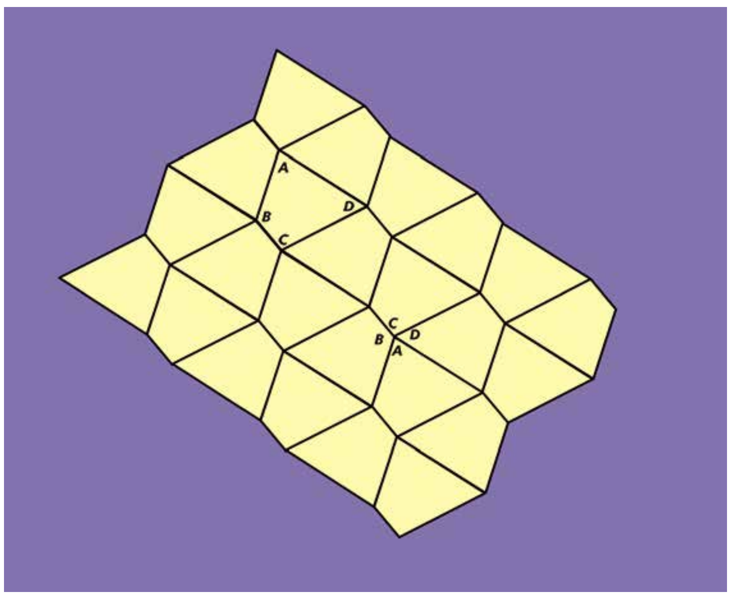


FIGURE 9

Tessellations can be made by combining more than one shape. The patterns are considered to extend forever in all directions by repeating the same units over and over.

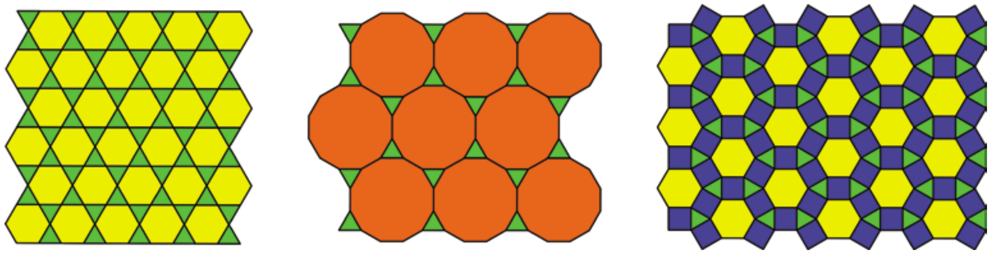
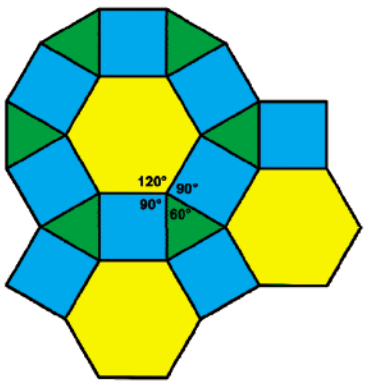


FIGURE 10

Students can verify that the angles around every vertex add to 360 degrees.



same time. Many fascinating tilings can be made with *regular polygons* (polygons where all sides and angles are equal) (see **fig. 9**). NCTM Illuminations has an excellent tessellation-creator online activity for guiding children to explore such tessellations (http://illuminations.nctm.org/ActivityDetail.aspx?ID=202). Even limiting tilings to just squares and triangles offers an infinite number of possibilities, some simple and some complex. These more complex tessellations give richer opportunities for exploring symmetry (CCSSM content standard 4.G.3). Challenge older students to study how the angles in regular polygons add up around each type of vertex (see **fig. 10**).

- Encourage students to defend the validity of their tilings (SMP 3). Some students who previously agreed that tilings should not have gaps may change their minds when they struggle with complicated tilings where small gaps appear. This provides opportunities for whole-class discussions about mathematical validity and precision as well as whether mathematical definitions are merely guidelines or hard-and-fast rules. Is it really a proper tessellation if it has a few small gaps? Is it possible to restructure a tiling so the gaps go away? Many students are surprised to discover that regular pentagons cannot make a tiling. No matter how you try to put them together, gaps appear (see **fig. 11a**). Because most students focus on holistic shapes instead of analyzing properties, they will keep trying. They will need help to see that the angles of the regular pentagon are all the same (108 degrees) and cannot add up to 360 degrees. What about other types of pentagons? If the angles are just right, some pentagons may work (see **fig. 11b**).

