“I think I understand stuff when I’m in class. When I try to do homework I get confused and then I try to get help from the book – but none of it makes sense. I get frustrated and give up” – Julian

Writing and Mathematical Rigor: Supporting the Mathematical Practices

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Agenda

• Overview and context for writing in mathematics: rigor; goals and purposes; practices in action
• Discussion of writing and the mathematical practices with examples
• Implementing a writing agenda in your math class: some basic issues
Some Preliminary Thoughts…..

What is rigor?
Write down a few thoughts.

Learning experiences that involve rigor …

challenge students
require effort and tenacity by students
focus on quality (rich tasks)
include entry points and extensions for all students
are not always tidy, and can have multiple paths to possible solutions

provide connections among mathematical ideas
contain rich mathematics that is relevant to students
develop strategic and flexible thinking
encourage reasoning and sense making
expect students to be actively involved in their own learning

Research on Writing and Mathematics

- Writing in the mathematics classroom promotes a deeper understanding of concepts and procedures. Writing helps students extend their critical thinking abilities as well as the ability to link a new idea to relevant prior knowledge (e.g., Bicer, Capraro, & Capraro, 2013; Craig, 2011; Powell, 1997; Pugalee, 2004, 2001).
- Encourages students to create their own problem solving knowledge (Carr & Biddlecomb, 1998; Steele, 2007)
- Promotes a metacognitive frameworks that extends students' reflection and analysis (Pugalee, 2004; Pugalee, 2001; Boscolo and Mason, 2001)
- Students' mathematical knowledge is extended as a result of their writing in mathematics (Reilly, 2007) and reflects changes in their understanding of mathematical concepts changed over time.

Goal, Purposes and Types of Writing

- **Exploratory**
  - To personally make sense of a problem, situation, or one's own ideas
- **Informative/Explanatory**
  - To describe
  - To explain
- **Argumentative**
  - To construct an argument
  - To critique an argument
- **Mathematically Creative**
  - To document original ideas, problems, and/or solutions
  - To convey fluency and flexibility in thinking
  - To elaborate on ideas

Exploratory

In words, write what you would do to solve this problem where appropriate, tell why you are doing what step to the problem:

First of all, you would look at the problem to see if there is anything in common. If there is, divide the problem out by (2x). In this case it will break down anymore so you would use the trial and error method. It ends up being 2x(3x+4)^2. Finally you would set each factor equal to zero. Solving x=0, 3x+4=0. If you like you could plug in little answers and come up with 0, because anything multiplied by zero is zero.

Informative/Explanatory

**B) Bestsize Cans (continued)**

I kind of did a "guess and check" problem solving method.

At first, I thought that the larger the radius, the smaller the surface area. The goal of this task is to find the smallest surface area for a can that can hold 200 cm$^3$ in volume.

After many "guess and check" trials, I came to a conclusion that a radius of $\frac{3}{2}$ and a height of $\frac{7}{2}$ in a can can have a volume of about 201. That means it can hold 200 cm$^3$ of liquid. And, it was the smallest amount of aluminum possible, which is about 190.5 cm$^2$. 
Argumentative

Susie is organizing the printing of tickets for a show her friends are producing. She has collected prices from several printers and these two seem to be the best.

<table>
<thead>
<tr>
<th>SURE PRINT</th>
<th>BEST PRINT</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ticket printing</td>
<td></td>
</tr>
<tr>
<td>28 tickets for $2</td>
<td></td>
</tr>
<tr>
<td>$10 setting up plus $1 for 25 tickets</td>
<td></td>
</tr>
</tbody>
</table>

Susie wants to go for the best buy.

She doesn't yet know how many people are going to come.

Show Susie a couple of ways in which she could make the right decision, whatever the number.

Illustrate your advice with a couple of examples.

The cost of sure print is represented by 2(x). Best print is 10 + \frac{x}{2}.

For sure print to cost less than best print, \(0 < 10 + \frac{x}{2} \Rightarrow x > 200\). So if the # of people is less than 250, use sure print.

