## Beliefs about Teaching and Learning Mathematics

<table>
<thead>
<tr>
<th>Beliefs about teaching and learning mathematics</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Unproductive beliefs</strong></td>
<td><strong>Productive beliefs</strong></td>
</tr>
<tr>
<td>Mathematics learning should focus on practicing procedures and memorizing basic number combinations.</td>
<td>Mathematics learning should focus on developing understanding of concepts and procedures through problem solving, reasoning, and discourse.</td>
</tr>
<tr>
<td>All students need to learn and use the same standard computational algorithms and the same prescribed methods to solve algebraic problems.</td>
<td>All students need to have a range of strategies and approaches from which to choose in solving problems, including, but not limited to, general methods, standard algorithms, and procedures.</td>
</tr>
<tr>
<td>Students can learn to apply mathematics only after they have mastered the basic skills.</td>
<td>Students can learn mathematics through exploring and solving contextual and mathematical problems.</td>
</tr>
<tr>
<td>The role of the teacher is to tell students exactly what definitions, formulas, and rules they should know and demonstrate how to use this information to solve mathematics problems.</td>
<td>The role of the teacher is to engage students in tasks that promote reasoning and problem solving and facilitate discourse that moves students toward shared understanding of mathematics.</td>
</tr>
<tr>
<td>The role of the student is to memorize information that is presented and then use it to solve routine problems on homework, quizzes, and tests.</td>
<td>The role of the student is to be actively involved in making sense of mathematics tasks by using varied strategies and representations, justifying solutions, making connections to prior knowledge or familiar contexts and experiences, and considering the reasoning of others.</td>
</tr>
<tr>
<td>An effective teacher makes the mathematics easy for students by guiding them step by step through problem solving to ensure that they are not frustrated or confused.</td>
<td>An effective teacher provides students with appropriate challenge, encourages perseverance in solving problems, and supports productive struggle in learning mathematics.</td>
</tr>
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Principles to Action, Ensuring Mathematics Success for All (NCTM, 2014) p 11
## Characteristics of Mathematical Tasks

### Levels of Demands

<table>
<thead>
<tr>
<th>Lower-level demands</th>
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</thead>
<tbody>
<tr>
<td>(memorization):</td>
<td>(procedures without connections):</td>
</tr>
<tr>
<td>- reproducing previously learned facts, rules, formulas, definitions or committing them to memory</td>
<td>- are algorithmic</td>
</tr>
<tr>
<td>- Cannot be solved with a procedure</td>
<td>- require limited cognitive demand</td>
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<tr>
<td>- Have no connection to concepts or meaning that underlie the facts rules, formulas, or definitions</td>
<td>- have no connection to the concepts or meaning that underlie the procedure</td>
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<tr>
<td></td>
<td>- focus on producing correct answers instead of understanding</td>
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<td></td>
<td>- require no explanations</td>
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<table>
<thead>
<tr>
<th>Higher-level demands</th>
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<tr>
<td>(procedures with connections):</td>
<td>(doing mathematics):</td>
</tr>
<tr>
<td>- use procedure for deeper understanding of concepts</td>
<td>- require complex non-algorithmic thinking</td>
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<tr>
<td>- broad procedures connected to ideas instead narrow algorithms</td>
<td>- require students to explore and understand the mathematics</td>
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<tr>
<td>- usually represented in different ways</td>
<td>- demand self-monitoring of one’s cognitive process</td>
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<tr>
<td>- require some degree of cognitive effort; procedures may be used but not mindlessly</td>
<td>- require considerable cognitive effort and may involve some level of anxiety b/c solution path isn’t clear</td>
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# Productive Struggle Reflection Survey

Rate the frequency of each statement in your classroom.

1. I anticipate what students might struggle with during a lesson and I prepare to support them.
   
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2. I give students time to struggle with tasks and ask questions that scaffold thinking without doing the work for them.

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3. I help students realize that confusion and errors are a natural part of learning. We talk about mistakes, misconceptions, and struggles.

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4. I praise students for their efforts more frequently than the correct answers.

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5. My students struggle at times but they know breakthroughs come from struggle.

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6. My students ask questions that are related to their struggles to help them understand.

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7. My students persevere in solving problems. They don’t give up.

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8. My students help one another without telling their classmates what the answer is or how to solve it.