- Asking students to describe one of their tessellations in such a way that another student can re-create it without seeing it offers a challenging activity that involves using geometric vocabulary to communicate with precision (SMP 6).

- Have students analyze and compare their tessellations. Second-grade students learn that square tilings have a precise mathematical structure (CCSSM content standard 2.G.2). Older students learn that all parallelograms, including rhombi and rectangles, can be tiled with this same row-and-column

structure. This knowledge can enrich their understanding of the shared attributes of these special quadrilaterals (CCSSM content standards 3.G.1 and 5.G.3).

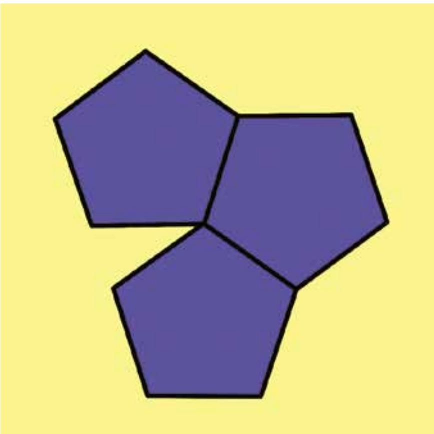
- When students make a variety of tessellations, ask them which one is the best. This may seem subjective, but as we will see below, aesthetic arguments are important for not only motivating students but also revealing what students are focusing on. One way to determine which tessellation is best is to have a contest to judge all the tilings that students have created. If three prizes are awarded in each category, every child can possibly win a prize in at least one tiling category. Aesthetic categories can include such popular ones as best color or best for a kitchen tiling, as well as more mathematical categories, such as most complex, best use of shape, best three-dimensional illusion, best use of symmetry, most symmetrical, most surprising, and most creative.

- Students might also enjoy studying professional tilings, such as those made by Dutch artist M. C. Escher. Discussion of tilings like the one in **figure 12** can inspire students

FIGURE 11

When pentagons have angles adding up to 360 degrees, this characteristic alone is no guarantee that a tessellation will work.

(a) When students try to put regular pentagons together, gaps inevitably appear.



(b) Other types of pentagons, such as the "house shape" obtained by putting a square and triangle pattern block together, *can* successfully tile the plane.

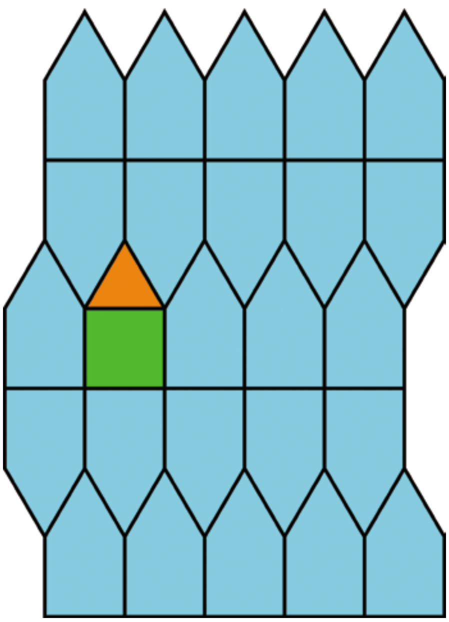


FIGURE 12

This pavement tiling in Hawaii was inspired by the work of artist M. C. Escher, who explored geometric tilings in his many works.

SUSAN EMPSON USED WITH PERMISSION

FIGURE 13

A few of the infinitely many lines of reflection symmetry are marked in red on this tessellation.

