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The Model Method: Singapore Children's Tool for Representing and Solving Algebraic Word Problems

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Solving arithmetic and algebraic word problems is a key component of the Singapore elementary mathematics curriculum. One heuristic taught, the model method, involves drawing a diagram to represent key information in the problem. We describe the model method and a three-phase theoretical framework supporting its use. We conducted 2 studies to examine teachers' perceptions and children's application of the model method. The subjects were 14 primary teachers from 4 schools and 151 Primary 5 children. The model method affords higher ability children without access to letter-symbolic algebra a means to represent and solve algebraic word problems. Partly correct solutions suggest that representation is not an all-or-nothing process in which model drawing is either completely correct or completely incorrect. Instead, an incorrect solution could be the consequence of misrepresentation of a single piece of information. Our findings offer avenues of support in word problem solving to children of average ability.

Key words: Algebra; Elementary, K–8; Problem-solving; Representation, modeling; Teaching practice

Solving arithmetic and algebraic word problems is a key component at every level of the Singapore primary mathematics curriculum (Curriculum Planning Division, 1990; Curriculum Planning & Development Division (CPDD), 2000a). However, the solution of word problems is one of the most problematic areas in the mathematics curriculum. Why are children challenged by word problems? What sort of instructional strategies can be provided to support children in such tasks?

Success in solving elementary word problems requires more than sound conceptual knowledge (Riley, Greeno, & Heller, 1983). Findings from the solution of physics word problems show how skilled physics problem solvers solved problems first by constructing elaborate representations of the problem, rather than solving directly from the text-based problem description. These representations often include diagrams that make certain relationships and constraints conspicuous

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(Larkin, McDermott, Simon, & Simon, 1980). Other studies, conducted with both adults and children, show that it is the ability to represent a given word problem and the ability to identify the appropriate computation, rather than algorithmic skill, that determines success in solving word problems (De Corte, Verschaffel, & De Win, 1985; Lee, Ng, & Ng, in press; Lewis, 1989).

The representation of problems should provide opportunities to (a) reflect on the representation, (b) make modifications, and (c) select a solution strategy (Briars & Larkin, 1984; Riley et al., 1983; Willis & Fuson, 1988). Of particular importance to our study are the challenges involved in children's acquisition and usage of various forms of algebraic representations.

Children encounter a variety of obstacles in using formal symbolic algebra to represent word problems. These include (a) understanding the meaning of letters used in symbolic algebra (Küchemann, 1981); (b) translating natural language into equations (e.g., Carpenter, Corbitt, Kepner, Lindquist, & Reys, 1981; Clement, 1982; Herscovics, 1989; MacGregor, 1991; Stacey & MacGregor, 2000); (c) understanding the semantic structure of word problems, in particular the nature of the relationships between quantities and how they are linked (Bednarz & Janvier, 1996); and (d) using text-based semantic cues in the construction of equations (Martin & Bassok, 2005).

In view of the cognitive demands posed by algebraic representations, with what other form of representation could young children work? A number of studies have shown that the use of visual and concrete representations improves performance in solving word problems. In a study conducted with adults, Lewis (1989) taught 96 college students in the United States to use a diagramming method for representing compare word problems. This method involved drawing a number line and asking students first to place on it a known value, then an unknown value related to the known value. Students' representation skills improved after the teaching experiment. Willis and Fuson (1988) taught two classes of second graders of average and above-average ability to use schematic drawings to solve word problems. Again, the results were positive. Further evidence comes from research on the Russian elementary curriculum by Davydov (Davydov, 1962; Davydov & Steffe, 1991; Freudenthal, 1974). In this curriculum, children learn to use lines to model part-whole relationships between quantities in their 1st year of primary schooling. They also use letters to express relationships between quantities and solve for missing wholes and parts using addition and subtraction. Children then learn to use equations to represent and solve two-step word problems. By the 3rd year of primary school, they are able to represent and solve word problems requiring proportional reasoning. Davydov's curriculum develops children's skill in solving complex word problems by drawing upon their ability to use visual models to analyze and express quantitative relationships, and to manipulate these relationships symbolically.

In Singapore, primary school children are taught a visual and concrete approach to solve *arithmetic word problems* and *algebraic*—the term we use for start-unknown—*word problems*. In 1983, the Singapore Ministry of Education (MOE) officially introduced a heuristic involving diagram or model drawing known as the model method into the primary mathematics curriculum. The model method can be used as a tool for solving both arithmetic and algebraic word problems involving whole numbers, fractions, ratios, and percents (Kho, 1987). It was believed that if children were provided with the means to visualize a word problem—be it a simple arithmetic word problem or an algebraic word problem—the structure underlying the problem would be made overt. Once children understood the structure of the problem, they were more likely to be able to solve it (Kho, 1987).

The model method focuses on the heart of the word problem solution—the importance of representation. The model method also requires children to work with three modes of representation: text, pictorial, and symbolic. It builds upon children's knowledge of part-whole relationships in numbers. In the lower grades, objects, pictures, and symbols are used to model combinations of numbers. In the higher grades, children are taught to use appropriately sized rectangles to represent the information presented in word problems. In arithmetic word problems, the rectangles represent specific numbers. By changing the role of the rectangles and using them to represent unknown quantities, the model method also can be used to depict algebraic word problems. Such representations effectively serve as "pictorial equations" (Cai et al. 2005, p. 8). Although the model method is very similar to Davydov's approach, there are two main differences. Davydov emphasized the use of letters as variables and the construction of equations to solve word problems. These are not the foci of the Singapore primary mathematics curriculum. In the model method, the rectangles represent unknown values, but the solution for the unknown is still grounded in children's knowledge of arithmetic.

In this article, we first provide details of the model method and show how it is used to solve arithmetic and algebraic word problems that require the construction of only one model drawing. Second, we discuss the theoretical framework highlighting the three phases of problem solving in which children engage when they use the model method to solve word problems. Third, we provide evidence from two studies showing how teachers and children use the model method to solve algebraic word problems. Fourth, we use examples of children's partly correct solutions to show that representation is not an all-or-nothing process. Such solutions are used to illustrate what may have prevented children from using the model method successfully. In the concluding section we discuss the implications of the findings from these two studies for mathematics education, in particular, how average-ability children can be supported in the model representation process of word problem solving.

THE MODEL METHOD—A HEURISTIC FOR SOLVING ARITHMETIC AS WELL AS ALGEBRAIC WORD PROBLEMS

In Singapore, a cartoon-like version of the model method is introduced in Primary 1 (entrance age of 6) or Primary 2. Children are taught to use pictures of bears and other familiar objects to model information presented in arithmetic word problems. Later, to increase the level of abstraction, rectangles are used to replace pictures.

Beginning in Primary 3, children are taught to apply the same heuristic to solve algebraic word problems involving whole numbers. This heuristic—the model method—gives children early access to such problems by circumventing the need to construct and solve linear algebraic equations. Algebraic word problems involving fractions and ratios are introduced in Primary 5 and 6.

The concept of letters as variables and the construction of algebraic expressions with one unknown are taught to Primary 6 children (CPDD, 2000a). The construction and solution of algebraic equations with one variable are taught in the 1st year of high school: Secondary 1. However, even younger children (9- to 11-year-olds) are taught to solve algebraic word problems that in algebra texts would be represented by equations such as $x \pm a = c$, $x \times a = c$, $(c > a,\ a \in \mathrm{N})$, $x \div a = c$, $(a,\ c \in \mathrm{N})$. (In this section, the letters *a, b, c,* and *n* are used to represent known values, and *x* is used to represent unknown values.)