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Productive Struggle Images
The Fraction Game

- Make a game board with halves, thirds, fourths, fifths, sixths, eighths, and tenths.
- Take turns rolling fractions with dice to create a fraction.
- Players remove the fraction or an equivalent of what they have rolled.
- The first player to remove all of their pieces wins.

Adapted from Illuminations’ The Fraction Game
http://illuminations.nctm.org/Activity.aspx?id=4148

Equivalency with Representations

Julie shares that
\[ \frac{3}{4} \text{ and } \frac{6}{8} \] are equivalent.

Do you agree?

Use 2 different representations to justify your thinking.
1-CENTIMETER GRID PAPER
Fraction of the Day – Option 1

What does it look like?

Place it on a number line.

The Fraction of the Day is

Is it close to 0, ½, or 1.
Tell how you know.

Is it a unit fraction?
If not, break it apart into unit fractions.
Fraction of the Day – Option 2

What are 3 equivalent fractions?

Place it on a number line.

The Fraction of the Day is

What are 3 fractions greater than it?

What fraction could you add to it for a sum greater than 1?

What are 3 fractions less than it?
Equivalency with Cuisenaire

White is $\frac{1}{3}$ of light green.
Find pairs of colors that show $\frac{1}{3}$?
What did you find?

More with Cuisenaire

True or False
- Light green is $\frac{1}{3}$ of brown.
- Red is $\frac{1}{3}$ of dark green.
- Yellow is $\frac{1}{4}$ of (orange + orange).
- Purple is $\frac{1}{4}$ of (orange + dark green).
- Red is $\frac{1}{7}$ of (orange + yellow).

Fill in the Missing Part
- $\frac{1}{4}$ of (orange + dark green) = ???
- $\frac{1}{5}$ of ??? = red
- $\frac{1}{2}$ of (orange + brown) = ???
- $\frac{1}{9}$ of ??? = white
- $\frac{1}{10}$ of (orange + orange) = ???

Fill in the Missing Fraction
- 3 whites/beige = ??? of black
- 5 light greens = ??? of (orange + brown)
- 3 dark greens = ??? of (2 orange + pink/purple)
Comparing Fractions with Representations

1. Fold a piece of paper in fourths.
2. Choose 4 forms of representations to compare

\[
\begin{array}{ll}
\frac{6}{8} & \frac{4}{6} \\
\end{array}
\]

Compare these Fractions without Finding Common Denominators

A. \( \frac{2}{7} \bigcirc \frac{3}{5} \)  
B. \( \frac{8}{9} \bigcirc \frac{2}{6} \)  
C. \( \frac{14}{18} \bigcirc \frac{14}{20} \)  
D. \( \frac{13}{14} \bigcirc \frac{15}{16} \)  
E. \( \frac{3}{6} \bigcirc \frac{15}{20} \)  
F. \( \frac{5}{6} \bigcirc \frac{5}{8} \)  
G. \( \frac{16}{17} \bigcirc \frac{30}{34} \)  
H. \( \frac{7}{12} \bigcirc \frac{9}{20} \)
Making Tangram Directions

1. We start with a square piece of paper.
2. Fold the square in half across the diagonal. Then cut along this line to make 2 isosceles triangles that are congruent. Set one of these aside.
3. Fold the triangle in half and cut this into 2 pieces. (These are the 2 large triangles of the set.)
4. With the triangle we set aside we will make the other 5 pieces. To do this, make a “mid-point pinch” along the base of the large triangle.
5. To make the mid-sized triangle hold the large triangle so that the base is close to you and there is a vertex pointing away from you. Bring that vertex down so that it touches the “mid-point pinch.” Fold and cut out this triangle. (You should now have an isosceles trapezoid)
6. To make 1 of the 2 small triangles, bring the vertex from one side of the trapezoid to the “mid-point pinch.” Fold and cut this triangle from the trapezoid.
7. To make the square, fold along the “mid-point pinch.” Then cut the square from the trapezoid. (You will be left with a small right-angled trapezoid.)
8. From this last piece you will need a small triangle (that is congruent to the other small triangle and a parallelogram (that is not a rectangle). Fold and cut to get these shapes.

Tangram Prompts

- What fraction of the whole does each piece represent?
- If A, C, E make the whole, what does C represent?
- If D is the whole, what does G represent?
- Find as many sets of equivalencies as you can.
Additional Equivalency and Comparison Problems

1. What are 3 fractions between $\frac{3}{5}$ and $\frac{9}{10}$?

   Use pictures, numbers, and/or words to explain your findings.