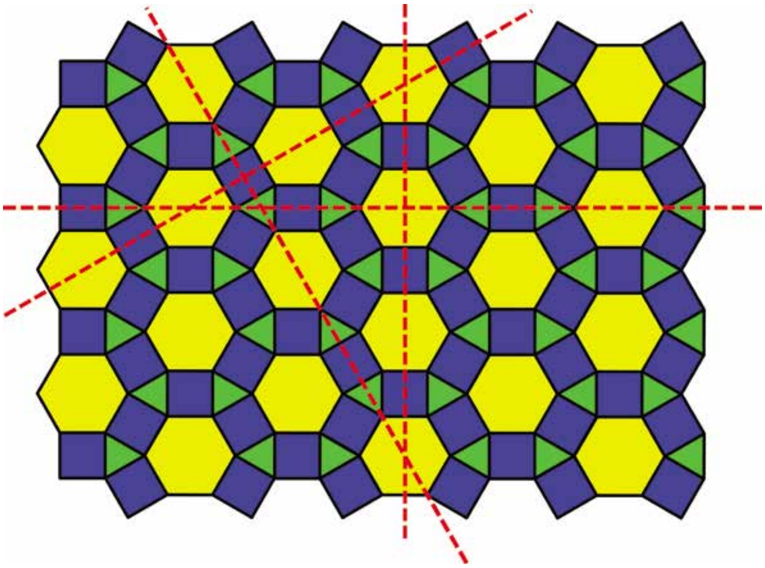
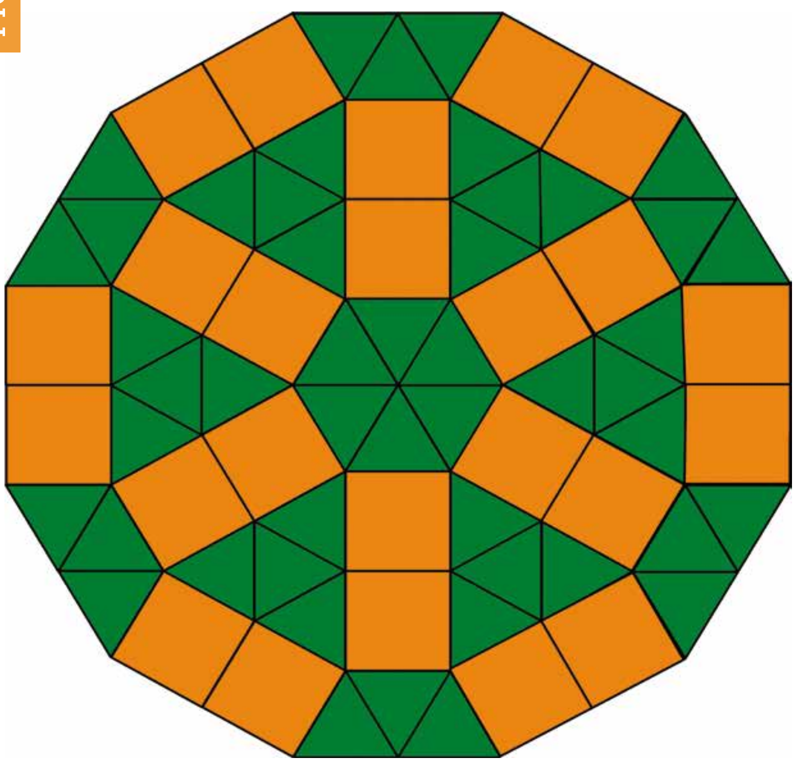


FIGURE 14

Trying something new, Moses expanded radially on a central hexagon of six triangles, beginning a tessellation without a repeating unit but with strong rotational symmetry.



to make increasingly complex tilings as well as elicit dialogue about what makes a tiling beautiful.

Aesthetics

Recent research has found that paying attention to children's mathematical aesthetics is important (Sinclair 2006). Aesthetics is the force that motivates and guides exploration in open-ended inquiry. For tessellations, children tend to get excited when they see connections to the real world, such as when a yellow hexagon pattern reminds them of honeycombs. Children will also say tessellations are "cool" when they have appealing colors or pleasing complexity. Real-world connections and color are not often helpful for seeing important mathematical content, but some other criteria are. Pleasing complexity, for example, is a helpful aesthetic quality because it encourages students to make tessellations that are neither too simple nor too complicated. Such moderately complex tessellations often contain just the right amount of mathematical structure to create interesting insights into geometric concepts.

As students continue to explore and gain familiarity with tessellations, they should begin to pay less attention to real-world connections and color and more attention to complexity and other important mathematical aesthetic criteria, such as symmetry. Students can begin to analyze the symmetry in their patterns by using folding or reflective devices (CCSSM content standard 4.G.3). Most student tessellations contain many lines of reflection symmetry and may also contain rotational symmetry (see **fig. 13**). Some students will use symmetry to help them develop their own tessellations.

