Name: ____Date: ________________

Painting a Wall-Area and Fractions

Predict: When you multiply a fraction by another fraction, will the product be larger than, smaller than or equal to the factors (original fractions)? ____________________________

Why? ____

____

____

Part 1: Show the fraction of the wall with play-doh. Draw a picture using the rectangles.

1. Paint $\frac{1}{3}$ of the wall.

2. Paint $\frac{1}{5}$ of the wall.

3. Paint $\frac{2}{3}$ of the wall.

4 Paint $\frac{3}{4}$ of the wall.

5. Paint $\frac{3}{5}$ of the wall.

Part 2: Painting a Fraction of a Fraction

1. Look back at # 1 from page 1. Build your play-doh to show $\frac{1}{3}$ again. What if on the first day, you only painted half of that amount?

a. How many equal pieces is the whole divided into?_____

b. What fraction of the wall did you paint? _________

c. Write what you painted in words:

I painted _________ of _________, which was __________ of the wall.

---

2. Look back at # 2 from page 1. Build your play-doh to show $\frac{1}{5}$ again. What if on the first day, you only painted half of that amount? Show what fraction you painted using the play-doh and with the picture below.

a. How many equal pieces is the whole divided into?_____

b. What fraction of the wall did you paint? _________

c. Write what you painted in words:

I painted _________ of _________, which was __________ of the wall.

---

3. Look back at # 3 from page 1. Build your play-doh to show $\frac{2}{3}$ again. What if on the first day, you only painted half of that amount? Show what fraction you painted using the play-doh and with the picture below.

a. How many equal pieces is the whole divided into?_____

b. What fraction of the wall did you paint? _________

c. Write what you painted in words:

I painted _________ of _________, which was __________ of the wall.

---

4. Look back at # 4 from page 1. Build your play-doh to show $\frac{3}{4}$ again. What if on the first day, you only painted one-third of that amount? Show what fraction you painted using the play-doh and with the picture below.

a. How many equal pieces is the whole divided into?_____

b. What fraction of the wall did you paint? _________

c. Write what you painted in words:

I painted _________ of _________, which was __________ of the wall.

Part 3: Area Model to Multiply Fractions

For each problem below, draw a picture to show the area as the product of the factors (dimensions). The square has a length and width of 1.

1. $\frac{1}{4} \times \frac{1}{2}$ This means I need to find ___________ of ____________________.

a) How many equal parts is the whole divided into? _________

b) What fraction of the whole represents the area or product?________

c) So, $\frac{1}{4} \times \frac{1}{2} =$ __________

2. $\frac{1}{2} \times \frac{1}{5}$ This means I need to find ___________ of ____________________.

a) How many equal parts is the whole divided into? _________

b) What fraction of the whole represents the area or product?________

c) So, $\frac{1}{2} \times \frac{1}{5} =$ __________

3. $\frac{1}{3} \times \frac{1}{4}$ This means I need to find ___________ of ____________________.

a) How many equal parts is the whole divided into? _________

b) What fraction of the whole represents the area or product?________

c) So, $\frac{1}{3} \times \frac{1}{4} =$ __________

4. $\frac{2}{3} \times \frac{1}{3}$ This means I need to find ___________ of ____________________.

a) How many equal parts is the whole divided into? _________

b) What fraction of the whole represents the area or product?________

c) So, $\frac{2}{3} \times \frac{1}{3} =$ __________

---

5. $\frac{1}{3} \times \frac{3}{4}$ This means I need to find ___________ of ____________________.

a) How many equal parts is the whole divided into? _________

b) What fraction of the whole represents the area or product?________

c) So, $\frac{1}{3} \times \frac{3}{4} =$ __________

---

6. $\frac{2}{3} \times \frac{1}{5}$ This means I need to find ___________ of ____________________.

a) How many equal parts is the whole divided into? _________

b) What fraction of the whole represents the area or product?________

c) So, $\frac{2}{3} \times \frac{1}{5} =$ __________

7. $1\frac{1}{3} \times \frac{1}{2}$ This means I need to find ___________ of ____________________.

a) How many equal parts is the whole divided into? _________

b) What fraction of the whole represents the area or product?________

c) So, $1\frac{1}{3} \times \frac{1}{2} =$ __________

8. $2\frac{1}{4} \times \frac{1}{2}$ This means I need to find ___________ of ____________________.

a) How many equal parts is the whole divided into? _________

b) What fraction of the whole represents the area or product?________

c) So, $2\frac{1}{4} \times \frac{1}{2} =$ __________

Part 4: Drawing Conclusions by Looking at the Math

Record the factors and product for each problem you solved in **part 3**.