2. Different improper fractions can be written as a mixed number in the form $\square \frac{1}{\square}$. What might the improper fractions be? Use pictures, numbers, and/or words to explain your findings.

3. Groups used Cuisenaire rods to show that $\frac{1}{2}$ is equivalent to $\frac{2}{4}$. But they used different rods. Is this possible? What different rods might they have used?
Additional Equivalency and Comparison Problems

Jackson shares that \( \frac{7}{10} \) and \( \frac{5}{8} \) are equivalent because they both are missing 3 pieces. Do you agree? Use 2 different representations to justify your thinking.

- Write 2 fractions.
- What is a fraction that is between these 2 fractions?
- Use pictures, numbers, and/or words to justify your thinking.

These fractions are written in order from least to greatest. What are the missing numbers?

\[
\begin{array}{cccccc}
\_ & 2 & 2 & \_ & \_ & \\
3 & 3 & 5 & 8 & \\
\end{array}
\]

Use pictures, numbers, and/or words to explain your findings.
Additional Equivalency and Comparison Problems

Fill in each box with a number from the NUMBER BANK to make the inequality true.

Use pictures, numbers, and/or words to justify your thinking.

Is more than $\frac{1}{2}$ of the square shaded?
Is more than $\frac{1}{4}$ of the square shaded?
# Types of Questions

<table>
<thead>
<tr>
<th>Question Type</th>
<th>Description</th>
<th>Examples</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Gathering information</strong></td>
<td>Students recall facts, definitions, or procedures.</td>
<td>• How many shaded parts are there?</td>
</tr>
<tr>
<td></td>
<td></td>
<td>• What is a numerator?</td>
</tr>
<tr>
<td><strong>Probing thinking</strong></td>
<td>Students explain, elaborate, or clarify their thinking, including articulating the steps in solution methods or the completion of a task.</td>
<td>• Can you explain more about how you created your new examples of one-half?</td>
</tr>
<tr>
<td></td>
<td></td>
<td>• Why is there more than one example of one-half?</td>
</tr>
<tr>
<td><strong>Making the mathematics visible</strong></td>
<td>Students discuss mathematical structures and make connections among mathematical ideas and relationships.</td>
<td>• What are some other ways to show one-half that are different than these models?</td>
</tr>
<tr>
<td></td>
<td></td>
<td>• How is your new model similar to the other models of one-half?</td>
</tr>
<tr>
<td><strong>Encouraging reflection and justification</strong></td>
<td>Students reveal deeper understanding of their reasoning and actions, including making an argument for the validity of their work.</td>
<td>• How might you prove that a fraction is equal to one-half?</td>
</tr>
<tr>
<td></td>
<td></td>
<td>• How can you tell if a new model represents one-half?</td>
</tr>
</tbody>
</table>

Types of Questions and Descriptions are from National Council of Teachers of Mathematics (NCTM). *Principles to Actions: Ensuring Mathematics Success for All*. Reston, VA: NCTM, 2014: 36-37. Question types may reflect all levels of Depth of Knowledge. Those noted are the most frequent examples of DOK levels.
# Patterns of Questions

<table>
<thead>
<tr>
<th>Pattern Type</th>
<th>Characteristics of Pattern Type</th>
</tr>
</thead>
</table>
| **Initiate-Response-Evaluate (IRE)** | • Teacher asks a question with a response in mind.  
• Student responds.  
• Teacher evaluates response.  
• Low-level questions with little wait time.  
• Limited opportunity to assess how students are thinking. |
| **Funneling**                | • A set of questions that lead students to a desired response.  
• Limited attention to responses that veer from the path.  
• Students’ ability to make their own connections is inhibited.  
• Similar to I-R-E, but higher-level questions may be part of the pattern |
| **Focusing**                 | • Teacher attends to what students are thinking.  
• Process is open to different approaches.  
• Teacher presses students to communicate clearly.  
• Teacher plans questions to outline key points. |


Beliefs About Mathematics Assessment

Select 2 statements about assessment. Write **P** if it is a productive belief or **U** if it is an unproductive belief. Be sure to explain your reasoning.