For best print to be the best choice, \(10 + \frac{x}{2} < 12 \Rightarrow x > 200\). So if the # of people is over 250, choose best print.

Mathematically Creative

Congruency is when figures are the same size and shape. After I did a sequence of rigid transformations, triangle UVW, I noticed it was exactly the same as it was in the beginning. Throughout the sequence, the triangle shape and size didn't change. The triangle only moved. That means all of the triangles on the right are congruent.
Problem Solving Seven on Angles in a Triangle.

<table>
<thead>
<tr>
<th>Problem Solving Seven</th>
</tr>
</thead>
<tbody>
<tr>
<td>Determine the measure of the missing angle. (Triangle is not drawn to scale.) Show all your work.</td>
</tr>
</tbody>
</table>

1. \[ \angle 40^\circ \]

<table>
<thead>
<tr>
<th>Key Words</th>
<th>Actions</th>
</tr>
</thead>
<tbody>
<tr>
<td>Measure angle</td>
<td>( \text{add and subtract} ) always ( 90^\circ )</td>
</tr>
<tr>
<td>Right angle</td>
<td></td>
</tr>
</tbody>
</table>

| End Product: |
| Find measure of third angle |
| Similar Problems: |
| Triangle problems angles add \( 90^\circ + 180^\circ \) |

<table>
<thead>
<tr>
<th>Plan</th>
<th>Execute</th>
<th>Verify</th>
</tr>
</thead>
<tbody>
<tr>
<td>Add 3 angles to get ( 180^\circ )</td>
<td>( 90^\circ + 40^\circ + \underline{\hspace{2cm}} = 180^\circ )</td>
<td>( 50^\circ + 40^\circ = 90^\circ ) Answer correct! Right triangle</td>
</tr>
<tr>
<td>( 180^\circ + \underline{\hspace{2cm}} = 180^\circ )</td>
<td>( 170^\circ + 50^\circ = 180^\circ )</td>
<td></td>
</tr>
</tbody>
</table>
Question 5: In a class three-fifth of the pupils were girls and the rest were boys. If the number of boys were doubled, and 6 more girls joined the class, there will be equal number of boys and girls. How many pupils were in the class at the start?

Algebra
Let no. of boys be $x$
Let no. of girls be $y$
Let no. of students in class be $3a$
\[ \frac{3}{5}a = y \]
\[ \frac{2}{5}a = x \]
\[ \therefore \frac{2}{5}a = \frac{3}{5}a + 6 \]
\[ \frac{4}{5}a - \frac{3}{5}a = 6 \]
\[ \frac{1}{5}a = 6 \]
\[ a = 30 \]
There were 30 pupils.

Fig 3(a)

Algebra + diagram

\[ \frac{1}{5}x + 6 = 2 \left( \frac{1}{5}x \right) \]
\[ \frac{1}{5}x = 6 \]
\[ \frac{3}{5}x + 6 = \frac{3}{5}x \]
\[ x = 6 \times 5 \]
There were 30 pupils at the start.

Fig 3(b)

Reason abstractly and quantitatively.

Write a story for the mathematical equation

\[ \frac{1}{2} \times 4 \]

DeJuan exercises 1/2 hour a day for 4 days. How many total hours does he exercise? Think what makes sense.
Here is a reminder of the data you saw in the video with a few extra cups added:

<table>
<thead>
<tr>
<th>Cup</th>
<th>Wide diameter</th>
<th>Narrow diameter</th>
<th>Slant length</th>
<th>Roll radius</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>3 1/2</td>
<td>3</td>
<td>3</td>
<td>28 1/2</td>
</tr>
<tr>
<td>B</td>
<td>3</td>
<td>2</td>
<td>3 1/4</td>
<td>10 1/2</td>
</tr>
<tr>
<td>C</td>
<td>2 1/2</td>
<td>2</td>
<td>6</td>
<td>28 1/2</td>
</tr>
<tr>
<td>D</td>
<td>3</td>
<td>3</td>
<td>4</td>
<td>Infinite</td>
</tr>
<tr>
<td>E</td>
<td>3</td>
<td>2</td>
<td>6</td>
<td>18</td>
</tr>
<tr>
<td>F</td>
<td>3 1/2</td>
<td>2</td>
<td>3 1/2</td>
<td>8 1/2</td>
</tr>
<tr>
<td>G</td>
<td>3 1/2</td>
<td>3</td>
<td>3 1/2</td>
<td>18 1/2</td>
</tr>
<tr>
<td>H</td>
<td>3 1/2</td>
<td>0</td>
<td>3 1/2</td>
<td>3</td>
</tr>
</tbody>
</table>

