Core Math Tools Lesson

Title: An Interactive Introduction to Transformational Geometry

Goal

In this lesson, students explore properties of three congruence transformations of the plane: translations, line reflections, and rotations. This lesson is meant to be an informal introduction to these transformations as functions that map the plane onto the plane and to key properties that define each of the three transformations.

The Common Core State Standards for Mathematics (CCSSM) include:

**Geometry–Congruence**
2. Represent transformations in the plane using, e.g., transparencies and geometry software; describe transformations as functions that take points in the plane as inputs and give other points as outputs.
4. Develop definitions of rotations, reflections, and translations in terms of angles, circles, perpendicular lines, parallel lines, and line segments.

**Mathematical Practices**
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
8. Look for and express regularity in repeated reasoning.

Supporting NCTM Materials

*Developing Essential Understanding of Geometry for Teaching Mathematics in Grades 9–12*

Lesson Outline

This investigation is designed to give students a general introduction to three types of geometric transformations: translation, line reflection, and rotation. The materials include a four-page student investigation sheet.

*Note:* In various venues, students might encounter translation, reflection, and rotation as three types of transformations in general and as three of types of isometries in particular. Two points might need to be clear. Translation, reflection, and rotation are not the only types of isometries. Glide reflection is the fourth type of isometry. In addition, “reflection” in this setting is *line reflection* and not another kind of reflection, such as point reflection. Although
line reflection and reflection might be used interchangeably later in this lesson, the focus on line reflections should be clear from the beginning of the lesson.

You may wish to have the students work in pairs, in which one student uses the Interactive Geometry Tools and the other records their conclusions. If students are working in pairs, you may wish to have them reverse roles as they move from one type of transformation to the next.

**Part A. Getting Started**

Part A is designed to help students become familiar with the Interactive Geometry Tools. If students have already used these tools, this part could be omitted.

The first item provides directions and practice in how to create a scalene triangle. The second task guides students through procedures to measure lengths and angles. Students might produce something like the following:

\[
\begin{align*}
CA &= 9.69 & m\angle ABC &= 111.12 \\
AB &= 6.69 & m\angle BCA &= 40.11 \\
BC &= 5 & m\angle CAB &= 28.77
\end{align*}
\]

In the parts of the lesson that follow, students create the image of a scalene triangle using translations, reflections, and rotations. For each type of transformation, students observe relationships between the original triangle (pre-image) and its images. One goal is for them to use the dynamic environment to produce examples and notice what is invariant across any transformation of the given type and then use those invariant elements to produce definitions. In subsequent lessons, they might formally justify their observations and refine their definitions.

**Part B. Translations**

In this part, students explore the effects of translating a triangle. They begin with a scalene triangle, and then create a translation vector. To create the vector, they need to use the blue vector tool ( ), not the black ray tool ( ). After clicking the icon, they need to press and drag to form the vector. Simply drawing two points will not work. After they have created the triangle (pre-image) and its image under the translation given by the vector, the screen might look something like the following:
If students have difficulty in describing the two measurements that appear by the vector (see item #3), you might encourage them to drag objects, including those that would move the vector to a new location.

*Note:* Vectors might be new to students. Students might benefit from conversation about how vectors embody both direction and magnitude. In particular, students might not distinguish between rays and vectors; vectors can have finite length but rays do not, although both vectors and rays are represented by “arrows” of finite length. In addition, vectors can also be represented as ordered pairs. For example, a translation that maps the point (1, 2) to the point (3,5) is a translation represented by the vector (1,3). In later lessons, students might use Core Math Tools to explore transformations using matrix representations of them.

*Note:* This lesson illustrates well the centrality of variance and invariance in geometry and draws on teacher understanding of the role of invariance in geometry, as emphasized in the book, *Developing Essential Understanding of Geometry for Teaching Mathematics in Grades 9–12* through

“**Big Idea 2.** Geometry is about working with variance and invariance, despite appearing to be about theorems.”

In particular, the lesson draws on:

“**Essential Understanding 2a.** Underlying any geometric theorem is an invariance - something that does not change while something else does.”

“**Essential Understanding 2b.** Invariances are rare and can be appreciated only when they emerge out of much greater variation.

“**Essential Understanding 2c.** Examining the possible variations of an invariant situation can lead to new conjectures and theorems.”

While stating observations in item #5, students might benefit from dragging the points around and make as many observations as possible. The students might also drag a side of the triangle or the entire triangle. You might want to bring the class together or otherwise gather students’ observations, which may be in the form of rough descriptions, such as:

- The figure moved to a new location without turning.
- The figure moved in the same direction as the vector.
- The two triangles are the same size.

After students add segments and measures as directed in items #5 and #6, ask them to suggest any additional observations they may have; observations may include:

- The distances from the pre-image vertices to their images are all the same, which is the same as the length of the vector.
• The image of each side is parallel to its pre-image.
• The sides and angles of the image have the same measures as the pre-image.