Problem #	1st Fraction	2nd Fraction	Product
1			
2			
3			
4			
5			
6			
7			
8			

1) Looking across each problem, what pattern do you notice about the two numerators of the factors compared to the numerator in the product?

I noticed that ____

____

____

2) Looking across each problem, what pattern do you notice about the two denominators of the factors compared to the denominators in the product?

I noticed that ____

____

____

3) So, when I am multiplying two fractions together, I ______________________________

the numerators and I ____________________________ the denominators to get the product.

Revisiting the Prediction

Is it possible to multiply two fractions and have the product be smaller than either of the two factors? When might this happen and why?

When we multiply two fractions together, the product is less than the two factors when

____ because ________________________

____

____

Teacher Directions

Materials:

- ◊ Play-doh (1 can per person or pair), Plastic Knife and Plate
- ◊ Colored Pencils or highlighters

Objective

Students will represent a fraction of a rectangle using play-doh and then show what a fraction of that fraction would represent. Students will connect this representation to a visual model of the area model and use this to multiply fractions. Students will study patterns to generalize a rule for multiplication of fractions.

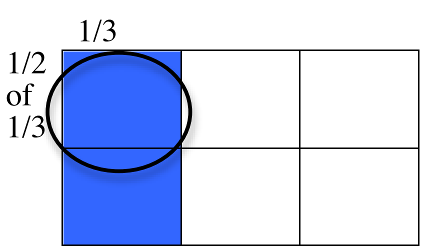
Directions:

Pass out the activity sheet and give the students two minutes to read and complete the **prediction** section (we will come back to this later in the lesson.)

Pass out play-doh, plastic knives (or dental floss) and a plate to each person or pair of students. Ask the students to form their play-doh into a rectangle or rectangular prism. Ask them to use their knives to show $\frac{1}{3}$. Call on a few students to come show how they represented one-third and how they know the piece is really one-third. Have the students use colored pencils to show one-third on the picture model.

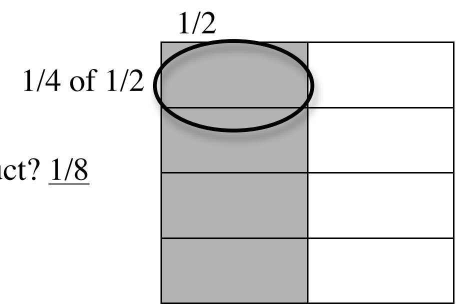
Repeat this process for the remaining problems in part 1. It is okay if students represent their fraction differently, as long as it really is the correct fraction of the wall and they can explain. For part 2, however, students will soon find that shading like above will be easier to do with all models (generalizing!). Thebig ideas for part 1 are: 1) 1/3 can llook different ways…cuts can be horizontal or vertical or other ways too and 2) we need equal parts if we are going to use the fractional names – 2 pieces that are unequal cannot be called halves).

Direct the class' attention to **part 2**. Have each student build their wall again and show one-third. Then ask, what if you only painted half of that amount on day 1? Have the students use their play-doh to show one-half of the one-third. Ask, how many equal pieces is the whole divided into (shoud be 6, regardless of which method students used). Ask how many pieces represent what they will paint that day (should be one of the sixths, or $\frac{1}{6}$). Have students share methods, and if possible, guide the students to agree upon a method where you mark halves on one side and thirds on the other side (as shown below). If other methods seem equally easy, you can agree upon this during a later problem (such as number 4 where other methods get too messy). See example.



If students did well with #1, let them continue using play-doh and then drawing a picture to represent the fraction of the wall painted each time. Have students come present their play-doh model and picture for each problem, making sure to ask for alternate ways to build or draw. Discuss the alternate ideas, noting which are the same (commutative property) and which are different and if they still work and what advantages or disadvantages might be. Make sure to support all correct methods, but agree, by the end of #4, to use the model above (or the same model with the dimensions reversed).