These models serve as visual analogues that capture all the information in a word problem. Their structure consists of a series of rectangles in which the relationships of the rectangles are specified and are presented globally. Children use the structure of the model to help them construct appropriate sets of step-by-step arithmetic procedures to solve given problems.

Some Generic Examples of Models Commonly Taught to Primary Children

In this section, we present three different types of model drawings commonly taught to children. These are the (a) part-whole, (b) comparison, and (c) multiplication and division models. In the model method, letters are not used to represent numerical values, known or unknown.

The Part-Whole Model

The part-whole model can be used to solve arithmetic problems similar to the following Cruise Problem or algebraic word problems of the form depicted by the following Orchid Problem.

> Cruise Problem: On Saturday, 1050 people went on a cruise. On Sunday, 1608 people went on a cruise. How many people went on the cruise over the 2 days?

> Orchid Problem: There were 2659 visitors at Orchid Gardens. 447 of them were adults and the rest were children. How many children visited Orchid Gardens?

Such problems are commonly found in Primary 3 textbooks (for examples, see Collars, Koay, Lee, & Tan, 2007, p. 43).

The part-whole model can be used to represent the arithmetic situation represented by $a + b = x$ or the algebraic case of $x + a = b$. In the arithmetic situation, $a + b = x$, the model consists of two rectangles of different lengths representing the quantities *a* and *b* (see Figure 1, at left). The unknown total, *x*, is represented by a brace that links the two rectangles together. The unknown is found by summing *a* and *b*.

Hence, for the Cruise Problem, *a* and *b* represent the number of people for Saturday and Sunday and *x*, the total number of people who went on the cruise.

The model drawing for the algebraic word problem (see Figure 1, at right), has the same structure as the model drawing for the arithmetic word problem. The algebraic model can be used to solve for x in $x + a = b$. Where *a* and *b* are given, *x* could be found by subtracting *a* from *b*. In the algebraic model drawing, no letters are used to represent unknowns. Instead, a rectangle with a question mark within it signifies an unknown. Such a rectangle is known as one unit, and this unit rectangle assumes the role of letters as variables. Rectangles also can be used to represent specific numerical values, such as the numerical relationships between two variables, but such relationships are indicated using appropriately sized rectangles. In Figure 1 at right, the rectangle with the letter *a* represents the given numerical relationship between *b* and the unknown. In the Orchid Problem, an example of a start-unknown problem, the rectangle with the letter *a* could represent the number of adults and the rectangle with the letter *b* the total number of visitors. The number of children, represented by *x*, can be found by subtracting *a* from *b*.

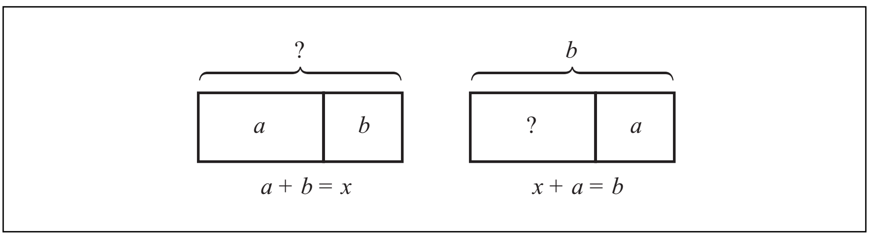

Figure 1. Part-whole models: Arithmetic model (at left) and algebraic model (at right).

The Comparison Model

The comparison model can be used to solve the following problems. The first, identified as the Enrollment Problem, is an arithmetic word problem; the second, the Animal Problem, is an algebraic or start-unknown word problem.

> Enrollment Problem: Dunearn Primary School has 280 pupils. Sunshine Primary School has 89 pupils more than Dunearn Primary. Excellent Primary has 62 pupils more than Dunearn Primary. How many pupils are there altogether?

> Animal Problem: A cow weighs 150 kg more than a dog. A goat weighs 130 kg less than the cow. Altogether the three animals weigh 410 kg. What is the mass of the cow?[1]

The comparison model shows the relationship between two or more quantities when they are compared. The varying lengths of the rectangles show that one quan-

[1] Given the age of the students in the study, we did not distinguish between mass and weight.

tity is bigger than another, and the difference between the quantities is indicated by the difference in lengths of the rectangles. The arithmetic model (see Figure 2, at left) represents the situation in which $a = b + d$ and the sum of a and b is x. With all the information made overt, the unknown total can be found by summing a and b. An algebraic model drawing with the same structure as the arithmetic model drawing can be used to solve the algebraic problem of the form $x + b = c$, where the rectangle with the question mark is the unit rectangle that assumes the role of the letter x (see Figure 2, at right).

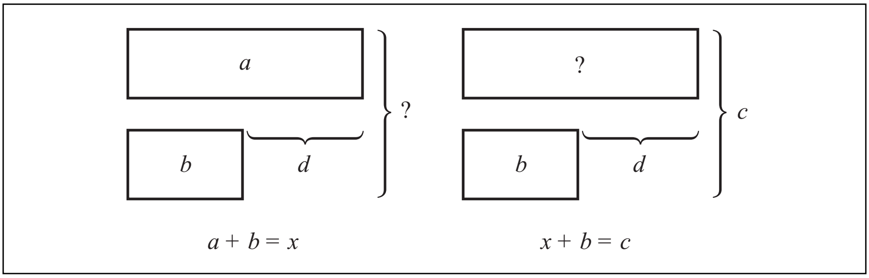

Figure 2. Comparison models: Arithmetic model (at left) and algebraic model (at right).

The model at left in Figure 3 depicts a more complex model, which is used to solve the Enrollment Problem, a type of problem commonly found in Primary 4 textbooks. This model drawing has three sets of rectangles. Each set of rectangles is labeled, for example, DP for Dunearn Primary. The numerical value represented by each rectangle is noted and what has to be calculated is indicated with a question mark. The total enrollment of the three schools is found by summing all the numbers in the rectangles. The algebraic model (see Figure 3, at right) with the same structure can be used to solve the Animal Problem.

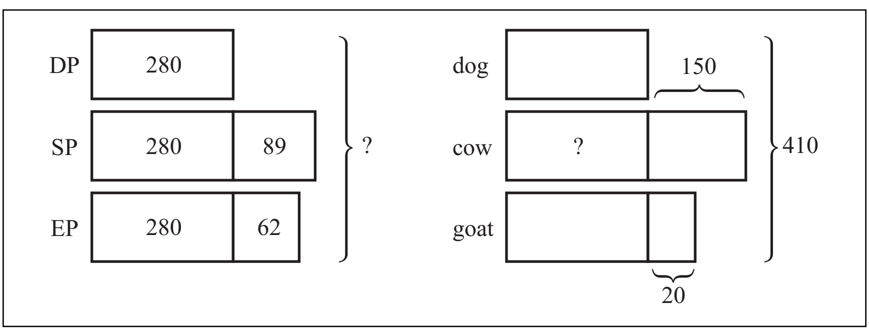

Figure 3. Comparison model used to represent the Enrollment Problem (at left) and the Animal Problem (at right).

In the Animal Problem, the comparison model is used to show the relationships among the masses of the three animals. Here the mass of the dog is used as the *generator* (Bednarz & Janvier, 1996) of the model drawing. In the model method, the rectangle representing the mass of the dog is the unknown unit. Representation of the masses of the two remaining animals is based on that of the dog. The differences in mass between the animals are recorded by rectangles indicating the known values. The total mass of the three animals is indicated on the right. The model drawing becomes a pictorial equation representing 3 *units* + 170 = 410. The value of one unit can be found by solving the equation 3 *units* + 170 = 410. The mass of the cow is found by summing the value of one unit and the difference between the mass of the cow and the dog. The pictorial equation represents an algebraic equation of the form $x + x + 150 + x + 20 = 410$, where x represents the mass of the dog.