1. Describe how each of the three lengths on the picture affect the roll radius. Show how you used the data to explain your ideas.

2. Show how you can use math to predict the radius of the circle rolled by any size of cup. Show all your reasoning, including any diagrams and calculations.

In Prague some sidewalks are made of small square blocks of stone. The blocks are in different shapes to make patterns that are in various sizes.

\[
\begin{array}{c}
\text{Pattern 1} \\
\text{Pattern 2} \\
\text{Pattern 3}
\end{array}
\]

How many blocks of each kind will pattern 3 need?

\[\begin{align*}
\text{Grey: } & 4n^2 + 2n + 1 \\
\text{White: } & 8n(2n + 1) = 2n(4n + 4)
\end{align*}\]

Which pattern has a total of 841 grey blocks?

\[n = 10 \quad \text{while} \quad \frac{2n(4n + 4)}{2} = 840 \quad \checkmark\]

How many white blocks has that pattern?

\[341 \cdot 1 = 340 \quad \checkmark\]

Explain your work and show your calculations.

Taking into account the total number of grey and white blocks in the diagrams above, as well as the pattern numbers of the diagrams, I used intuitive reasoning to create the following:

- **Grey:** \(4n^2 + 2n + 1\)
- **White:** \(8n(2n + 1) = 2n(4n + 4)\)

where \(n\) is the pattern number. I then substituted 841 for grey and isolated a using factoring (shown on next page). The pattern number is 10. Since the number of grey blocks is 1 more than the number of white blocks in a pattern if pattern 310 has 841 grey blocks, pattern 30 also has 840 white blocks.
<table>
<thead>
<tr>
<th>Rubric for Assessing Five Mathematical Facets</th>
</tr>
</thead>
<tbody>
<tr>
<td>Conceptual Understanding</td>
</tr>
<tr>
<td>4   Identifies and provides information about major concepts; supplies examples or illustrations with explanations when appropriate.</td>
</tr>
<tr>
<td>3   Identifies and provides information about major concepts but may omit minor details. May use examples or illustrations when appropriate but may not effectively relate them to mathematical concepts.</td>
</tr>
<tr>
<td>2   ...</td>
</tr>
<tr>
<td>1   ...</td>
</tr>
<tr>
<td>0   ...</td>
</tr>
</tbody>
</table>

For your consideration:

IF a square with a side length of 12 cm is revolved around a diagonal axis then the object that will be made will be 2 cones connected at their bases.

[Diagram of cones]

I will find the volume of one cone and then multiply that volume by 2 because there are 2 cones.

To find volume of one cone: 
\[ V = \frac{1}{3} \pi r^2 h \]

to find \( r \):

\[ a^2 + b^2 = c^2 \]
\[ a^2 = 12^2 = 144 \]
\[ b^2 = 12^2 = 144 \]
\[ c = 15 \]

because \( r \) is half of \( c \) \( r = \frac{15}{2} = 7.5 \).

[Diagram of cone]

\[ h \] will equal the same value as \( r \) because the height of the cone is also its radius.

\[ h = r = 4.242 \]

\[ V = \frac{1}{3} \pi r^2 h \]

\[ V = \frac{1}{3} \pi (4.242)^2 4.242 \]

\[ V = 79.93 \text{ cm}^3 \]

Multiply by 2 because there are two cones:

\[ (79.93)(2) = 159.86 \text{ cm}^3 \]

The volume of the solid created by the revolution is:

\[ 159.86 \text{ cm}^3 \]

0 Answer is unresponsive, unrelated, or inappropriate. Nothing is correct.

1 Answer addresses item but is only partially correct; something is correct related to the question.

2 Answer deals correctly with most aspects of the question, but something is missing. Answer may deal with all aspects but have minor errors.