Part 3
Allow students to continue with play-doh, if they would like, but require them to draw using an area model. (See below for #1). Have the students try #1, come back to have students present and discuss and then repeat this for #2. After this, allow students time to work on their own for about 20 minutes (while you circulate and ask questions) before coming back and having students share their thinking and work.



a) How many equal parts is the whole divided into? 8

b) What fraction of the whole represents the area or product? 1/8

c) So, $\frac{1}{4}\times\frac{1}{2}=\frac{1}{8}$

Part 4
Some students may have already recognized a pattern for how to multiply without the area model- great! To help all students see this, have them begin by filling in the table, using the answers for part 3. (To save time, you can pre-fill in the table and put up the correct answers). Use think-pair-share to have students answer the conclusion questions. The goal of this section is for students to use logic in repeated reasoning to see that the product of two fractions can be found by calculating the product of the numerators and the product of the denominators. Additionally, the “revisiting the prediction” section is intended for students to explain the concept of the multiplication of fractions as finding a part of a part. You should hihglight answers that reveal that the product can be less when multiplying two fractions less than one as you take a fraction of a whole and then only a fraction of that fraction. Note that we will deal with the concept of when the product of two fractions is less than one, equal to one or greater than one in a subsequent lesson.

Name: _______________________________________ Date:_______________

Money for Children's Research Hosptial

Background: Each year near the Holidays, businesses will work to collect money for different charities. Children's Research Hospital is a place where kids with serious medical issues come get treatment for free and where doctors try to find cures for some of these diseases. This year, Children's Research Hospital is trying to raise 1 million dollars with the help of a toy store.

In the past, the toy store has asked each shopper to donate $1, $5 or $10 to the Children's Research Hospital. While they were able to collect a lot of money, they did not reach $1,000,000. This year they have a new plan. The toy store thinks a lot of people say no to giving dollars, but they think that people would be willing to give up change. So, they have decided to ask each person who shops if they would donate their "change" from each order. What this means is that they will ask each person to donate the amount of money needed to make their bill reach the next whole dollar amount. So, if a bill was $34.50, they would ask the shopper to donate $.50 to make their total bill $35.00. If a bill was $34.72, they would ask the shopper to donate $.28 cents to make their total bill $35.00.

The toy store tried this approach for a week and found that almost every customer said yes to donating the "change" it would take to get to the next full dollar. But, the toy store and charity are worried that they won't collect very much money, as each shopper is only donating between $.01 and $.99.

Prediction

1. How long do you think it will take the toy store to collect $1,000,000 using this method? Why?

____

____

2. Will this new method allow the toy store to collect more or less money than when they just asked for donations of $1, $5 or $10?

____

____

Assumptions: As a group, discuss and record what assumptions you are making to determine your answers above. Below are some questions to consider.

1. How many toy stores are there in the United States who are participating in raising funds for the Research Hospital?

2. About how many people buy something from each toy store each day?

3. How many people will say, “yes” to donating their “change” each day?

4. How many days will the toy store be trying to collect money for the Research Hospital?

5. What is the average amount of money each person will donate?

6. How many people donated money with the old system of asking for $1, $5 or $10?

Preparing Your Groups Presentation

After discussing all of your assumptions, as a team come up with an official answer to the two prediction questions. Be prepared to share the following with the class or a small group:

How long do you think it will take the toy store to collect $1,000,000 using this method?

- ◊ State your assumptions
- ◊ Show calculations used
- ◊ Use complete sentences to explain your answer and reasons

Will this new method allow the toy store to collect more or less money than when they just asked for donations of $1, $5 or $10?

- ◊ State your assumptions
- ◊ Show calculations used
- ◊ Use complete sentences to explain your answer and reasons

Teacher Directions

Materials:

- ◊ Optional: Video of Children's Research Hospital
- ◊ Calculators

Objective: In this modeling lesson which serves as a hook for this unit, students consider how much money can be raised for a charity by asking each shopper to donate the change needed to increase their total bill to the next full dollar amount. Students will discuss what assumptions they are making to determine whether or not the store can raise $1,000,000 and whether or not this method will be more effective than asking shoppers to simply donate $1, $5, or $10.

Directions

Note: This is lesson involves mathematical modeling, which means students must assume or infer missing information in making a decision and thus there can be different correct answers, justified by math and assumptions!