Although a model with a similar structure can be used to solve arithmetic as well as algebraic word problems, the solution to the algebraic word problem is more challenging. Using the model to solve arithmetic problems is a direct process since it involves summing the numerical inputs represented by the rectangles, as demonstrated by the solution to the Enrollment Problem depicted in Figure 3, at left. In algebraic word problems, the outputs are given and the intent is to solve for the input. Solving for the unknown input requires a different way of thinking. In the Animal Problem depicted in Figure 3, at right, masses of the animals cannot be calculated using step-by-step procedures with a selected input. Instead, to solve for the unknown input, all three unknown states or rectangles have to be considered together. Also, any of the three unknown states can be used as an entry point to the problem. The mass of the cow can be found directly if it is chosen as the generator, and the cow rectangle is the unit rectangle. Alternatively, it can be found indirectly if the mass of the dog or the mass of the goat is chosen as the generator. This alternative solution is discussed further in the context of children's responses.

The Multiplication and Division Model

Models can also be used to represent problems involving multiplication and division as well as fractions. The model at left in Figure 4 represents the arithmetic situation $3a = b$ and $a + b = x$. The following is an example of a Primary 3 word problem that can be solved using this arithmetic model.

> Bala took 24 pictures. David took 3 times as many pictures as Bala. How many pictures did the two boys take in all? (Collars, Koay, Lee, & Tan, 2003, p. 84)

Figure 4, at right, shows the model representing the algebraic situation $x + b = a$ and $b = nx$, where n represents the multiplicative relationship between a and b. In this instance, the total is given but one of the parts is an unknown. This algebraic model can be used to solve this word problem introduced at Primary 5. The question mark in the bottom rectangle indicates the base used for comparison, in this case, the sum of money held by Mary.

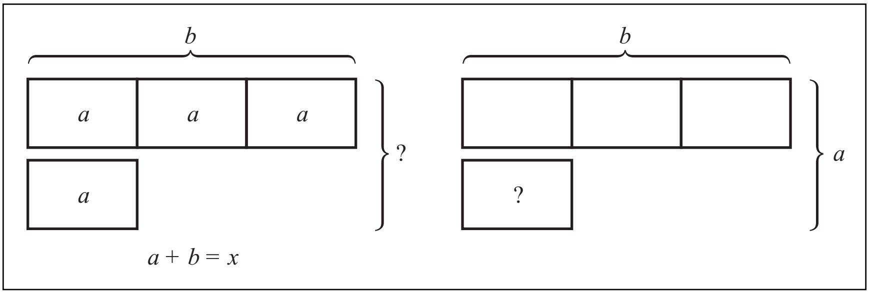

Figure 4. Multiplication and division models for arithmetic word problem (at left) and algebraic word problem (at right).

> Mary and John have $48 altogether. John has three times as much money as Mary. How much money has Mary? (Collars, Koay, Lee, Ong, & Tan, 2008, p. 33).

The multiplication and division model illustrates the efficacy of the model method in solving problems involving fractions. With the model method, rather than operating on fractions, children are provided the option of using concepts of, and operating on, whole numbers to solve problems involving fractions. For example, using the model method, children need not operate on fractions to solve the following word problem involving fractions introduced at Primary 5.

> If the volume of water in container A is 1/4 the volume of water in container B, and the total volume of water in both containers is 250 litres, find the volume of water in container A.

By working backward, children identify the volume of B as four times that of A. Hence, one rectangle is used to represent the volume of A while B is represented by four rectangles of the same size. The algebraic model in Figure 5 illustrates the relationship between A and B and all the information provided in the question. The model drawing shows that since there are five equal parts, then 5 *units* = 250 *l*. Hence, 1 *unit* = 50 *l* and container A's volume is 50 *l*. As with the other models, the multiplication and division model can be used to solve algebraic problems in which $x + b = c$ and $x = (1/n)b$.

THE EFFICACY OF THE MODEL METHOD: THEORETICAL CONTEXT

We hypothesized that children who use the model method to solve word problems are guided through three phases of problem solution presented schematically in Figure 6. This theoretical framework is constructed based on the processing model for solving arithmetic word problems presented by Kintsch and Greeno (1985) and on our observations of teachers and children who used the model method to solve word problems. The theoretical model we present is for consistent-language

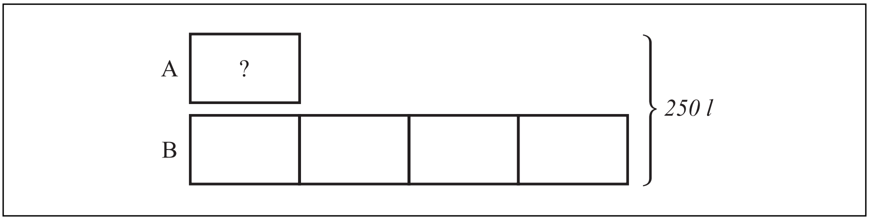

Figure 5. Example of an algebraic multiplication and division model.

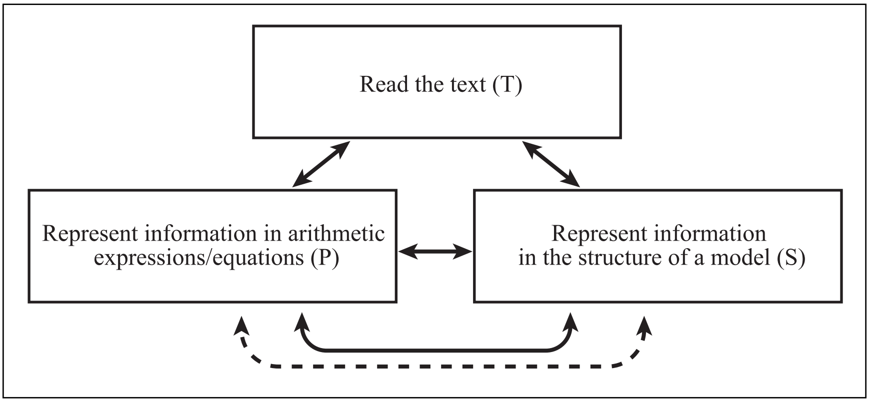

Figure 6. Possible phases of problem solution by children who used the model method to solve word problems.

Note: In the TS route, children represent information embedded in the text in the structure of a model. In the SP route (the solid line), to solve the problem, children represent the relationships embedded in the model in a series of arithmetic equations. In the TP route, some children bypass the TSP route, moving directly from T to P. In the TPS route, children represent the textual information in a set of arithmetic expressions or equations, which they then use to draw the model. This alternative route is represented by the path in dotted lines. Double-headed arrows are used to signify that children may alternate between representations to check the accuracy of the representations.

compare word problems that require the construction of a single model drawing that typify word problems for Primary 5 children in Singapore. Consistent-language word problems are those in which the comparative relationships such as *more than* are in agreement with the arithmetic operation for solution.

Kintsch and Greeno's (1985) processing model for solving arithmetic word problems works with two modes of representation, text-based information and abstract-problem representation. The text-based representation comprises sets of information abstracted from the word problem, which are then organized into a meaningful macrostructure highlighting pertinent concepts and relationships between sets of information. The abstract-problem representation contains relevant information derived from the text but presented in an abstract form that with application of suitable calculation strategies will result in the solution of the problem.