3 All parts of the question are answered accurately and completely. All directions are followed.

Source: Adapted from the 1997-98 North Carolina Open-Ended Assessment, Grade 8
I solved this problem by putting the triangle into corresponding parts. Then I used cross products. And I multiplied fifteen by sixty which gave me a thousand. Then I multiplied twenty by x to give me 20x. So then I took my thousand and put that it equals 20x (900=20x) Then I divided nine hundred by twenty. My answer was x=45.
How would you calculate the number of cubes needed for a tower n cubes high?

No of cubes in each wing is \(1 + 2 + 3 + \ldots + (n-1)\) which is \(\frac{n(n-1)}{2}\).

There are 4 wings so the number of cubes is \(4 \times \frac{n(n-1)}{2}\).

This does not include the number of cubes down the center so the complete formula for the number of cubes is

\[
\frac{4n(n-1) + n}{2} = 2n^2 - 2n + n = 2n^2 - n.
\]

Check:

\[
\begin{align*}
n &= 5 & 2(5)^2 - 5 &= 72 - 5 = 67 \checkmark \\
n &= 12 & 2(12)^2 - 12 &= 288 - 12 = 276 \checkmark
\end{align*}
\]
How do you check Q and R on calculator
Always true
Infinite Solutions: R
2(3x-1) = 6x - 2
y_1 = 2(3x-1)
y_2 = 6x - 2
On your graph, it's the same line

No solution
2(3x-1) = 6x - 1
y_1 = 2(3x-1)
y_2 = 4x - 1
2x
Graph the lines are parallel

You must zoom out to see it.

The slope is 2/5 because
the change from one point
to the next is up 2 and over 3.

The slope is -1 since it is
down 1 and over 2 which is
\(-\frac{1}{2}\) or -1. This one goes
down to be negative. First
one goes up. Both show
change between two points
on the line.
### Think-Pair-Share

**Linear Systems/Parallel and Perpendicular lines**

<table>
<thead>
<tr>
<th>Think</th>
<th>Pair</th>
<th>Share</th>
</tr>
</thead>
</table>
| **1.** Find the equation for each of the lines on the graph. Determine whether the lines are parallel, perpendicular, or neither. Explain your answer. | **We agree the problem should be solved like this:**

\[ L_1: y = 3x + 1 \]

\[ L_2: y = \frac{1}{3}x + 4 \]

Perpendicular because their slopes are **opposite** and **reciprocals**. |

**Notes from class discussion:**

- When graphing, don’t count down or over if the lines don’t cross through.
- Pick 2 points on the line and count up or down and over.
- **Perpendicular**

\[ m_1 \cdot m_2 = -1 \]

- **Parallel**

\[ m_1 \cdot m_2 = 0 \]

- **Perpendicular**

\[ m_1 \cdot m_2 = \text{undefined} \]

- **Parallel**

\[ m_1 \cdot m_2 \in \mathbb{R} \] |
1. Given the equation:

\[3x + 9 = 21\]

Write your own story problem without using the words add, subtract, multiply, or divide. Also, do not use \(x\) or any other variable \((a, b, c, \ldots, y, z)\) to describe something in your story problem.

Katie is expecting 21 people at her birthday party. There are already 9 people there. All of her guests are coming in groups of 3. How many more groups of 3 is Katie expecting in order for all of her guests to be there?
I’m an Equals Sign…..

