Piecwise Functions and the Mathematics Teaching Practices

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Tell a Story that Goes with this Graph
Tell a Story that Goes with this Graph
Considering this task

1. What could the teacher learn from this?

2. Did all students have an entry point?

3. What is one goal you could have for this task?
Goals for Today

1. Define and use Piecewise functions in context

2. Examine Domain and Range in a Piecewise context

3. Model Effective Teaching Practices from *Principles to Actions*

4. Apply mathematical practices: 1 - Problem Solving, 2 - Reasoning, and 7 - Structure
Introduction to Piecewise Functions

On every mathematical level, one of the key to students’ success is visualization.

Visualization – A way that students can utilize not only their knowledge gained from their other classes, but also can see a “real world” situations of piecewise functions.
Typical introduction:

The “EASY” one:

\[ f(x) = \begin{cases} 
2x - 3 & x < 2 \\
\frac{1}{3}x + 7 & x \geq 2 
\end{cases} \]
Setting Goals

What are your goals when teaching Piecewise Functions?
Selecting and Enacting a Task

Multiple Entry Points
High Cognitive Demand
Meets the Goals You Have Set
<table>
<thead>
<tr>
<th>Lower-Level Demands</th>
<th>Higher-Level Demands</th>
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</thead>
<tbody>
<tr>
<td><strong>Memorization Tasks</strong></td>
<td><strong>Procedures with Connections Tasks</strong></td>
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<tr>
<td>- Involve either reproducing previously learned facts, rules, formulae, or definitions OR committing facts, rules, formulae, or definitions to memory.</td>
<td>- Focus students’ attention on the use of procedures for the purpose of developing deeper levels of understanding of mathematical concepts and ideas.</td>
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<td>- Cannot be solved using procedures because a procedure does not exist or because the time frame in which the task is being completed is too short to use a procedure.</td>
<td>- Suggest pathways to follow (explicitly or implicitly) that are broad general procedures that have close connections to underlying conceptual ideas as opposed to narrow algorithms that are opaque with respect to underlying concepts.</td>
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<tr>
<td>- Are not ambiguous—such tasks involve exact reproduction of previously seen material and what is to be reproduced is clearly and directly stated.</td>
<td>- Usually are represented in multiple ways (e.g., visual diagrams, manipulatives, symbols, problem situations). Making connections among multiple representations helps to develop meaning.</td>
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<tr>
<td>- Have no connection to the concepts or meaning that underlie the facts, rules, formulae, or definitions being learned or reproduced.</td>
<td>- Require some degree of cognitive effort. Although general procedures may be followed, they cannot be followed mindlessly. Students need to engage with the conceptual ideas that underlie the procedures in order to successfully complete the task and develop understanding.</td>
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<tr>
<th><strong>Procedures Without Connections Tasks</strong></th>
<th><strong>Doing Mathematics Tasks</strong></th>
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<tr>
<td>- Are algorithmic. Use of the procedure is either specifically called for or its use is evident based on prior instruction, experience, or placement of the task.</td>
<td>- Require complex and non-algorithmic thinking (i.e., there is not a predictable, well-rehearsed approach or pathway explicitly suggested by the task, task instructions, or a worked-out example).</td>
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<tr>
<td>- Require limited cognitive demand for successful completion. There is little ambiguity about what needs to be done and how to do it.</td>
<td>- Require students to explore and understand the nature of mathematical concepts, processes, or relationships.</td>
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<td>- Have no connection to the concepts or meaning that underlie the procedure being used.</td>
<td>- Demand self-monitoring or self-regulation of one’s own cognitive processes.</td>
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<td>- Are focused on producing correct answers rather than developing mathematical understanding.</td>
<td>- Require students to access relevant knowledge and experiences and make appropriate use of them in working through the task.</td>
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<td>- Require no explanations, or explanations that focus solely on describing the procedure that was used.</td>
<td>- Require students to analyze the task and actively examine task constraints that may limit possible solution strategies and solutions.</td>
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Amy’s Bicycle Trip
What questions do you have about the graph?
Selecting and Enacting a Task

What cognitive level do you think this task has?

Justify your answer.
Amy’s Bicycle Trip

Questions that we had:

1) Are there different speeds/rates along Amy’s journey?
2) What is the Amy’s distance at 5 minutes? 20 minutes? 95 minutes?
3) Write the equations for the different parts of Amy’s trip.
4) Can we write a single function that summarizes Amy’s Distance vs. Time
Amy’s Bicycle Trip

How can you write a rule for this graph? Why might you want a rule(s)?