Although the Kintsch and Greeno model discusses the processes and the knowledge involved at each phase of representation, the exact form of the abstract-problem representation is not specified. With the model method, however, the abstract-problem representation is delineated into two specific phases, the structural phase and the procedural-symbolic phase, with the nature of each phase clearly specified. The model drawing becomes a macrostructure capturing the inputs, the relationships between the inputs, and the output of the problem. This information is then translated into a series of arithmetic equations. The model drawing is one layer of the multilayered representation necessary for the solution of word problems. Thus, this model extends the work by Kintsch and Greeno.

Phase 1: Text Phase (T)

Children read the information presented in text form.

Phase 2: Structural Phase (S)

In this phase, children represent the text information in the structure of the model. Children can alternate between text and the model to check that the model accurately depicts the textual information.

Phase 3: Procedural-Symbolic Phase (P)

Once they have constructed a model, children then use the model to plan and develop a sequence of logical arithmetic equations, which allows for the solution of the problem. Again, alternating between the two representations, structural and procedural (SP), allows for the accuracy of the arithmetic equations to be checked against the model.

In the text phase, many iterations of the reading are conducted. The first reading provides the context of the story. In subsequent readings of the text, information is further processed to ascertain what the givens are and what is to be found. Each bit of information is termed a chunk. Each chunk may refer to a specific numerical quantity or specific information related to an unknown quantity. In the Enrollment Problem, the information “Dunearn Primary School has 280 students” is a chunk. The chunk “Sunshine Primary School has 89 more students than Dunearn Primary” stipulates the comparative relationship between the two schools. Each chunk is represented by at least one rectangle. If the comparative relationship between two chunks is one of more than, then two rectangles may be used to represent that information. For example, two rectangles can be used to represent the enrollment of Sunshine Primary or the mass of the cow in the Animal Problem. With the model method, there is continual coordination between the text phase and the structural phase. After representing one chunk of information, the child returns to the text for the next chunk of information to be represented. In the structural phase, the focus is on how to represent each chunk; hence, alternations between the text phase and the structural phase will continue until all the information has been represented. Each rectangle or set of rectangles becomes a pictorial chunk representing text information. Translation between the structural phase and the procedural-symbolic phase takes place once the model drawing is completed. The arithmetic equations are the foci of the final phase, and the information processing takes place in several steps. Each alternation between the structural phase and the procedural phase may result in an arithmetic equation that captures the information of a pictorial chunk or two chunks.

Rectangles of arithmetic word problems represent specific numerical inputs. If the model method is used to solve algebraic word problems, the rectangles can either represent unknown inputs, such as the mass of the animals in the Animal Problem, or specific numerical values, such as the difference in mass between two particular animals. Whether the model method is used to solve arithmetic or algebraic word problems, children go through these three phases of problem solution, albeit not necessarily in a fixed order. Where appropriate, children may first translate the text information into a set of arithmetic expressions (TP) and then use these to construct a model (TPS). The objects of the procedural phase could either be a series of arithmetic equations or a set of algebraic equations, the latter being dependent on children's cognizance with equations. Regardless of the routes taken, the products at the second phase provide evidence as to whether children have understood the word problem. Similarly, the arithmetic expression presented in the final phase provides evidence of children's interpretation of the model.

Solution of word problems is contingent upon, inter alia, sound conceptual knowledge (Vergnaud, 1982), knowledge of part–part–whole relationships of numbers (Carpenter & Moser, 1982), an integrated and well-organized knowledge base, and metacognitive processes (Verschaffel, Greer, & De Corte, 2000). Sound conceptual knowledge is vital because though different word problems may have the same mathematical structure, their underlying conceptual structures may be very different. The discussion of generic examples of models shows that although certain arithmetic and algebraic word problems may share the same model, models for algebraic word problems have very different underlying conceptual structures from those underpinning arithmetic word problems.

The model method utilizes knowledge of the part–part–whole relationship of numbers and that a rectangle is used to represent each part, a relatively longer rectangle for a bigger number, a shorter rectangle for a smaller number, or a rectangle of arbitrary length for an unknown unit. An integrated and well-organized knowledge base of how to represent information such as a known unit, an unknown unit, and comparative relationships such as *more than*, *less than*, and *as many as* are vital for the construction of models.

Successful utilization of the model method is also contingent upon metacognitive processes. For example, it may be necessary first to represent the text information with an appropriate set of arithmetic expressions that can be used to draw the model (T → P → S). Also, as any unknown could be used to solve an algebraic word problem, a decision has to be made to select the appropriate generator. In both of these cases, metacognition is necessary (a) to decide whether to use a TSP route or a TPS route and (b) in generator selection.

The affordance of a visual and concrete receptacle for given numbers or unknowns means that primary children without prior knowledge of letter-symbolic algebra can use the model method to solve algebraic word problems. The large numbers involved make it unlikely that problem-solving procedures such as counting all or counting on can be used. Rather the method capitalizes on children's facility with arithmetic operations to solve for the value of the unknown unit represented by rectangles. By a process of undoing, the value of the unknown unit is found. Hence, children circumvent the cognitive demands that are inherent in solving algebraic equations, which normally require the construction of a system of equivalent equations. Children can manipulate and operate on the rectangles without having to engage with many of the abstractions and difficulties associated with symbolic representations reported in the research literature (Behr, Erlwanger, & Nichols, 1980; Booth, 1988; Kieran, 1981; Küchemann, 1981; Schoenfeld & Arcavi, 1993; Stacey & MacGregor, 1999; Steinberg, Sleeman, & Ktorza, 1991; Usiskin, 1988).

THE PRESENT STUDIES

Singapore has a centralized education system with the Ministry of Education spearheading curriculum development and implementation. Hence, all schools follow a common mathematics curriculum and a common syllabus provided by the Ministry. Prior to 2006, all schools used a common textbook series. Although schools can now select textbooks from a number of commercially produced series, all these series are approved by the Ministry. All textbook series introduce and develop the model method as a problem-solving tool.

There is a dearth of research on the delivery of the model method and on children's use of the model method. Two studies were conducted: Study 1 examined teachers' perception of and facility with the model method, and Study 2 explored how children used the model method. Although it would have been preferable to observe teachers in their teaching of the model method, funding and timing constraints made this impractical. Instead, Primary 5 teachers were asked to demonstrate and explain how they used the model method to solve a set of algebraic word problems that were used in the second study. Because data for the second study were collected as part of a larger study on the relationship between working memory and problem-solving proficiency (Lee, Ng, Ng, & Lim, 2004), constraints on the amount of time we had with each child precluded detailed interviews with the children. Instead, teachers were asked to help us understand children's responses.

Study 1: Teachers

We interviewed four heads of department (HODs) and 14 Primary 5 mathematics teachers, 4 of whom were from the five schools participating in Study 2. The objectives of interviews with the HODs were to find out how and when the model method was introduced to children. Interviews with the teachers were conducted to help us understand the key ideas on which they believed children had to focus when using the model method.

Method

Participants

Contact with schools was made via teachers who participated in professional development courses conducted by the first author. All HODs and one Primary 5 mathematics teacher from each of the five participating schools were invited to take part in the interview, and all but those from one school agreed to participate. Ten primary teachers, two from each of five other neighborhood schools, also were interviewed. All the HODs had held their positions for more than 10 years and all 14 teachers had been teaching Primary 5 mathematics for at least 5 years. Primary 5 mathematics teachers were chosen because it was at this grade that the model method was used to solve a wider variety of word problems.