Questions to ask to encourage students to articulate the invariances or properties that they observe and to build on these ideas to formulate definitions:

➔ How are the pre-image, image, and vector related? (When the pre-image is a triangle, the image is congruent to the pre-image. When the pre-image is a point or a triangle, the image is located at a distance and direction from the pre-image that matches the magnitude and direction of the vector.)

➔ Think about picking a point that is not on the triangle, and translating it using the same vector. What do you think would happen?

Does it matter which point in the plane you pick? Why or why not?

➔ How could we define a translation? (A translation is a transformation that matches a point in the plane [pre-image] with a point in the plane [image] that is a specific distance in a specific direction from the original point. The direction and distance can be represented by a vector.)

Note: Students are asked to develop definitions of translation, and of reflection and rotation later in this lesson. In this introductory lesson, their ideas likely will be incomplete and at best informal definitions. The conversation about proposed definitions would benefit from consideration of:

“Big Idea 3. Working with and on definitions is central to geometry.”
Particularly useful might be:

“Essential Understanding 3a. Geometric objects can have different definitions. Some are better than others, and their worth depends both on context and values.”

“Essential Understanding 3b. Definitions in geometry are of two distinct types: definition by genesis (how you can create the object) and definition by property (how you can characterize the object in terms of certain features.)

Part C. Line Reflections

In this part, students will explore the effects of reflecting a triangle across a line following a similar format to their work with translations. The screen might look as follows:

Students’ initial observations #3 might include:

• The figure is flipped over.
• The image and pre-image are right across from each other.
• The image and pre-image are the same distance from the line.

Examples of additional observations are:
• The segments joining the images and pre-images are perpendicular to the line of reflection.
• The line of reflection bisects the segments (splits them in half). (You may wish to ask students what measurements they could add to be sure that is correct.)
• The sides and angles of the image have the same measures as the pre-image.

Questions to ask could be similar in form to those asked about translations. A description or informal definition of reflection might be:
• A reflection is a transformation that pairs a pre-image, which could be any point in the plane, with an image that is a point on a line through the pre-image that is perpendicular to the given reflection line and is the same distance from the reflection line as the pre-image.

**Part D. Rotations**

Students explore the rotations using a triangle and its images, following the same general process as that used for reflections. The figure might look like the following:

Students’ observations may include:
• The figure is rotated or turned.
• Everything turns around the center point.

They might suggest additional observations such as:
• The segments joining the images and pre-images to the center are the same length.
• The image is the same distance from the center as the pre-image.
• The line of reflection bisects the segments (splits them in half). (You may wish to ask them what measurements they could add to be sure that is correct.)
• The sides and angles of the image have the same measures as the pre-image.

Debriefing on their work with rotations could benefit from a set of questions similar to that used with translations and reflections. In this case, an informal definition might be:
• A rotation is a type of transformation that, for a given angle and center of rotation, assigns an image point to a pre-image point such that the angle formed by the image, pre-image with the center of rotation as its vertex has the same directional measure as the given angle.

Conclusion

Translation, reflection, and rotation can be defined using familiar concepts such as length, distance, perpendicular, and angle measure. Students might be asked to compare properties of the three types of transformation with questions such as:
• Each of the three types of transformations involved distance. Which of the three require directed distance? Why do these require directed distance but the others do not? (Ideas might include use of “distance between” a point and a line for a reflection means it does not matter whether we think about measuring from the line to the point or from the point to the line. Rotation has directed angle measure, which is related to directed distance. Vectors for translations imply directed distance.)
• Angle measure is obviously important for rotations. How is angle measure involved in the properties of reflections and translations? (Some ideas are right angles for perpendicular to reflection line and connecting the direction of a vector with angle measure for a translation.)

Extension

You might pose the challenge of determining which, if any, vector, line of reflection, or angle and center of rotation would have the image of any point be the original point. This could lead to an exploration of the identify transformation and form the basis for a subsequent lesson on inverse transformations. The experience would be important in that it provides students with opportunities to argue why something is not possible and to make important distinctions while justifying their claims. For example, students untangle a difference between symmetry and identity transformation when they note that, although one can find some figure for which a given reflection line maps the figure to itself, one cannot find a reflection line that maps every point of the plane to itself.

Credit:
W. Gary Martin, Auburn University
An Interactive Introduction to Transformational Geometry
Part A. Getting Started

1. Draw a scalene triangle—that is, a triangle, whose sides and angles all have different measures.
   a. Click on the polygon tool in the top menu bar.
   b. Click on the screen to mark the first vertex.
   c. Click on the screen to mark the second vertex.
   d. Click on the screen to mark the third vertex.
   e. Double click on the first vertex to close the polygon.

2. Check that it really is a scalene triangle.
   a. Click on the triangle (not on its points) so that it is outlined in red.

   ![Triangle](image)

   b. Choose Lengths from the Measurements menu at the top of the screen.