In Function Land I have an important job. You see - there has always been a rift between the left and right sides of this great state. My job is to guarantee that regardless of what happens, both sides are the same. This is difficult because my friends who deal in operations (like addition, subtraction, multiplication, and division – but there are others) are always changing the way things look. I have to make sure that when something gets changed on one side that another operation on the other side keeps things the same. Just the other day, two units were added to the right side of Function Land, I had to arrange for someone to add two units to the left side. I’m no longer an equals sign if I allow something done on one side to not balance the other. Can you imagine how it would be to lose your identity – not to mention your job…. The hardest part of my work is making sure the Variables keep things balanced. You see, I have to constantly check to make sure their value hasn’t changed. I can do this by getting my friends in operations to do their jobs until I have nothing but variables on one side and numbers on the other…. Whew, sometimes that is a difficult task. So, you can see my job is important. Maybe the most important job in Function Land.
Implementing Writing in Math Class

1. Write often: writing is part of learning not an add-on.
2. Create classroom expectations.
3. Have a goal in mind and select appropriate writing prompts and tasks.
4. Weave assessment into your process.
Model for Creating Expectations for Writing

Key Terms for Clear Writing Tasks

**Compare and/or Contrast:** Write about what is different and/or what is alike.

**Describe:** Provide step-by-step details using key terms, graphs, charts, diagrams, and other illustrations. Illustrations and diagrams are referred to in descriptions.

**Explain:** Elaborate on solutions, steps, ideas, concepts, and conjectures using numbers, symbols, illustrations, and examples to make your explanations clear.

Key Terms for Clear Writing Tasks

Interpret: Provide mathematical reasoning to describe relationships (such as mathematical information from data tables or charts, graphs, illustrations, models, diagrams, symbolic representations, and other representations and interactions).

Provide Reasons or Justification: Give supporting evidence from mathematics to support your thinking including examples, mathematical concepts and definitions, theorems, and other reasons that support what you say.

List: Provide ideas in numbered or bulleted format.

Show all work: Include all calculations, steps, and ideas that you thought about and used to reach your conclusion: all information that shows your thinking.

Modifying Text Features to Support Writing

<table>
<thead>
<tr>
<th>Text Feature</th>
<th>Modification for Promoting Writing</th>
</tr>
</thead>
<tbody>
<tr>
<td>Conceptually Dense Information</td>
<td>Restating and/or summarizing important concepts or ideas.</td>
</tr>
<tr>
<td>Diagrams, Cases, Models &amp; Illustrations</td>
<td>Descriptive writing that requires students to incorporate diagrams and illustrations into a problem or explain how a diagram or illustration supports the mathematical concept or idea being emphasized.</td>
</tr>
<tr>
<td>Definitions, Rules, Formulas</td>
<td>Stating definitions, rules, or formulas in one’s own words. Explaining why a rule or formula works.</td>
</tr>
<tr>
<td>Examples Problems &amp; Procedures</td>
<td>Describing how examples are similar and different. Describing how examples demonstrate definitions, rules, formulas, etc..</td>
</tr>
</tbody>
</table>
Writing Prompts Aligned to Learning Targets

Developing Students’ Knowledge of Mathematics

- Write personal definitions of terms, rules, theorems, etc.
- Write explanations of mathematical concepts and ideas
- Write a summary of a lesson or task
- Write explanations of errors (What went wrong? How can the error be addressed?)
- Offer examples and justify selection
- Describe rules, their application, and mathematical importance

Developing Problem Solving Methods

- Write problems, applications, and provide solutions
- Describe how to solve a problem
- Compare and contrast alternative approaches to a problem
- Describe how technology helped in finding a solution (Describe the mathematical processes required for the output.)
- Write a formal report for approaching a problem situation.