<table>
<thead>
<tr>
<th>Statement</th>
<th>Reasoning</th>
</tr>
</thead>
<tbody>
<tr>
<td>The primary purpose of assessment is accountability for students through report card marks or grades.</td>
<td>Only multiple-choice and other &quot;objective&quot; paper-and-pencil tests can measure mathematical knowledge reliably and accurately.</td>
</tr>
<tr>
<td>Assessment is an ongoing process that is embedded in instruction to support student learning and make adjustments to instruction.</td>
<td>Assessment in the classroom is an interruption of the instructional process.</td>
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<tr>
<td>Mathematical understanding and processes can be measured through the use of a variety of assessment strategies and tasks.</td>
<td>A single assessment can be used to make important decisions about students and teachers.</td>
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<td>Assessment is a process that should help students become better judges of their own work, assist them in recognizing high-quality work when they produce it, and support them in using evidence to advance their own learning.</td>
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<td>Stopping teaching to review and take practice tests improves students’ performance on high-stakes tests.</td>
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<td>Multiple data sources are needed to provide an accurate picture of teacher and student performance.</td>
<td>Ongoing review and distributed practice within effective instruction are productive test preparation strategies.</td>
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Write >, <, or = in the circle to compare the fractions. Explain how they could be compared without using common denominators.

\[ \frac{5}{6} \quad \bigcirc \quad \frac{2}{8} \]

Tell how you compared them without common denominators.

First, the denominator in \( \frac{5}{6} \) is 6, so I compared 6 and 8.

The denominator in \( \frac{2}{8} \) is 8, so \( \frac{5}{6} \) is greater than \( \frac{2}{8} \).

Write >, <, or = in the circle to compare the fractions. Explain how they could be compared without using common denominators.

\[ \frac{5}{6} \quad \bigcirc \quad \frac{2}{8} \]

Tell how you compared them without common denominators.

\[ \frac{5}{6} \text{ and } \frac{2}{8} \text{ are not equivalent. So } \frac{5}{6} \text{ is greater than } \frac{2}{8} \]

Write >, <, or = in the circle to compare the fractions. Explain how they could be compared without using common denominators.

\[ \frac{5}{6} \quad \bigcirc \quad \frac{2}{8} \]

Tell how you compared them without common denominators.

I used pictures to help, so I drew 5 and \( \frac{3}{8} \) and \( \frac{5}{8} \) covered more area.
1. The point shows \( \frac{3}{4} \) on each number line. Write the missing endpoint for each number line in the box.

\[ \begin{align*} 
0 & \quad 0 & \quad 0 
\end{align*} \]

\[ \begin{align*} 
\frac{3}{4} & 
\end{align*} \]

Explain how you found your endpoints.

I counted the line to see how many it was.

2. The point shows \( \frac{3}{4} \) on each number line. Write the missing endpoint for each number line in the box.

\[ \begin{align*} 
0 & \quad 0 & \quad 0 
\end{align*} \]

\[ \begin{align*} 
\frac{3}{4} & 
\end{align*} \]

Explain how you found your endpoints.

Well, I first counted how many lines there were, then how many lines were there. This would be in the box.

3. The point shows \( \frac{3}{4} \) on each number line. Write the missing endpoint for each number line in the box.

\[ \begin{align*} 
0 & \quad 0 & \quad 0 
\end{align*} \]

\[ \begin{align*} 
\frac{3}{4} & 
\end{align*} \]

Explain how you found your endpoints.

I counted how many units there were on each number line, then how many wholes there were and wrote the number of wholes there were in the box.

4. The point shows \( \frac{3}{4} \) on each number line. Write the missing endpoint for each number line in the box.

\[ \begin{align*} 
0 & \quad 0 & \quad 0 
\end{align*} \]

\[ \begin{align*} 
\frac{3}{4} & 
\end{align*} \]

Explain how you found your endpoints.

I lined up the wholes all to equal \( \frac{3}{4} \).
Bubble Gum Task

Four friends each bought a roll of bubble gum tape. Carlos chewed \( \frac{3}{4} \) of his gum. Helen chewed \( \frac{5}{6} \) of her gum. Jamal chewed \( \frac{6}{8} \) of his gum. Lizbeth chewed \( \frac{3}{5} \) of her gum.

1. Use the number line below to illustrate:
   - Which friend chewed the most gum?
   - Which friend chewed the smallest piece?
   - Which 2 friends chewed the same sized piece?

   ![Number Line]

2. Explain how you can use \( \frac{3}{4} \) to help you determine which value is greater, \( \frac{5}{6} \) or \( \frac{3}{5} \), if each fraction refers to the same whole.

3. Explain why \( \frac{a}{b} = \left( \frac{a}{b} \right) \times \left( \frac{n}{n} \right) \) and connect this equation to a visual diagram.

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