Redefining “Help”: How to Support Conceptual Understanding and Procedural Fluency

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TIMED TEST

Answer as many of the problems on the right in one mad minute.

Letter your paper A through L.

Which Fraction is Greater?

- A. $\frac{4}{5}$ or $\frac{4}{9}$
- B. $\frac{4}{7}$ or $\frac{5}{7}$
- C. $\frac{3}{8}$ or $\frac{4}{10}$
- D. $\frac{5}{3}$ or $\frac{5}{8}$
- E. $\frac{3}{4}$ or $\frac{9}{10}$
- F. $\frac{3}{8}$ or $\frac{4}{7}$
- G. $\frac{7}{12}$ or $\frac{5}{12}$
- H. $\frac{3}{5}$ or $\frac{3}{7}$
- I. $\frac{5}{8}$ or $\frac{6}{10}$
- J. $\frac{9}{8}$ or $\frac{4}{3}$
- K. $\frac{4}{6}$ or $\frac{7}{12}$
- L. $\frac{8}{9}$ or $\frac{7}{8}$

How to Help Students

#1

Develop a Common Language:
Conceptual Understanding and Procedural Fluency
What are we *talking about?*

Let’s speak the same language!
Build procedural fluency from conceptual understanding.

Effective teaching of mathematics builds fluency with procedures on a foundation of conceptual understanding so that students, over time, become skillful in using procedures flexibly as they solve contextual and mathematical problems.
What are we talking about?
Let’s speak the same language!

Conceptual Understanding
Conceptual Understanding

Comprehension of:

✓ Mathematical Concepts (Conceptual Knowledge)

✓ Operations and

✓ Relations

Conceptual Understanding ≠ Math Concepts
Conceptual Understanding

Comprehension of:

✓ Mathematical Concepts (Conceptual Knowledge)

✓ Operations and

✓ Relations

Conceptual Understanding ➔ Math Concepts
What classroom practices positively impact conceptual understanding?
1. Allowing students to struggle with important mathematical ideas

2. Making mathematical relationships explicit

Hiebert & Grouws, 2007, Lambdin, 2010
What are we talking about?
Let’s speak the same language!

+ Procedural Fluency
What are we talking about?

Procedural Knowledge
Skill
Understanding
Fluency
Flexibility
Efficiency
What are we talking about?

Procedural

- Knowledge
- Skill
- Understanding
- Fluency
- Flexibility
- Efficiency
Procedural Fluency is “skill in carrying out procedures flexibly, accurately, efficiently and appropriately”

(CCSSO, 2010, p. 6)
<table>
<thead>
<tr>
<th></th>
<th>A.  $\frac{4}{5}$ or $\frac{4}{9}$</th>
<th>G.  $\frac{7}{12}$ or $\frac{5}{12}$</th>
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</thead>
<tbody>
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<td>B.</td>
<td>$\frac{4}{7}$ or $\frac{5}{7}$</td>
<td>H. $\frac{3}{5}$ or $\frac{3}{7}$</td>
</tr>
<tr>
<td>C.</td>
<td>$\frac{3}{8}$ or $\frac{4}{10}$</td>
<td>I.  $\frac{5}{8}$ or $\frac{6}{10}$</td>
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<tr>
<td>D.</td>
<td>$\frac{5}{3}$ or $\frac{5}{8}$</td>
<td>J. $\frac{9}{8}$ or $\frac{4}{3}$</td>
</tr>
<tr>
<td>E.</td>
<td>$\frac{3}{4}$ or $\frac{9}{10}$</td>
<td>K. $\frac{4}{6}$ or $\frac{7}{12}$</td>
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<td>F.</td>
<td>$\frac{3}{8}$ or $\frac{4}{7}$</td>
<td>L.  $\frac{8}{9}$ or $\frac{7}{8}$</td>
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</tbody>
</table>

Which strategies did you use for which problems?

1. Same sized pieces (denominator)
2. Same number of parts (numerator)
3. Over/Under $\frac{1}{2}$ or 1 (Benchmark)
4. Closeness to $\frac{1}{2}$ or 1
5. Finding equivalent fractions

Aspects of Procedural Fluency

How to Help Students #2

Make connections explicit
Concepts & Procedures Develop Iteratively

Weak

Procedural Knowledge

Strong

Conceptual Knowledge

Weak

Strong
2. Make connections explicit
2. Make connections explicit

Find how many in ONE bag (shared fairly)...

- 4 bags and 12 counters
  - $12 \div 4$

- 3 bags and 18 counters
  - $18 \div 3$

- 5 bags and 100 counters
  - $100 \div 5$
2. Make connections explicit

Find the number in ONE whole bag if...