Instrument

An instrument comprising 10 questions was designed for Studies 1 and 2. Except for Question 1, which is an arithmetic word problem, the remaining questions are algebraic. One arithmetic word problem was used to ensure that as many children as possible were able to participate in the test. Curricular materials such as textbooks and worksheets used by different schools guided construction of the items. Various versions of the test instrument were designed and piloted, and instruments that resulted in extreme scores were rejected. The items listed in Table 1 were chosen because the piloted version showed that children gave varied responses to the same items. A group of primary mathematics teachers attending a professional development course, but who were not involved in the Study with Teachers, also was asked to comment on the suitability of the final set of items. The final set of items was chosen because they required the application of a variety of mathematical concepts taught to children by the end of Primary 4. Solution of the problems required proficiency with concepts of fraction and number theory up to and including factors and multiples of numbers. No questions tested the concepts of ratio and percent because the study was conducted before such concepts were taught. The first 5 problems can be solved with the construction of a single model drawing. The last 5 problems are more challenging than the first 5, in part because more than one model drawing would be needed if the model method were used to solve the last 5 problems.

Interview With the HODs and the Teachers

Although the mathematics questions were e-mailed to the respective teachers ahead of the interview, the teachers chose to solve the questions during the interview itself. In this article, we report teachers' responses to two questions: (a) What are the

Table 1
Word Problems Presented to Children and Success Rates as a Function of Group

The problems	Success rates (percents)		
	Entire group ($n = 151$)	EM1 ($n = 31$)	EM2 ($n = 120$)
1. Enrollment Problem:[a] Dunearn Primary School has 280 pupils. Sunshine Primary School has 89 more pupils than Dunearn Primary. Excellent Primary has 62 more pupils than Dunearn Primary. How many pupils are there altogether?	63	87	57
2. Furniture Problem: At a sale, Mrs. Tan spent \$530 on a table, a chair, and an iron. The chair cost \$60 more than the iron. The table cost \$80 more than the chair. How much did the chair cost?	44	84	33
3. Animal Problem: A cow weighs 150 kg more than a dog. A goat weighs 130 kg less than the cow. Altogether the three animals weigh 410 kg. What is the mass of the cow?	37	74	28
4. Water Problem: A tank of water with 171 liters of water is divided into three containers, A, B, and C. Container B has three times as much water as container A. Container C has 1/4 as much water as container B. How much water is there in container B?	21	58	11
5. Book Problem: A school bought some mathematics books and four times as many science books. The cost of a mathematics book was \$12 while a science book cost \$8. Altogether the school spent \$528. How many science books did the school buy?	21	55	12
6. Cat Problem: Mae Ling bought a new box of cat biscuits. In the first week, she gave the cat half the biscuits and three more. In the second week, she gave the cat half of the remaining biscuits and 3 more. In the third week, she gave the cat half of the remaining biscuits and 3 more. There was only 1 piece of biscuit left. How many pieces of biscuits were there in the new box?	11	42	3
7. Age Problem: Mr. Raman is 45 years older than his son, Muthu. In 6 years time, Muthu will be 1/4 his father's age. How old is Muthu now?	16	42	9
8. Stamp Problem: Vincent bought a total of 62 pieces of \$3 and \$5 stamps. Altogether Vincent spent \$254. How many \$3 stamps were there?	0	0	0
9. Money Problem: Each week, Yah Hui gets \$6 more pocket money than Philip. Each week, Yah Hui and Philip each spend \$19 on books. After some weeks, Yah Hui saved \$98 while Philip saved \$56. How much pocket money does Philip get each week?	10	16	8
10. Class Problem: During a class game, Peter threw a ball to 4 times as many boys as girls. Mei Lin threw the ball to 5 times as many boys as girls. If Peter and Mei Lin threw the ball to every student in the class, how many boys were there in the class?	5	16	2

[a] Problems are named for ease of reference.

important points to look for when drawing the model for a given problem? and (b) How are children taught to solve for the value of the unknown unit if they are neither taught to construct nor taught to transform equations? The HODs and the mathematics teachers were interviewed on separate occasions. Each interview session was audiotaped and required approximately 1 hour to complete.

Findings

Both the HODs and the teachers reported that the model method was taught to their students. All but one school introduced the model method at the Primary 2 level. The remaining school did so at Primary 1. Other problem-solving heuristics were introduced to children as they progressed through the primary years. The HODs explained that an even less abstract precursor activity to the model method, such as pictures, was used to introduce the part-whole model to Primary 1 children. For example, pictures of five brown teddy bears and three white teddy bears could be used to represent the part-whole concept of addition of 5 + 3 at Primary 1. With older children, the pictures were replaced by rectangles within the models.

What are the important points to look for when using the model method to solve a given problem? The teachers explained that care had to be exercised when drawing the rectangles so that the entire model drawing was meaningful. For example, when the model method is used to solve arithmetic word problems, the lengths of the rectangles should be in proportion to each other. This was possible for arithmetic word problems because the rectangles represent specific numbers. In the Enrollment Problem, for example, the length of the rectangle representing the difference in enrollment between Dunearn Primary and Sunshine Primary is 89, so the length of this rectangle should be about one third of the rectangle representing the enrollment of Dunearn Primary. Also, all the known information should be recorded onto the model and question marks used to identify unknown values that are to be evaluated. When it was necessary to partition a rectangle into smaller rectangles, dotted lines were used. The use of different types of lines helped children keep track of the problem structure, whether the multiplication or division operation was involved.

How are children taught to solve for the value of the unknown unit if they are neither taught to construct nor taught to transform equations? Because primary children were neither taught to construct nor taught to transform equations, they were taught the unitary method as a strategy to evaluate the value of the unknown unit. This strategy, which takes place in the procedural-symbolic phase of the model method, involves the construction of a series of arithmetic equations that represent the unknown units. This set of arithmetic equations is solved by undoing related operations. In the Animal Problem, for example, if the mass of the dog was the chosen generator, then the rectangle representing the mass of the dog is the base for comparison. Children were taught to find the sum represented by the three rectangles by undoing the differences from the total mass (410 – 170). This suggests that the model drawing was a form of pictorial equation in which if a certain known amount was removed from the left side of the brace, then the same amount needed to be removed from the right of the brace. The brace functions as if it were the pivot of a balance or the equal sign of an algebraic equation. If three unknown units were equivalent to 240, then the value of one unknown unit was found by dividing the difference by three (240 ÷ 3). In this way, children were required to undo the multiplication operation. The mass of the cow was found by summing the value of one unknown unit with its difference with the dog (80 + 150 = 230). At no time were children required to indicate what each individual expression represented, but they were expected to know the meaning of each expression (e.g., that the expression 410 – 170 represented the total value of three units).

Discussion

Because we did not have access to classroom teaching, we cannot comment on the nature of the delivery of lessons across the years. This is a limitation of this study. Nevertheless, the four schools in this study had a structure to ensure the teaching of heuristics. Although HODs and teachers were from different schools, they agreed that the visual and concrete nature of the model method made it a useful problem-solving tool. Because they used the unitary method strategy to solve for the unknown unit, they avoided the need to construct equivalent equations. The objects of the procedural phase were a series of logical arithmetic equations, not a set of equivalent algebraic equations. Such a strategy circumvented the need to teach transformation of equations, which the teachers explained was beyond the primary school syllabus.

Study 2: Children

In this study, we examined how 151 children from five primary schools chose to use the model method to solve five word problems of increasing difficulty.