Begin the lesson by asking students if they have ever seen a store ask shoppers to donate money for a charity or cause. Allow a few students to share. Then ask if anyone has heard of "children's research hospitals". Optional: Show a brief video of what places like St. Jude's Children's Research Hospital do. Note- the videos are very emotional, so consider this when sharing with your class. Here is one option of which you can show a few minutes: https://www.youtube.com/watch?v=IhuE8K25TWA
Instead of showing a video, you can also share some quick facts about what St. Jude does as listed below:

1. Families never receive a bill from St.Jude for treatment, travel, housing and food - because all a family should worry about is helping their child live.
2. St.Jude is working to drive the overall survival rate for childhood cancer to 90 percent in the next decade. We won't stop until no child dies from cancer.
3. Treatments invented at St. Jude have helped push the overall childhood cancer survival rate from 20 percent to more than 80 percent since it opened 50 years ago. We won't stop until no child dies from cancer.
4. Because the majority of St. Jude funding comes from individual contributors, St. Jude has the freedom to focus on what matters most - saving kids regardless of their financial situation.

Pass out the activity sheet and have a few volunteers read the background. Ask a few questions to make sure all the students understand how the fund-raising works. Using think-pair-share, ask the class how much money the toy store would ask someone to donate if their bill was $3.90. Ask this for 2-3 more examples to make sure students realize that each shopper would donate between $.01 and $1.00. Make sure the class understands that not every shopper will say yes. Once the class understands the scenario, give the students 2-3 minutes to silently and individually complete the prediction questions. Do not say anything about how many stores there might be or answer any questions about details, but let the students know it is up to them to decide. After every student has recorded a prediction, take a quick class poll. Begin by asking who thinks the toy store will raise more money with this new method vs. asking for $1, $5 or $10 (use thumbs up or down to have the class vote and allow 2-3 students to share their reasoning). The take a poll to see how long the class thinks it will take to raise a million dollars. Ask who thinks they can raise this in 1 week, 1 month, 2 months, 3 months, 1 year, 5 years, 10 years, more than 10 years.

Put the students into groups of 2-4. Explain to the class that each student had to make some assumptions when answering the prediction question. Give the groups a few minutes to discuss what assumptions each student made in answering question #1. Use roundtable to allow each student 30-60 seconds to share. Bring the class back together and have them turn to page 2 of the activity sheet. Let them know they will have about 10 minutes to discuss the questions listed under the assumptions section and record their ideas. Make sure the class knows there are not exact answers for any of these! Option: allow students to do some research to help them answer the assumptions.

Once the 10 minutes are up, explain to the class that each group will now need to fully answer the two prediction questions. In their answer, each group must:

- ◊ State their assumptions
- ◊ Show calculations used
- ◊ Use complete sentences to explain their answer and reasons

Give groups about 10-20 minutes to prepare their solutions and answers. At the end of 20 minutes, you can either have groups present to the class and lead a discussion or you can pair up two groups and have them present to each other. Provide a few sentence frames for the groups to use to question the presenting group. See below for some examples.

- ◊ I agree with _______, but wonder if you considered __________.
- ◊ What were you assuming _________ would be?
- ◊ Did you consider ______ in your answer?

Close out the lesson by letting the students know that there is a store that raises just over $1,000,000 each year during the month of December using this method: Gymboree. Note that this is not to tell students they are right or wrong, as Gymboree is not the same as a Toy store, but just to let them know it is possible. Let the class know that this next unit will involve a study of "change" or decimals and that by the end of the unit, they should be able to make more precise calculations to show how much money could be raised.

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Math Modeling: An Introduction

Task 1: What is Modeling? Think-Pair-Share
In the CCS-M, math practice #4 is about students "modeling with mathematics" .
What do you think "modeling" means in mathematics?

Task 2: Reading

- The room will be divided into 1's and 2's.
- 1's will read a definition found in the appendix of the CCS
- 2's will read a definition by Polluck
- Using inside-outside line, 1's and 2's will teach and learn from one another.

Task 3: From what you learned, add to, delete or modify your initial definition of modeling from Task 1. What is Mathematical Modeling?

Task 4: Do a Modeling Task

Task 5: From what you learned, add to, delete or modify your initial definition of modeling from Task 1. What is Mathematical Modeling?

Task 6: Read about what Modeling is "not" from CA Framework Appendix D.
From what you learned, add to, delete or modify your initial definition of modeling from Task 1. What is Mathematical Modeling?

Reflection Questions:
1. Have you ever done "modeling" with your students? If so, how did it go? If not, why?

2. How would your students do if presented with today's modeling problem?

3. What will you need to do to prepare your students for success with mathematical modeling, such as the task we did today?