What will your students struggle with? How can you address those struggles?
Student Work

0 \leq x \leq 15
30 \leq x \leq 60

Amy's Trip 4 mph
24 mph

Distance in Miles

Time in Minutes
Effective teaching requires being able to support students as they work on challenging tasks without taking over the process of thinking for them. Asking questions that assess student understanding of mathematical ideas, strategies, or representations provides teachers with insights into what students know and can do. The insights gained from these questions prepare teachers to then ask questions that advance student understanding of mathematical concepts, strategies, or connections between representations.

Pose Meaningful Questions

What questions do you want to ask to assess student learning?
What does an “assessing” question look like?
Assessing Questions

- Based closely on the work the student has produced.
- Clarify what the student has done and what the student understands about what s/he has done.
- Provide information to the teacher about what the student understands.

Pose Meaningful Questions

What questions do you want to ask to advance your students?

What makes an “advancing” question?
Advancing Questions

Move students beyond their current thinking about the mathematic ideas or strategies under study by pressing students to extend what they know to a new situation to illuminate mathematical ideas.

Advancing Questions

Use what students have produced as a basis for making progress toward the mathematical learning goal.

Press students to think about mathematical ideas or strategies that they are not currently thinking about.

### Student Work

<table>
<thead>
<tr>
<th>MINS</th>
<th>mpm</th>
<th>mph</th>
</tr>
</thead>
<tbody>
<tr>
<td>0-15</td>
<td>4 mpm</td>
<td>24 mph</td>
</tr>
<tr>
<td>15-30</td>
<td>0 mpm</td>
<td>0 mph</td>
</tr>
<tr>
<td>30-60</td>
<td>0.06 mpm</td>
<td>4 mph</td>
</tr>
</tbody>
</table>

[Graph showing a shaded area representing a trip.]

The values in the table represent the speed in miles per mile (mpm) and miles per hour (mph) for different time intervals.
Student response

\[ y = \frac{3}{5} x \]

\[ y = 6 \]

\[ y = \frac{2}{15} x + 2 \]
Possible Extensions:

1) Create a graph of Amy’s speed vs. time during her trip.

2) How could we write a function that summarizes Amy’s speed vs. Time?

3) What is Amy’s speed at 30 minutes? 95 minutes?
Another Look at Piecewise Functions

What if we start with a context?

- How does that differ from the Amy problem?
- How might a task start with a context?
The Slide

Watch the following video and set up your piece of paper accordingly.

While watching, think about what questions your students may have/need in order to graph this function.
The Slide

Video available at: graphingstories.com/4my

Video made by: Adam Poetzel, University of Illinois
The Slide

Make a graph of the story.

Individual time
Small Group time
?Whole group?  Sequencing
The Slide

Now that you have graphed the story, what decisions did you have to make?

Where do you think your students would struggle?
Concepts

How can you use this task to learn what students understand about domain and range? About the meaning of function?

Could your students make a “rule” for this function?
Student Work
Student Work
The Slide

What solution paths would you anticipate for your students?
What questions would you have planned to assess student understanding?
What Is Your Next Goal?

What else do we want students to be able to do with a piecewise function?
Consider “procedural fluency” from “conceptual understanding”. Is that a goal for this concept?
Graphing

Sign function is defined by:

\[
f(x) = \begin{cases} 
-1 & x < 0 \\ 
0 & x = 0 \\ 
1 & x > 0 
\end{cases}
\]

This is how we can introduce step functions.

Question for students:
How can you rewrite \( g(x) = |x| \) as a piecewise function.
How can we get students to be able to discuss this and understand how to graph it?

\[ g(x) = \begin{cases} 
  x & x < -1 \\
  x^2 & -1 \leq x < 2 \\
  -x & x > 3
\end{cases} \]
Consider

Create three piecewise functions that contain the points (1, 2), (2, 4) and (4, 8).
Discourse

How have we encouraged discourse?
How have we considered the use of representations?
Discourse

How can these tasks Promote Productive Struggle?
Evidence of Student Thinking

How can you elicit evidence of student thinking in these tasks? How does that influence what you are going to do next?
Thanks