Method

Participants

Of the 151 Primary 5 children recruited from five neighborhood schools—non-selective schools situated in housing estates—in Singapore, 77 (51%) were boys, and 74 were girls (49%). The children's average age was 10.7 years (*SD* = 0.65). All children participated with parental consent. Most children in this study were drawn from middle-class to lower middle-class areas. In Singapore, Primary 5 classes are streamed, or ability-tracked, according to performance in languages and mathematics (Ministry of Education, 2005). In this study, 21% of the children were from the top stream (henceforth known as EM1 children), and the remainder was from the middle stream (EM2 children). In all subjects, EM1 and EM2 children shared the same syllabus and the same Primary School Leaving Examination. In addition, EM1 children completed a more demanding mother tongue examination. The lowest stream, EM3, was not recruited because these children followed a different mathematics syllabus. Many children in this study were nonnative speakers of English but had at least 7 years of education, with English as the medium of instruction. All examinations were written in English.

Instrument

Table 1 lists the 10 problems presented to the children in Study 2. Because teachers and children shared the same instrument, the discussion on instrument design appears in the description of Study 1.

Administration of the Test

Using the pilot studies as a guide, the children were given 1 hour to complete the 10-item mathematics test. They sat for the test in a classroom assigned by the teacher. Although children may have been familiar with other problem-solving heuristics, they were asked to use only the model method to solve these problems because the use of this method was the focus of this study. Some children asked the researcher what actions they should take if they were unable to use the model method. These children were advised to use any method they could think of to solve the word problems.

Findings

Because the children were not interviewed, interpretations are based on analysis of children's written work. Compared to the first 5 problems, children's rate of response to problems 6 through 10 is low ($M = 0.63$). Because of the low rate of response to problems 6 through 10, only responses to the first 5 questions (see Table 1) are discussed in detail. One reason for this low performance is that more than one model drawing is required to capture the problem state. The use of such models, which depict the before and after states, is more common after the introduction of ratio. When the children participated in this study, they had yet to be taught this concept.

Children's written work was coded at different levels. Responses were first coded question by question. Each question was coded for accuracy. The correct responses were then checked for variation in model drawings. Next, the errors were coded into four categories: (a) partly correct models with essential information missing or misrepresented, (b) changing generators midway, (c) correct model drawing and correct arithmetic equations but failure to keep the goal in mind, and (d) lack of the conceptual knowledge necessary for the solution of the five problems. Finally, we compared and contrasted the difference in responses between the EM1 and EM2 groups. Responses using other methods were coded as "Did not use the model method" and, for the purpose of this study, were coded as wrong. Even when model drawings were wrong, children's computations were checked to ascertain whether the children were proficient with arithmetic computations. In this section, we first provide descriptive statistics for the data. Second, we provide examples of how some children successfully used the model method to solve the five problems. Finally, we use examples of children's errors to illustrate how children failed to use the model method successfully.

Table 1 provides children's overall success rate in using the model method to solve each of the 10 problems, and success rates by academic stream. Children were more successful on the Enrollment Problem than on the other problems. This is not surprising, given that the former is an arithmetic word problem and the rest are algebraic. Children found the Furniture Problem to be the easiest of all the algebraic word problems. If symbolic algebra is used, then linear equations with one variable are needed to solve Questions 2, 3, and 4, whereas the last question would require solution of two simultaneous linear equations. That the EM1 children performed better than the EM2 children is also not surprising because they were streamed according to their mathematics and language performance. The mean score of EM1 children was 3.58 compared to a mean of 1.43 for the EM2 children. Analysis of the correct solutions by these two groups of children showed no differences in the models they constructed and the undoing method they used to solve for the unknown unit.

Table 2 shows the performance of children with the model method. About 23% of the children did not answer any of the first 5 questions correctly. Overall, 9% of the children obtained the full score. About 15% of the children used the model method to solve the Enrollment Problem, the only arithmetic problem, and at least one algebraic word problem.

Table 2
Children's Performance on the First Five Problems

Number correct	0	1	2	3	4	5
Percent of children	23	27	17	15	9	9

Children from one school were taught the model method in first grade, and children from the other four schools were taught it in second grade. Performance in this school did not differ significantly from the average performance of the other four, $t(58.106) = -0.798$, $p = .428$. Difference in the amount of experience with the model method did not seem to have an impact on children's success rates.

Successful Use of the Model Method

Arithmetic word problem. The Enrollment Problem is an arithmetic word problem involving the comparative term *more than*. Rectangles for Sunshine Primary and Excellent Primary were drawn based on the rectangle for Dunearn. Unlike the rectangles in the other four questions, all the rectangles in the Enrollment Problem carried the given values. About 63% of the children used the model method to answer this question correctly. Figure 7 shows the two types of correct responses offered by children. Although the model drawings were similar, the arithmetic solutions were not. Of the 63%, 26% presented the solution shown on the left, which is the more efficient of the two. Here, children first used multiplication to find the total of the values represented by the three identical rectangles and then summed this total with the differences in enrollment between pairs of schools. Solutions of the type in the right panel consisted of a series of independent arithmetic expressions that matched, step-by-step, the structure of the model. A total of 73% of the correct responses were of this type.

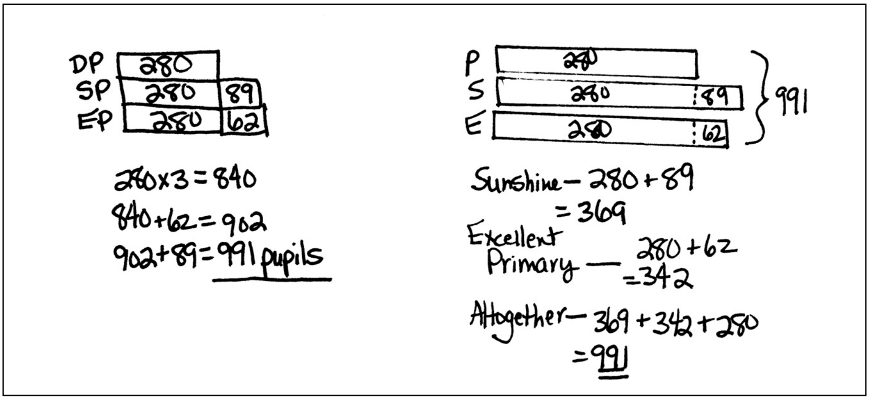

Figure 7. Examples of correct solutions to the Enrollment Problem: Efficient use of the arithmetic model (at left) and step-by-step use of the arithmetic model (at right).

The teachers were asked to comment on these two types of solutions. They confirmed that they would have presented similar model drawings and that both sets of solutions were acceptable. They explained that many children preferred the step-by-step solution shown on the right since it meant that they found the enrollment of each school first; its logic appeared to be more amenable to children. The solution on the left was more complicated because it no longer referred to the individual enrollment of the schools. Instead, it required children to move away from the context of the story and to focus on the structure of the arithmetic relationships—three groups of 280 and then the individual numbers giving the differences in enrollment.

Algebraic word problems involving whole numbers. Children offered two types of correct models to the Animal Problem. About 29% of the children provided the correct response as shown on the left in Figure 8, where the mass of the dog was taken as the generator. Nine percent of the children gave the correct solution in which the mass of the cow was used as the generator, as in right panel of Figure 8. A rectangle was drawn to represent the mass of the cow. Because the mass of both the dog and the goat were less than the mass of the cow, rectangles shorter than that for the cow were drawn to represent their respective masses, and the difference in mass was indicated on the respective rectangles. By using the mass of the cow as the generator, a homogeneous or common operation—addition of the differences in mass to the total mass—was used to solve for the mass of the cow. This solution is more efficient and elegant than the former, in which the mass of the dog was the generator, because the mass of the cow can be found directly without having first to refer to the mass of the dog.