Math Practices 3, 7, and 8
Reflection

CCSS.Math.Practice.MP3

Construct viable arguments and critique the reasoning of others.

Mathematically proficient students understand and use stated assumptions, definitions, and previously established results in constructing arguments. They make conjectures and build a logical progression of statements to explore the truth of their conjectures. They are able to analyze situations by breaking them into cases, and can recognize and use counterexamples. They justify their conclusions, communicate them to others, and respond to the arguments of others. They reason inductively about data, making plausible arguments that take into account the context from which the data arose. Mathematically proficient students are also able to compare the effectiveness of two plausible arguments, distinguish correct logic or reasoning from that which is flawed, and—if there is a flaw in an argument—explain what it is. Elementary students can construct arguments using concrete referents such as objects, drawings, diagrams, and actions. Such arguments can make sense and be correct, even though they are not generalized or made formal until later grades. Later, students learn to determine domains to which an argument applies. Students at all grades can listen or read the arguments of others, decide whether they make sense, and ask useful questions to clarify or improve the arguments.

Construct viable arguments and critique the reasoning of others.

I can:

- use definitions and previously established results in constructing arguments.
- communicate and defend my own mathematical reasoning using examples, drawings, or diagrams.
- distinguish correct reasoning from reasoning that is flawed.
- listen to or read the conclusions of others and decide whether they make sense.
- ask useful questions in an attempt to understand other ideas and conclusions.

Where do students have the opportunity to use math practice #3 in the fractions lesson? Explain.

CCSS.Math.Practice.MP7 Look for and make use of structure.
Mathematically proficient students look closely to discern a pattern or structure. Young students, for example, might notice that three and seven more is the same amount as seven and three more, or they may sort a collection of shapes according to how many sides the shapes have. Later, students will see 7×8 equals the well remembered $7 \times 5 + 7 \times 3$, in preparation for learning about the distributive property. In the expression $x^2 + 9x + 14$, older students can see the 14 as 2×7 and the 9 as $2 + 7$. They recognize the significance of an existing line in a geometric figure and can use the strategy of drawing an auxiliary line for solving problems. They also can step back for an overview and shift perspective. They can see complicated things, such as some algebraic expressions, as single objects or as being composed of several objects. For example, they can see $5 - 3(x - y)^2$ as 5 minus a positive number times a square and use that to realize that its value cannot be more than 5 for any real numbers x and y.

CCSS.Math.Practice.MP8 Look for and express regularity in repeated reasoning.
Mathematically proficient students notice if calculations are repeated, and look both for general methods and for shortcuts. Upper elementary students might notice when dividing 25 by 11 that they are repeating the same calculations over and over again, and conclude they have a repeating decimal. By paying attention to the calculation of slope as they repeatedly check whether points are on the line through (1, 2) with slope 3, middle school students might abstract the equation $(y - 2)/(x - 1) = 3$. Noticing the regularity in the way terms cancel when expanding $(x - 1)(x + 1)$, $(x - 1)(x^2 + x + 1)$, and $(x - 1)(x^3 + x2 + x + 1)$ might lead them to the general formula for the sum of a geometric series. As they work to solve a problem, mathematically proficient students maintain oversight of the process, while attending to the details. They continually evaluate the reasonableness of their intermediate results.

Look for and make use of structure.

I can:

- look closely to see a pattern or a structure in a mathematical argument.
- can see complicated things as single objects or as being composed of several objects.
- can step back for an overview and can shift my perspective.

Look for and express regularity in repeated reasoning.

I can:

- notice if calculations are repeated.
- look for general methods and more efficient methods to solve problems.
- evaluate the reasonableness of intermediate results.
- make generalizations based on results.

How does math practices #7 and #8 help students move from the concept/big idea to deriving the procedure. Explain.

Mathematics Modeling

Modeling links classroom mathematics and statistics to everyday life, work, and decision-making. Modeling is the process of choosing and using appropriate mathematics and statistics to analyze empirical situations, to understand them better, and to improve decisions. Quantities and their relationships in physical, economic, public policy, social, and everyday situations can be modeled using mathematical and statistical methods. When making mathematical models, technology is valuable for varying assumptions, exploring consequences, and comparing predictions with data.