Writing Prompts Aligned to Learning Targets

Developing Self-Monitoring and Reflective Behaviors

- Describe what made a problem or task easy and/or difficult
- Explain why an answer or solution is reasonable
- Identify and react to questions one may raise about your work (or respond to a question someone raised about your work)
- Analyze the quality of one’s work (process, methods, mathematical soundness, communication)
- Describe how different decisions might impact an answer
- Describe how problems are similar and/or different


Writing Prompts Aligned to Learning Targets

Promoting Affective Issues

- Write an autobiography about a mathematics experience
- Write about the role mathematics plays in your life or might play in the future.
- Describe how mathematics changes or changed one’s life.
- Explain what helps or hinders you in understanding mathematics
- Describe how you feel about your performance on a task or problem
Writing Prompts Aligned to Learning Targets

*Promoting Discourse*

- Write a note or notes to the teacher for additional information
- Specify lesson components which were not understood or components which you understood well
- Write a journal entry about some aspect of the day’s class
- Summarize and interview with a peer or other individual about a topic, problem, or other mathematically related idea
- Write a response for a group or team to a problem or task

**Five Mathematical Facets**

- Mathematical Content
- Conceptual Understanding
- Procedural Understanding
- Problem Solving Ability
- Mathematical Reasoning
50 Activities for Writing in Math Class

1. Summarize a part of class discussion or lecture.
2. Summarize a part of the text.
4. Use key words in an explanation or description.
5. Construct test or quiz questions.
7. Defend a decision or action.
8. Create a dialogue between student and another person.
9. Describe a graph or table.
10. List characteristics or steps.

Fifty Activities for Writing in Mathematics

1. Summarize a part of class discussion or lecture.
2. Summarize a part of the text.
4. Use key words in an explanation or description.
5. Construct test or quiz questions.
7. Describe a graph or table.
8. Create a dialogue between student and another person.
9. List characteristics or steps.
10. Compare understanding of a concept to what was known before a lesson or exercise.
11. Write a proposal for a project.
12. Write a children’s story using a math concept.
13. Summarize an interview related to a math topic.
14. Write a biographical sketch of a mathematician.
15. Paraphrase a section of text.
16. Write a memo dealing with a math topic or problem.
17. Create and defend projections of what might happen in (days, months, years).
18. Identify personal goals for mathematics learning.
19. Write critiques of a process or approach.
20. Create examples of a concept.
21. Write a response to a question, comment, quotation, etc.
22. Use mapping or another graphic organizer to identify key ideas and associations.
23. Write a summary of major points when using notes to study for a quiz or test.
24. Write an editorial or letter to the editor using mathematics to make a point.
25. Use metaphors or similes to write about a mathematics concept or idea.
27. Write a commercial ad using mathematics.
28. Write a mathematics “word” problem.
29. Write a letter explaining a mathematical idea, problem, or process.
30. Write an autobiography related to mathematics or to when one might have applied a particular concept (such as what mathematics have you used in sports, travel, a hobby…).
31. Develop an argument supporting an approach.
32. Find and describe a contradiction.
33. Prepare an outline of a class lesson.
34. Find an example of mathematics in a novel, newspaper, or other reading. Summarize the problem and describe the mathematics being used.
35. Describe how two problems are similar and/or different.
36. Write the definition of a term given in class or the text. Revise it in your own words. Tell how your own definition includes all the important information.
37. Analyze another student’s work. Tell whether you agree or disagree with their conclusion and why.
38. Select a homework problem that gave you some difficulty. Write about what gave you difficulty and how you understand the idea/concept/process now.
39. Write a poem or song about a math topic.
40. List what you know about a math topic (such as odd numbers, exponents, adding fractions, graphing equations).
41. Write several questions that are important to ask yourself when doing a particular type of problem (such as dividing two fractions, solving a word problem, graphing a quadratic equation).
42. Create a drawing or illustration to a problem. Describe how they are related.
43. Tell how to use a calculator or other tool to help solve a problem.
44. Compile a list of unfamiliar terms in a lesson. Follow-up with definitions and examples.
45. Use Venn diagrams or concept cards.
46. Describe how a concrete example or manipulative helped in understanding a concept. (Ex. ….)
47. Write a summary of major points when using notes to study for a quiz or test.
48. Write an editorial or a letter to the editor using mathematics to make a point.
49. Use metaphors or similes to write about a mathematics concept or idea.
50. Write a description of how you located an error in a problem or misconception in an idea and how you corrected it.