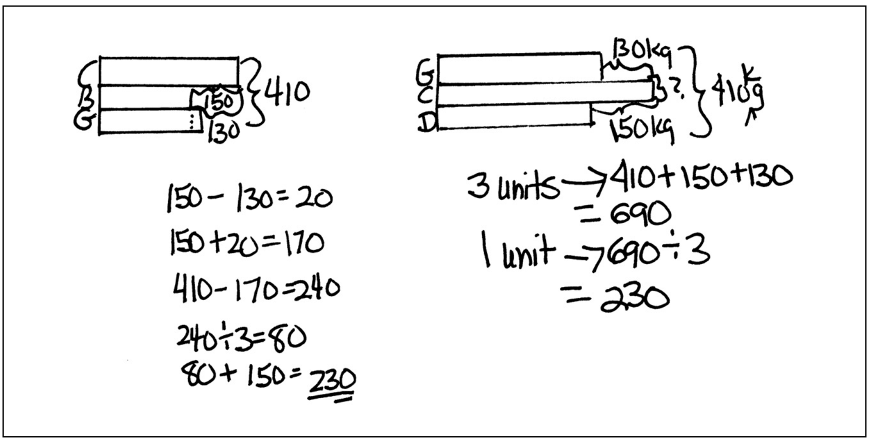

Figure 8. Solutions to the Animal Problem: Use of the mass of the dog as generator (at left) and use of the mass of the cow as generator (at right).

It is noteworthy that while 9% of the children chose the mass of the cow as a generator, which gave the solution of the mass of the cow directly, none of these children used the cost of the chair as a generator for the furniture problem. If they had done so, they would have found the cost of the chair directly. This choice, however, would necessitate the use of nonhomogeneous operations—addition and subtraction of the difference in cost between the table and the chair and the difference in cost of the iron and the chair, respectively, to and from the total cost. These children who selected the mass of the cow as the generator were able to recognize which generator to use and for which type of questions.

Children's models showed that they represented all the unknown units and the known values as specified rectangles to the left of the brace. The total known output was kept to the right of the brace. Children could then solve for the unknown unit in one of two ways. The procedures on the right in Figure 8, offered by one child, showed that by equalizing the lengths of the rectangles representing the mass of the goat and the dog to that of the cow there were three rectangles of equivalent length. This equalizing process was represented numerically by incrementing the total mass of the three animals by their differences (410 + 150 + 130). This allowed the direct computation of the mass of the cow. Figure 8 shows how children worked part by part from the model until they found the value of the unknown unit.

The most common form of solution was of the type shown on the left in Figure 8. Here because no algebraic equation was constructed to represent the information captured by the models, the unknown to be found was not made explicit. These children, however, knew they had to calculate the value of the unit, which was knowledge they held mentally. The children then proceeded to solve for the unknown value by undoing, step by step, the operations suggested by the relationships of the rectangles to the left of the brace. A rectangle with a specific rectangle appended would suggest subtraction of that specified value from the given total while three identical rectangles would suggest division of the total on the right. The work of these children suggests that they exhibited good metacognition skills because they knew what to do next after each operation and knew when they had the answer. When asked to comment on these solutions, the teachers explained that the solution on the left was the easier of the two and hence they preferred to teach this to solve similar problems. The teachers explained that children who gave the solution in the right may have received extra tutoring from either their parents or private tutors.

Algebraic word problems involving rational numbers. In the Water Problem, concepts of fractions are involved. With such problems, children are taught to use the inverse of operations, thus converting the relationship involving unit fractions into a relationship involving whole numbers. Which unknown is selected as the generator depends on which relationship is under consideration. We have selected three children's solutions to demonstrate how the model method can be used to solve the Water Problem.

A total of 57% of the children who answered this question correctly drew models with the exact number of rectangles in which the number of rectangles for each container was ascertained by evaluating the relationship between the volume of water in A, B, and C. For example, the solution of Student 126 (see top, left of Figure 9) suggested that the child's thinking alternated between the volume of B and A, then B and C. Twelve is a common multiple of 3 and 4. Hence, if B has 12 rectangles, then A has 4 rectangles and C, 3. It is possible that the numerical information could have been refined as the model was being drawn—alternating between the model drawing and the numerical relationships. The volume of container B could be found once the value of one unit was ascertained.

A comparison of Student 126's solution with that of Student 147 (see bottom, left of Figure 9) suggests that the latter might have processed the information as follows: (a) If A is the base, then there should be three rectangles for B; (b) If C is one quarter of B, then using its inverse, the volume of B is four times C; (c) How many parts must C be so that B is 4 times C? (d) Look for the lowest common

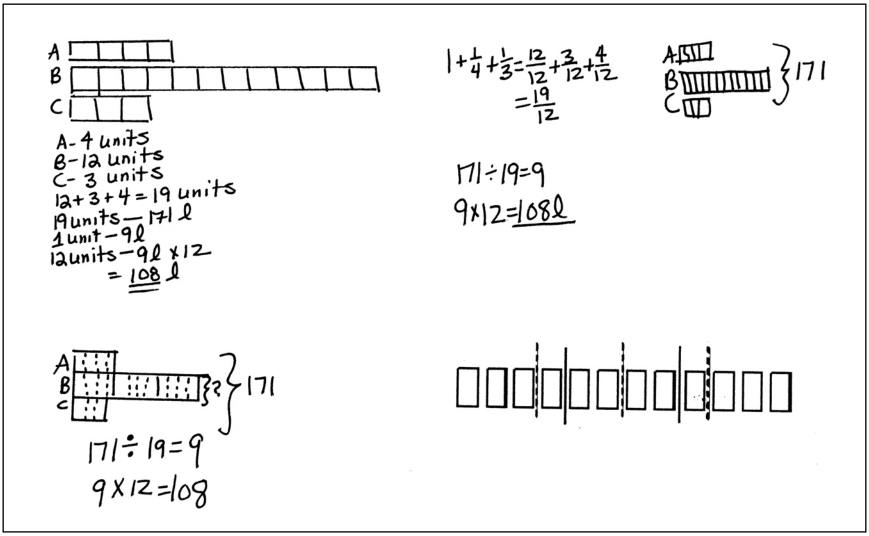

Figure 9. Solutions to the Water Problem: Student 126 (at top, left), Student 105 (at top, right), and Student 147 (at bottom, left) and global relationships among 3, 4, and 12 (at bottom, right).

multiple of 3 and 4. If it is 12, then B has 12 rectangles and A has 4; (e) The dotted lines are then worked into the original rectangles; and (f) Find the volume for B.

Although the solutions of Students 126 and 147 appear similar, the process used by Student 147 suggests that perhaps he or she was thinking more globally than Student 126, who was focusing more on the details. Perhaps Student 147 saw the structure in the bottom, right in Figure 9, which gave rise to the concise solution. A total of 23% of the children with the correct answer gave this particular solution.

Although the model drawings of Students 105 and 126 were accompanied by the corresponding arithmetic expressions, that of Student 147 showed only two arithmetic equations, one showing how the value of one subunit was found and the final equation, 171 □ 19 = 9, showing how the volume of container B was found. Comparing these three solutions suggests that Student 147 carried out many processes mentally, which meant that many meaningful and conceptual thought processes are hidden.

Of all the different solutions, Student 105's solution (see top, right in Figure 9) was the clearest, and 20% of those with correct answers used this approach. Student 105's structural phase comprised two representations—symbolic and model. He used the symbolic representation to help him draw the model. His solution showed that container B was taken as a whole and hence was selected as the generator. If B is one whole, then A is one third of B (working backward) and C is one quarter of B (using the given information). This mode of thinking allowed the construction of this mathematical sentence $1 + 1/4 + 1/3 = 12/12 + 3/12 + 4/12 = 19/12$. Once the symbolic representation was constructed, drawing the model was not difficult. With the aid of the model he found the value of one unit (171 ÷ 19) and then the volume of container B (9 × 12).