Other mathematically important aesthetic criteria that students will occasionally use include validity (satisfying the three conditions), uniqueness (being "creative"), units, and alternation (see **fig. 6b**). Each aesthetic contributes to fruitful inquiry. For example, valuing uniqueness pushes students into ever-newer discoveries, rather than simply rehearsing the same old patterns. With each new tiling, students expand their repertoire of possible patterns and gain new insights into possible geometric structures. Moses had created a variety of tessellations with reflectional symmetry. When challenged to make another tiling with squares and triangles,

he decided to be creative and try a new strategy. He discovered that he could make a tiling with radial structure (see **fig. 14**). This type of tiling can lead to a discussion of rotational symmetry.

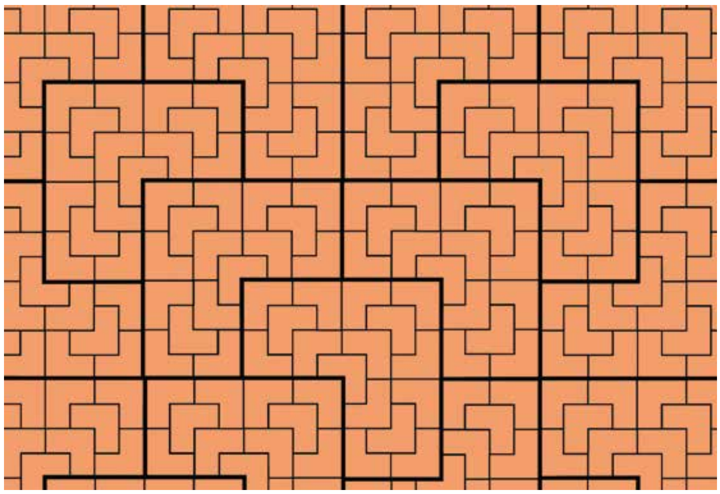
Mathematicians also value aesthetics in their work. Many mathematicians claim that surprising results and rich connections are important aesthetic criteria. Children do not always value surprise and connectedness in their mathematical explorations, so the teacher may need to model and encourage these aesthetic criteria. When Moses noticed a surprising tiling (see **fig. 15**), I asked the other students to stop and listen to his surprising discovery, and they also started to get excited about the unique mathematical structure.

All of these aesthetic criteria help guide children to make richer tessellations and to focus on the mathematical structure of the patterns they discover. Teachers will notice if students are gradually moving away from an unhelpful focus on color and real-world objects to a focus on richer mathematical aesthetic criteria, such as complexity, symmetry, validity, uniqueness, units, surprise, and connections.

Tiling provides one of the richest possible studies of geometric concepts as well as a powerful topic for open-ended mathematical exploration, fostering important mathematical practices. For students, mathematical aesthetics can be a guiding force for mathematical inquiry, just as it is for mathematicians.

FIGURE 15

With their teacher's prompting, classmates of Moses became excited about his surprising discovery: The small L-shape tiles made larger L-units, which combined to make still larger L-shapes, and so on.



Common Core Connections

K.G.6	3.G.1	SMP 3
1.G.2	4.G.3	SMP 6
2.G.2	5.G.3	SMP 7
	6.G.1	

REFERENCES

Common Core State Standards Initiative (CCSSI). 2010. Common Core State Standards for Mathematics (CCSSM). Washington, DC: National Governors Association Center for Best Practices and the Council of Chief State School Officers. http://www.corestandards.org/wp-content/uploads/Math_Standards.pdf

National Council of Teachers of Mathematics (NCTM). 2006. *Curriculum Focal Points for Prekindergarten through Grade 8 Mathematics: A Quest for Coherence*. Reston, VA: NCTM.

Outhred, Lynne N., and Michael C. Mitchelmore. 2000. "Young Children's Intuitive Understanding of Rectangular Area Measurement." *Journal for Research in Mathematics Education* 31 (2): 144–67. doi:http://dx.doi.org/10.2307/749749

Sinclair, Nathalie. 2006. *Mathematics and Beauty: Aesthetic Approaches to Teaching Children*. Foreword by William Higginson. New York: Teachers College Press.

Wheatley, Grayson H., and Anne Reynolds. 1996. "The Construction of Abstract Units in Geometric and Numeric Settings." *Educational Studies in Mathematics* 30 (1): 67–83. doi:http://dx.doi.org/10.1007/BF00163753

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