   ![Measurements Menu](image)

   c. Now choose Angles from the Measurements menu.
   d. If your triangle is not scalene, click on the arrow icon. Click on and drag the points to relocate them.
An Interactive Introduction to Transformational Geometry
Part B. Translations

1. Your scalene triangle from Part A should be on the screen. Otherwise, repeat #1 and #2 from Part A.

2. We will now draw a vector.
   a. Click on the Vector tool on the top menu bar.
   b. Click where you would like to start your vector. Without releasing the button, drag to the second point to form the vector.

3. Two measurements are given beside the vector. What does each of these mean?
   a. Drag the points that determine the vector in order to better understand.

4. We will now translate our triangle using that vector.
   a. Select the Arrow icon, and then click on the triangle so that it is outlined in red.
   b. Hold down the Shift-key, then click on the vector so it is also highlighted in red.
   c. Now click on the Translate icon on the side bar. An image of the triangle should appear.

5. Compare the triangle and its image. What do you notice? Make as many statements as you can.
   a. Drag the points that determine the vector to help you see what is happening.
   b. You may also wish to drag the vertices of the pre-image (original triangle).

6. Draw segments joining each vertex of the pre-image (original triangle) with its image.
   a. Choose the segment tool at the top. Click on the first endpoint, then without releasing the button, drag to the second endpoint.
   b. Measure the length of each of the segments: Click on the segment so that it is highlighted in red, and then choose Length from the Measurements menu.
   c. What do you notice? Add any new observations to your list.

7. Add measurements for the image of your triangle.
   a. As in Part A, click on the triangle so that it is highlighted in red, and then choose Lengths from the Measurement menu, as well as Angles.
   b. What do you notice? Add any new observations to your list.
An Interactive Introduction to Transformational Geometry
Part C. Line Reflections

1. Your scalene triangle from Part A should be on the screen. Otherwise, repeat #1 and #2 from Part A. You may erase the vector and image from Part B.

2. Now draw an angle.
   a. Choose the line tool at the top. Click on the first point, and then without releasing the button, drag to the second point.

3. We will now reflect our triangle using that line.
   a. Select the Arrow icon, and then click on the triangle so that it is outlined in red.
   b. Hold down the Shift key, and then click on the line so it is also highlighted in red.
   c. Now click on the Reflect icon on the side bar. An image of the triangle should appear.

4. Compare the triangle and its image. What do you notice? Make as many statements as you can.
   a. Drag the points that determine the line of reflection to help you see what is happening.
   b. You may also wish to drag the vertices of the pre-image (original triangle).

5. Draw segments joining each vertex of the pre-image (original triangle) with its image.
   a. Choose the segment tool at the top. Click on the first endpoint, and then without releasing the button, drag to the second endpoint.
   b. Measure the length of each of the segments: Click on the segment so that it is highlighted in red, and then choose Length from the Measurements menu.
   c. What do you notice? Add any new observations to your list.

6. Add measurements for the image of your triangle, as in the previous part.
   a. What do you notice? Add any new observations to your list.
An Interactive Introduction to Transformational Geometry
Part D. Rotations

1. Your scalene triangle from Part A should be on the screen. Otherwise, repeat #1 and #2 from Part A. You may erase the vector and image from Part B and line of reflection and image from Part C.

2. Now draw an angle.
   a. Choose the angle tool at the top. Click one three locations; the center point will be your vertex.
   b. Measure your angle by clicking one of the sides so that the angle is highlighted in red. Then choose Angles from the Measurements menu.
   c. Drag the points determining the angle to see how its

3. We will now rotate our triangle using that angle.
   a. Select the Arrow icon, and then click on the triangle so that it is outlined in red.
   b. Hold down the Shift key, and then click on the angle.
   c. Now click on the Rotate icon on the side bar. An image of the triangle should appear.

4. Compare the triangle and its image. What do you notice? Make as many statements as you can.
   a. Drag the points that determine the rotation angle to help you see what is happening.
   b. You may also wish to drag the vertices of the pre-image (original triangle).

5. Draw segments joining each vertex of the pre-image (original triangle) with its image.
   a. Choose the segment tool at the top. Click on the first endpoint, and then without releasing the button, drag to the second endpoint.
   b. Measure the length of each of the segments: Click on the segment so that it is highlighted in red, and then choose Length from the Measurements menu.
   c. What do you notice? Add any new observations to your list.

6. Draw segments joining each vertex of the pre-image to the center of rotation (the vertex of the angle) and measure their lengths.
   a. Also, add in segments joining each vertex of the image to center of rotation and measure their lengths.
   b. What do you notice? Add any new observations to your list.
   c. Finally, draw the angles going from a vertex of the pre-image, through the center of rotation, and finally through the image of that vertex.

7. Add measurements for the image of your triangle, as in the previous part.
   a. What do you notice? Add any new observations to your list.