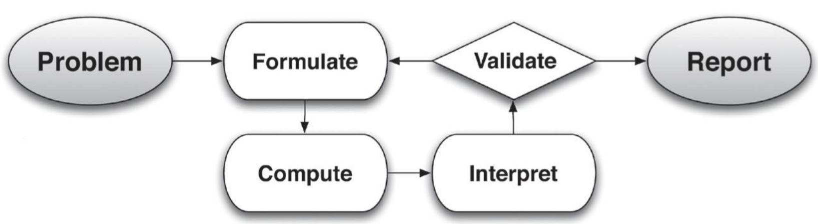
A model can be very simple, such as writing total cost as a product of unit price and number bought, or using a geometric shape to describe a physical object like a coin. Even such simple models involve making choices. It is up to us whether to model a coin as a three-dimensional cylinder, or whether a two-dimensional disk works well enough for our purposes. Other situations—modeling a delivery route, a production schedule, or a comparison of loan amortizations—need more elaborate models that use other tools from the mathematical sciences. Real-world situations are not organized and labeled for analysis; formulating tractable models, representing such models, and analyzing them is appropriately a creative process. Like every such process, this depends on acquired expertise as well as creativity.

Some examples of such situations might include:

- Estimating how much water and food is needed for emergency relief in a devastated city of 3 million people, and how it might be distributed.
- Planning a table tennis tournament for 7 players at a club with 4 tables, where each player plays against each other player.
- Designing the layout of the stalls in a school fair so as to raise as much money as possible.
- Analyzing stopping distance for a car.
- Modeling savings account balance, bacterial colony growth, or investment growth.
- Engaging in critical path analysis, e.g., applied to turnaround of an aircraft at an airport.
- Analyzing risk in situations such as extreme sports, pandemics, and terrorism.
- Relating population statistics to individual predictions.

In situations like these, the models devised depend on a number of factors: How precise an answer do we want or need? What aspects of the situation do we most need to understand, control, or optimize? What resources of time and tools do we have? The range of models that we can create and analyze is also constrained by the limitations of our mathematical, statistical, and technical skills, and our ability to recognize significant variables and relationships among them. Diagrams of various kinds, spreadsheets and other technology, and algebra are powerful tools for understanding and solving problems drawn from different types of real-world situations.

One of the insights provided by mathematical modeling is that essentially the same mathematical or statistical structure can sometimes model seemingly different situations. Models can also shed light on the mathematical structures themselves, for example, as when a model of bacterial growth makes more vivid the explosive growth of the exponential function.



The basic modeling cycle is summarized in the diagram. It involves (1) identifying variables in the situation and selecting those that represent essential features, (2) formulating a model by creating and selecting geometric, graphical, tabular, algebraic, or statistical representations that describe relationships between the variables, (3) analyzing and performing operations on these relationships to draw conclusions, (4) interpreting the results of the mathematics in terms of the original situation, (5) validating the conclusions by comparing them with the situation, and then either improving the model or, if it is acceptable, (6) reporting on the conclusions and the reasoning behind them. Choices, assumptions, and approximations are present throughout this cycle.

In descriptive modeling, a model simply describes the phenomena or summarizes them in a compact form. Graphs of observations are a familiar descriptive model— for example, graphs of global temperature and atmospheric CO2 over time.

Analytic modeling seeks to explain data on the basis of deeper theoretical ideas, albeit with parameters that are empirically based; for example, exponential growth of bacterial colonies (until cut-off mechanisms such as pollution or starvation intervene) follows from a constant reproduction rate. Functions are an important tool for analyzing such problems.

Graphing utilities, spreadsheets, computer algebra systems, and dynamic geometry software are powerful tools that can be used to model purely mathematical phenomena (e.g., the behavior of polynomials) as well as physical phenomena.

Modeling Standards *Modeling is best interpreted not as a collection of isolated topics but rather in relation to other standards. Making mathematical models is a Standard for Mathematical Practice, and specific modeling standards appear throughout the high school standards indicated by a star symbol* (★).

CA Framework:

What isn't mathematical modeling?

The terms "model" and "modeling" have several connotations, and while the term "model" has a general definition of "using one thing to represent something else," *mathematical modeling* is something more specific. Below is a list of some things that are *not* mathematical modeling in the sense of the CA CCSSM.