- 4 counters fill one-half of a bag
  \[4 \div \frac{1}{2} =\]

- 6 counters fill one-third of a bag
  \[6 \div \frac{1}{3} =\]

- 4 counters fill one-sixth of a bag
  \[4 \div \frac{1}{6} =\]
2. Make connections explicit

Find the number in the whole bag if...

- 6 counters are two-thirds of a bag
  \[6 \div \frac{2}{3} =\]

- 6 counters are three-halves of a bag
  \[6 \div \frac{3}{2} =\]

- 12 counters are three-fourths of a bag
  \[12 \div \frac{3}{4} =\]

- 10 counters are five-halves of a bag
  \[10 \div \frac{5}{4} =\]
As students connect tools to symbols, they…

✓ Strengthen conceptual knowledge (meaning of division)

✓ Strengthen procedural knowledge (learn different ways you can divide)

✓ Make connections (see relationship between whole number and fraction division)
**P.I.C.S. Page:** Supporting the relationship between procedural and conceptual knowledge

<table>
<thead>
<tr>
<th>Procedure</th>
<th>Illustration</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
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</table>

<table>
<thead>
<tr>
<th>Concept</th>
<th>Situation</th>
</tr>
</thead>
<tbody>
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</tbody>
</table>

Explicitly Teach Strategies
Making Mathematical Relationships Explicit

- Treat mathematical relationships between ideas, facts, and procedures in an explicit and public way.
- Remind students of the main point of the lesson and how it connects to previous and future learning.
- Discuss the mathematical meaning that underlies a procedure.
- Question students about how different solution strategies are similar to or different from each other.
- Discuss the ways in which mathematical concepts build on each other.

(Hiebert & Grouws, 2007)
2. Make connections explicit

Explicit Instruction ≠ Direct Instruction
2. Make connections explicit

Double and Double Again

(a) Double and double again (facts with a 4)

Fact | Also
---|---
\[ \frac{4}{6} \times 6 \quad \frac{6}{4} \times 4 \]

Double 6 is 12. Double again is 24.

Strategy
Explicitly Teach Strategies

What’s going on here?

\[
\frac{2}{3} + \frac{3}{5} = \frac{5}{8}
\]

Rush to fluency is an oxymoron!

Mastering Basic Facts

#3 Explicitly Teach Strategies

Phase 1: Counting
(counts with objects or mentally)

Phase 2: Deriving
(uses reasoning strategies based on known facts)

Phase 3: Mastery
(efficient production of answers)

Adapted from Baroody, 2006
CCSS-M Descriptions

Grade K (K.0A.A.5):

Fluently add and subtract within 5.

Grade 1 (1.0A.C.6):

Add and subtract within 20, demonstrating fluency for addition and subtraction within 10. Use strategies such as counting on; making ten; decomposing a number leading to a ten; using the relationship between addition and subtraction; and creating equivalent but easier or known sums.

Grade 2 (2.0A.B.2):

*Fluently* add and subtract within 20 using mental strategies (reference to 1.0A.C.6). By end of Grade 2, know from memory all sums of two one-digit numbers.
CCSS-M Descriptions

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Fluently add and subtract within 20 using mental strategies (reference to 1.0A.C.6). By end of Grade 2, know from memory all sums of two one-digit numbers.
Knowing from Memory ≠ Memorization

Outcome

Strategy
#3 Explicitly Teach Strategies

**Tens Go Fish**

Play *Go Fish*, a match is a combination that makes 10 (instead of a match).
How to Help Students #4

Scaffold Strategy Development
1. Same sized pieces (denominator)
2. Same number of parts (numerator)
3. Over/Under $\frac{1}{2}$ or 1 (Benchmark)
4. Closeness to $\frac{1}{2}$ or 1
5. Finding equivalent fractions

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A possible mental process (not a procedure to be memorized!)

First
- Common Numerator?
- Common Denominator?

Next
- Over/under a Benchmark
- Compare to a Benchmark

Otherwise
- Find Common Denominator
- Convert to Decimals
Reasoning Strategies for Addition Facts

**Foundational Fact Strategies**
- Sums within 5
- +/- 1 or 2
- Doubles
- Combinations of Ten

**Derived Fact Strategies**
- Near Doubles
- Making Ten

Examples:
- Near Doubles: (6 + 7, 8 + 7)
- Making Ten: (8 + 3, 9 + 5)

Developing Fluency in K-2, Jennifer Bay-Williams
## Reasoning Strategies for Multiplication Facts

<table>
<thead>
<tr>
<th>Foundational Fact Strategies</th>
<th>Derived Fact Strategies</th>
</tr>
</thead>
<tbody>
<tr>
<td>- 2s, 5s, 10s</td>
<td>- Adding or subtracting a group</td>
</tr>
<tr>
<td>- 0’s*, 1’s</td>
<td>- Halving and doubling</td>
</tr>
<tr>
<td>- Multiplication squares</td>
<td>- Nearby square</td>
</tr>
<tr>
<td>2 3</td>
<td>3 3</td>
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<tr>
<td>3</td>
<td>3 3</td>
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<td>3</td>
<td>3 3</td>
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</tbody>
</table>
# Addition Fact Fluency: A 3-Year Plan

