Seeing Parabolas Problem

The Common Core State Standards for Mathematics (CCSSM) include:

Translate between the geometric description and the equation for a conic section
2. Derive the equation of a parabola given a focus and directrix.
3. (+) Derive the equations of ellipses and hyperbolas given the foci, using the fact that the sum or difference of distances from the foci is constant.

We might use the Slicing a Double Cone custom app to develop underlying images for students to connect the focus–directrix relationship underlying standard #2. The general gist is to use the app and then graphs and sliders to associate multiple views of parabolas (i.e., plane slicing cone, graphs of quadratic functions), and then to develop the equation using the focus–directrix definition to emphasize that what makes the figure a parabola is more than just what it “looks like.” Students encounter, for example, the focus–directrix definition and the plane–cone interaction definition of a parabola and see their relative usefulness in different contexts.

Note: This lesson illustrates the use of teacher knowledge of geometry such as that promoted by the book Developing Essential Understanding: Geometry 9–12 through—
"Big Idea 3. Working with and on definitions is central to geometry."

In particular, the lesson draws on:
"Essential Understanding 3a. Geometric objects can have different definitions. Some are better than others, and their worth depends both on context and values."
"Essential Understanding 3b. Definitions in geometry are of two distinct types: definition by genesis (how you can create the object) and definition by property (how you can characterize the object in terms of certain features)."
"Essential Understanding 3c. Building definitions requires moving back and forth between the verbal and the visual."

Start with the plane intersecting the cone at a particular angle to a generating line gives us a parabola. In the app, angles of -45, -46, and -44 degrees illustrate the differences:
How would you define a parabola in terms of how it can be created by the intersection of a plane and a cone?

Once we have the angle that matches the generating line's angle with the axis, we can "slide" the plane up and down the axis to produce a family of parabolas. Sliding the plane shows how the parabola could be “narrower” or “wider,” including the degenerate case of a line.
We can produce a similar family of parabolas using the graphing tool:

**Command:**

\[ y = ax^2 \]

**Graph:**

\[ y = 0.5x^2 \]
Another way to think about parabolas requires identifying a particular point—called the *focus*—and a line—called the *directrix*—for each parabola. A parabola is the set of points that are equally distant from the focus and the directrix. Develop a synthetic sketch starting with a line and point (and then point on line, perpendicular to the line at that point, perpendicular bisector of segment joining that point and the focus, intersection of the two constructed perpendiculars):

Drag the point on the directrix (*D* in this case) to show how the final intersection point (*G* in this case) traces a parabola. To “see” the parabola emerge more easily, it helps to hide all but the directrix, focus, and the final intersection point (see below); projecting the image on a whiteboard and using a marker to mark various locations of the final intersection point can help students to “see” the parabola.
Make connections between this diagram and the other two images of parabolas. Some examples are:

- Associate the horizontal directrix with the images in the graph.
- Note that the focus is not the vertex of the cone.

Develop equations based on the focus-directrix definition.

Connect back to earlier observations. For example, use the equation and Synthetic tool sketch to make sense of when and why the line can happen.