- It is not modeling in the sense of, "I do; now you do."
- It is not modeling in the sense of using manipulatives to represent mathematical concepts (these might be called "using concrete representations" instead.)
- It is not modeling in the sense of a "model" being just a graph, equation, or function. Modeling is a process.
- It is not just starting with a real-world situation and solving a math problem; it is returning to the real-world situation and using the mathematics to inform our understanding of the world. (I.e. contextualizing and de-contextualizing, see MP.2.)
- It is not beginning with the mathematics and then moving to the real world; it is starting with the real world (concrete) and representing it with mathematics.

"The Definition of Mathematical Modeling"

By: Henry O. Pollak

What do we mean by a *model* in a scientific or mathematical setting (see Juncosa 1966)?

1. An engineer may make a mechanical "model," with weights and springs, of an electrical circuit with capacitances and resistances.
2. A switching circuit may be a "model" for a Boolean function.
3. A real number may be a "model" for a point on a line, and vice versa.
4. A group may serve as a "model" for the symmetries of a rectangle.
5. A logistic function may be a "model" for the growth of a bacterial population.

In the first of these, one real-world object is represented by another. In the second, a mathematical object is represented by a real-world object; in the third, a mathematical object by a different mathematical object; in the fourth, one mathematical object by a more abstract mathematical object. The first four all represent meanings of a *model* in a scientific or mathematical setting. This chapter, however, is concerned with the fifth and last meaning, a *mathematical* model, which represents a real-world situation by a mathematical one.

Every application of mathematics uses mathematics to understand, or evaluate, or predict something in the part of the world outside of mathematics. What distinguishes modeling from other forms of applications of mathematics are (1) *explicit* attention at the beginning to the *process* of getting from the problem outside of mathematics to its mathematical formulation and (2) an explicit reconciliation between the mathematics and the real-world situation at the end. Throughout the modeling process, consideration is given to both the external world and the mathematics, and the results have to be both mathematically correct and reasonable in the real-world context. The steps in the mathematical modeling have been described many times.

The present analysis follows that of Pollak (1997):

1. We identify something in the real world we want to know, do, or understand. the result is a question in the real world.
2. We select "objects" that seem important in the real-world question and identify the relations among them. The result is the identification of important concepts in the real world situation.
3. We decide what we will keep and what we will ignore about the objects and their interrelations. We cannot take everything into account. The result is an idealized version of the original question.
4. We translate this idealized version into mathematical terms and obtain a mathematical formulation of the idealized question. This is called a *mathematical model*.
5. We identify the field or fields of mathematics that are relevant to the model and bring to bear our instincts and knowledge about these fields.
6. We use mathematical methods and insights to get results. Out of this step may come new techniques, interesting examples, solutions, approximations, theorems, algorithms.
7. We take all these results and translate back to the real world. We now have a theory about the idealized question.
8. Now comes the reality check. Do we believe what is being said? Are these results practical, the answers reasonable, the consequences acceptable?

a. If yes, the real-world problem solving has been successful. Our next job- namely, to communicate with potential users- is both difficult and extraordinarily important.

b. If no, we go back to the beginning. Why are the results impractical, or the answers unreasonable, or the consequences unacceptable? Because the model was not right. We examine what went wrong, try to see what caused it, and start again.

The whole sequence, steps 1-8, forms mathematical modeling. When people speak of "word problems" in a mathematics textbook, they usually mean some vocabulary from step 4 plus some aspect of step 6. (Step 5 is usually determined from the title of the book, or chapter, in which the word problem appears.) "Problem formulation" refers to steps 1 through 4, and the mathematical model itself is the end result of step 4. "Applied mathematics" is traditionally a name for a collection of fields of mathematics that arise frequently in step 5. An "application of mathematics" typically encompasses steps 4 through 7: An idealized version of a real-world problem is given to the solver, who must then translate it into mathematical terms, carry out appropriate mathematics, and restate the results in the vocabulary of the real-world situation. What is usually missing is the understanding of the original situation, the process of deciding what to keep and what to throw away, and the verification that the results make sense in the real world.

These definitions of what are, after all, familiar terms and phrases do not represent consensus among the body of mathematicians or authors. In particular, *applied mathematics* tends to be used very freely, and with a variety of meanings. But the differences among these meanings are very important for understanding the history of the teaching of mathematical modeling.

Furthermore, practitioners of mathematical modeling do not inevitably follow steps 1-8 sequentially and without further thought. Just as in metamathematics, or any thoughtful activity for that matter, they continually look both ahead and back, and the actual process is far less linear than the write-up might suggest.