<table>
<thead>
<tr>
<th>K</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>+/- 1</td>
<td>Doubles</td>
<td>Near Doubles</td>
</tr>
<tr>
<td>+/- 2</td>
<td>Combo’s of 10</td>
<td>Making 10</td>
</tr>
<tr>
<td>Sums within 5</td>
<td>Sums within 10</td>
<td>Sums within 20</td>
</tr>
</tbody>
</table>

Developing Fluency in K-2, Jennifer Bay-Williams
Multiplication Fact Fluency: A Third Grade Priority

Products within 5 as repeated addition

- Foundational Fact Strategies
- Derived Fact Strategies
- Fluency
How to Help Students

#5 Assess Basic Facts Effectively
Assessing Basic Fact Fluency

What can we learn from assessments related to:

- Flexibility
- Accuracy
- Efficiency
- Appropriate Strategy Use
Aspects of Fluency

- Flexibility
- Accuracy
- Efficiency
- Appropriate Strategy Use

<table>
<thead>
<tr>
<th>Addition</th>
<th>9 + 2</th>
<th>6 + 3</th>
<th>8 + 4</th>
<th>5 + 5</th>
<th>7 + 6</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>4 + 7</td>
<td>2 + 8</td>
<td>3 + 9</td>
<td>1 + 8</td>
<td>9 + 8</td>
</tr>
<tr>
<td></td>
<td>8 + 8</td>
<td>7 + 8</td>
<td>6 + 8</td>
<td>5 + 6</td>
<td>4 + 5</td>
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<td></td>
<td>3 + 4</td>
<td>2 + 3</td>
<td>1 + 2</td>
<td>7 + 5</td>
<td>7 + 4</td>
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</tbody>
</table>
Timed Testing: Issues

Limitations as an assessment tool

A child finishes a 20-fact timed test in 60 seconds.

- Did the child spend 3 seconds on each fact?

  Or...

- Did the child spend 1 second on 16 facts and 10 seconds each on 4 of the facts?
Timed Testing: Issues

Can Impede Progress

A study of nearly 300 first graders found that children who were more frequently exposed to timed testing demonstrated lower progress towards knowing facts from memory than their counterparts.

Henry & Brown, 2008
Timed Testing: Issues

Negative psychological effects

- The stress that children experience with timed testing is not experienced when they complete the same tasks in untimed conditions.

- “Evidence strongly suggests that timed tests cause the early onset of math anxiety for students across the achievement range.”

Boaler, 2014
Timed Testing: Issues

Negative psychological effects

Anxiety over timed testing is often not related to achievement. Even high-achieving children share concerns such as “I feel nervous. I know my facts, but this just scares me.”

Boaler, 2012
Timed Testing: Issues

Negative psychological effects

Children experience math anxiety as early as first grade and this anxiety is not correlated with reading achievement. This suggests that the children’s anxiety is specific to mathematics, not general academic work.

Ramirez et al. 2013
Timed Testing: Issues

Negative psychological effects

Children who tended to use more sophisticated mathematical strategies experienced the most negative impact on achievement due to math anxiety. Thus, it appears that some of our best mathematical thinkers are often those most negatively impacted by timed testing.

Ramirez et al. 2013
# Observation Checklist

**Aspects of Fluency**

- Flexibility
- Accuracy
- Efficiency
- Appropriate Strategy Use

<table>
<thead>
<tr>
<th>Student</th>
<th>Models and counts all</th>
<th>Counts on Derived Fact</th>
<th>Recall (double or combo of 10)</th>
<th>Recall</th>
<th>Comments</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
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</table>

Addition Facts Fluency Chart  
Date:______________    Game:________________________
### Interviews

<table>
<thead>
<tr>
<th>Flexibility</th>
<th>Accuracy</th>
</tr>
</thead>
<tbody>
<tr>
<td>Solve $6 + 7$ using one strategy. Now try solving it using a different strategy.</td>
<td>What is the answer to $7 + 8$? How do you know it is correct (how might you check it)?</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Efficiency</th>
<th>Appropriate Strategy Selection</th>
</tr>
</thead>
<tbody>
<tr>
<td>For which facts did you <strong>just know</strong>?</td>
<td>Emily solved $6 + 8$ by changing it in her mind to $4 + 10$. What did she do? Is this a good strategy? Tell why or why not.</td>
</tr>
<tr>
<td>For which facts did you <strong>use a strategy</strong>?</td>
<td></td>
</tr>
</tbody>
</table>
#5 Assess Basic Facts Effectively

Strive to Derive by 5 & Subtraction Stacks

Play your choice of games

Discuss

- How can I provide strategy support using this game?
- How can I assess fluency as students play the game?
How to Help Students #6

Ask When and Why Questions
How did you solve #5?

What is the definition of a square?

How do we find the area of a rectangle?
## What do Bloom & Fluency have in Common?

<table>
<thead>
<tr>
<th>LEVEL</th>
<th>Fluency with Algorithms</th>
<th>Bloom’s Levels</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Level 3</strong></td>
<td>comparing different algorithms, judging the efficiency of an algorithm, constructing new algorithms (strategies), and generalizing</td>
<td>Create, Evaluate</td>
</tr>
<tr>
<td><strong>Level 2</strong></td>
<td>knowing why a procedure works and applying it in a complex situation</td>
<td>Analyze, Apply, Understand</td>
</tr>
<tr>
<td><strong>Level 1</strong></td>
<td>knowing the steps and carrying out the steps in a straightforward situation</td>
<td>Remember</td>
</tr>
</tbody>
</table>

*Fan & Bokhove, 2014*
My teacher says that order doesn’t matter. You can multiply these numbers in any order: \(8 \times 7 \times 3\).

My teacher says that you have to go from left to right. So for \(8 \times 7 \times 3\), you have to multiply \(8 \times 7\) first.
When would you use each of these strategies?

1. Same sized pieces (denominator)
2. Same number of parts (numerator)
3. Over/Under ½ or 1 (Benchmark)
4. Closeness to ½ or 1
5. Finding equivalent fractions

#6 Ask When and Why Questions

Strategize Steps (before solving)

**Ask:**
- Which step first and why?
- What options are possible?
- What options are reasonable?

\[ 4 \times 2 \frac{1}{3} \]
#6 Ask When and Why Questions

Analyze results (after solving)

Ask:

- When does that strategy work?
- When would you use each strategy?
- When is that strategy most efficient?
Ask When and Why Questions

3c. Use Worked Examples

- Why did Ella change $2$ to eight-fourths?
- When will this strategy work?
How to Help Students #7

Compare Problem Solutions
## What do Bloom & Fluency have in Common?

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<tr>
<td>Evaluation and Construction</td>
<td></td>
<td></td>
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<tr>
<td><strong>Level 2</strong></td>
<td>knowing why a procedure works and applying it in a complex situation</td>
<td>Analyze Apply Understand</td>
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<tr>
<td>Understanding and Comprehension</td>
<td></td>
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</tr>
<tr>
<td><strong>Level 1</strong></td>
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<td>Remember</td>
</tr>
<tr>
<td>Knowledge and Skills</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Fan & Bokhove, 2014
#6 Ask When and Why Questions

**Analyze results**
(after solving)

Ask:

- When does that strategy work?
- When would you use each strategy?
- When is that strategy most efficient?
You see the sign pictured here as you are driving. How far apart are these two cities?

- Elizabethtown: 164 miles
- Louisville: 209 miles
#7 Compare Problem Solutions

Solve this problem any way you like.

You see the sign pictured here as you ride in your car. How far apart are these two cities?

Possible strategies:
• Standard algorithm
• Counting up from 164 (multiple ways to do this)
• Counting down from 209 (multiple ways to do this)
How to Help Students

Compare Problem Types
Application: Pattern Rich Problem Sets

What do you notice across these problems?

Find each fractional amount of 12.

\[
\frac{1}{2} \times 12 = \quad \frac{1}{8} \times 12 = \\
\frac{1}{4} \times 12 = \quad \frac{3}{8} \times 12 = \\
\frac{3}{4} \times 12 = \quad \frac{8}{8} \times 12 = \\
\frac{12}{8} \times 12 = \quad \frac{16}{8} \times 12 = 
\]
Identify an important skill
Tell the pattern/property you have identified
Prepare a notecard with your problem set
How to Help Students #9

Focus Feedback
Teacher: How did you find the area of this problem?

Anita: I added each row of squares.

Teacher: Very good! That is a great strategy.
#9 Feedback

To...

- Improve/extend work

  Can you add a visual to help others better understand your strategy?

- Reinforce productive struggle.

  I see that you tried three strategies – that really shows good problem solving!

- Notice when a connection is made.

  Wow- I love that you see the connection between decimal place value and whole number place value!

- Encourage students to give peer feedback.

  Do you think Mac’s strategy is going to work?
In Sum...

Helping students ≠ Showing students how
Helping students means…

1. Developing a common language
2. Making connections explicit
3. Explicitly teaching strategies
4. Scaffolding strategy instruction
5. Assessing basic facts effectively
6. Asking when and why questions
7. Comparing solutions
8. Comparing problem types
9. Focusing feedback
10. Your idea!
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