The teachers explained that without the model drawing, children would not be able to solve for the volume of B because they had not been taught how to solve the equation 19 *units*/12 = 171. They had yet to learn how to transform equations and to divide whole numbers by a rational number. The model method provided children with a way to work with whole numbers rather than fractions and this simplified the solution process because it now involved dividing 171 by 19 and then multiplying the result by 12. This meant that children were able to circumvent the need to evaluate the expression 171 × 12/19, a skill taught only in secondary school mathematics (CPDD, 2000b).

Algebraic word problems involving proportional reasoning. The Book Problem involves two unknowns and is the most challenging of the questions under consideration because two levels of relationships are presented in it. First, there is a direct proportional relationship involving the number of mathematics books and science books: For every 1 mathematics book, there are 4 science books. Hence, the model shows the relationship between the number of mathematics books and the number of science books. Second, the costs of the books, and not the number of books, are represented by each rectangle. Therefore, rectangles of proportionate lengths should be used to represent the cost of the mathematics and science books. With the exception of the Book Problem, all the rectangles for models drawn for the Furniture, Animal, and Water Problems are of the same size because each rectangle represents a unit that is the unknown that needs to be determined.

In a standard algebra text, the Book Problem can be solved by construction and solution of two simultaneous linear equations. With the model method, however, children can dispense with this necessity. The solution presented at left in Figure 10 shows that children used the individual rectangles of the same size to reflect the costs of the mathematics and science books. Each set of 1 mathematics book and 4 science books has a total cost of $44. The problem is solved by asking how many such sets of $44 there are in $528. There are 12 such sets, meaning that there are 12 mathematics books and 48 science books. Perhaps these children had a mental picture (see the diagram at right in Figure 10) of the question “How many $44 are there in the total of $528?”

Although many of the children’s answers were correct, their models were not. All children drew erroneous models in which the rectangles representing the mathematics books were the same size as those used to represent science books. This suggests that children were using the model method as an algorithm to solve such problems. Teachers confirmed that for such problems, it was not necessary to show the relative lengths of the rectangles because the objective of the model drawing was to help children make the information overt. For such problems, the meaning of the rectangles need not be known so long as the method consistently gives the correct answer.

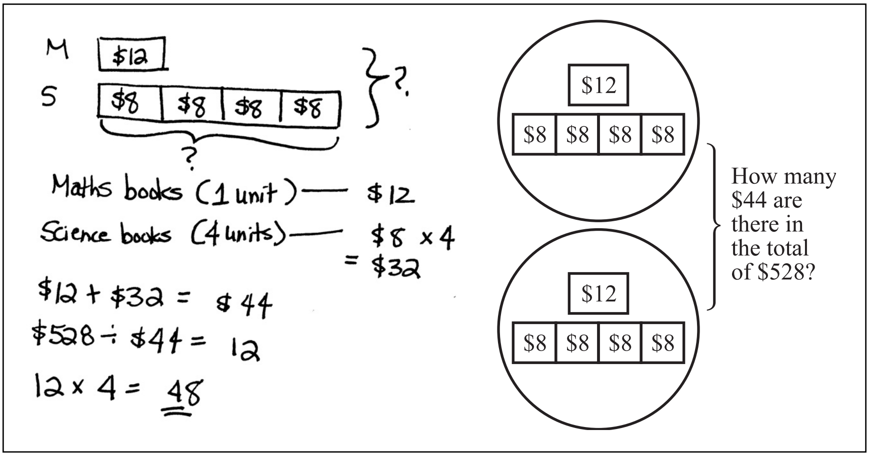

Figure 10. A common solution to the Book Problem (at left) and grouping method held mentally (at right).

How did children fail to use the model method successfully? Not all children acquiesced to our request to use the model method to solve the word problems. Some attempted to use other heuristics, but they were not always successful, even when using the method of their choice. Although some children who participated in the pilot studies used the model method to solve the Stamp Problem, none of the children in the Study 2 did. Instead, guess and check and systematic listing were the two heuristics of choice. The rate of success was low because they either failed to satisfy the condition of a total of 62 stamps or the total amount spent.

The visual nature of model drawing enabled us to probe which aspects of the translation and representation processes were problematic for children. We compared and contrasted model drawings of correct solutions with those that resulted in wrong answers. What differences exist between these model drawings? The model drawings that resulted in correct solutions accurately represented the information presented in that word problem. Teachers explained that care is needed to construct accurate models. To what level of care were they alluding?

Correct model drawings were elaborate in that they indicated clearly and precisely every piece of information presented in the text that was necessary for the solution of the word problem: the selected generator, the difference rectangles, and what was to be evaluated. In contrast, partly correct model drawing representations, which formed about 30% of the errors, showed how a lack of attention to a single detail or a misrepresentation of a single piece of information could result in an incorrect solution. Furthermore, construction of correct model drawings was underpinned by sound conceptual knowledge.

The discussion in this section is divided into four categories according to the type of error made: (a) partly correct models with essential information missing or misrepresented, (b) changing generators midway, (c) correct model drawing and correct arithmetic expressions but failure to keep the goal in mind, and (d) lack of conceptual knowledge necessary for the solution of the five problems.

Partly correct models with essential information missing or misrepresented. Errors of this nature were common among algebraic word problems. Correct model drawings identified and represented information accurately, but partly correct model drawings often misrepresented a detail or omitted it entirely. Partly correct model drawings appeared in about 7% of the Furniture Problem solutions and 11% of the Animal Problem solutions. The difference between a correct model drawing and one with missing or misrepresented information is very subtle. At a superficial level, the partly correct model drawings appear very similar to a correct representation, but upon closer inspection, missing details or misrepresented information can be noted. In the case of correct solutions for the Furniture Problem, since the cost of each item is based on the cost of the iron, each representation clearly shows the rectangle for the cost of the iron and the difference rectangle. Partly correct model drawings with missing information did not reflect this level of detail. The item that was furthest from the chosen generator carried fewer details. Although the representation of the chair was accurate, the cost of the table did not show how it was related to the generator. Because of the lack of detail at each subsequent level, the resulting arithmetic equation was incorrect.

In the Furniture Problem, the homogeneous relationship *more than* relates the remaining two items to the cost of the iron. In the Animal Problem, however, the nonhomogeneous relationships *more than* and *less than* relate the mass of the cow with the mass of the dog, and the mass of the goat with that of the cow, respectively. This difference between the two problems could account for the lower accuracy rate on the Animal Problem than on the Furniture Problem. Accurate model drawings clearly showed the generator and the difference rectangle. This was not the case for partly correct model drawings. Although the overall representation for 13% of the Animal Problem solutions was correct, the arithmetic equations were wrong.

The model drawing at top, left of Figure 11 shows a single rectangle was used to represent the mass of each animal. A dotted rectangle was used to represent the difference in the masses of the related animals. This model drawing could have resulted in a correct solution if the child had equalized the length of the rectangles for the mass of the dog and the mass of the goat to that of the cow, effectively treating the mass of the cow as the generator. The first arithmetic equation, $150 - 130 = 20$, suggests that the child was trying to find the difference in mass between the goat and the dog, indicating an attempt to use the mass of the dog as the generator. Perhaps this child forgot which was the selected generator, and thus the subsequent set of arithmetic equations was incorrect (see Figure 11, top left). Another 11% of the solutions clearly identified the mass of the dog as the generator for the mass of the cow, but because there was no clear distinction between the representations for less than and more than, the goat is then 130 kg heavier than the dog. Hence, the set of arithmetic equations was incorrect (see Figure 11, top right).

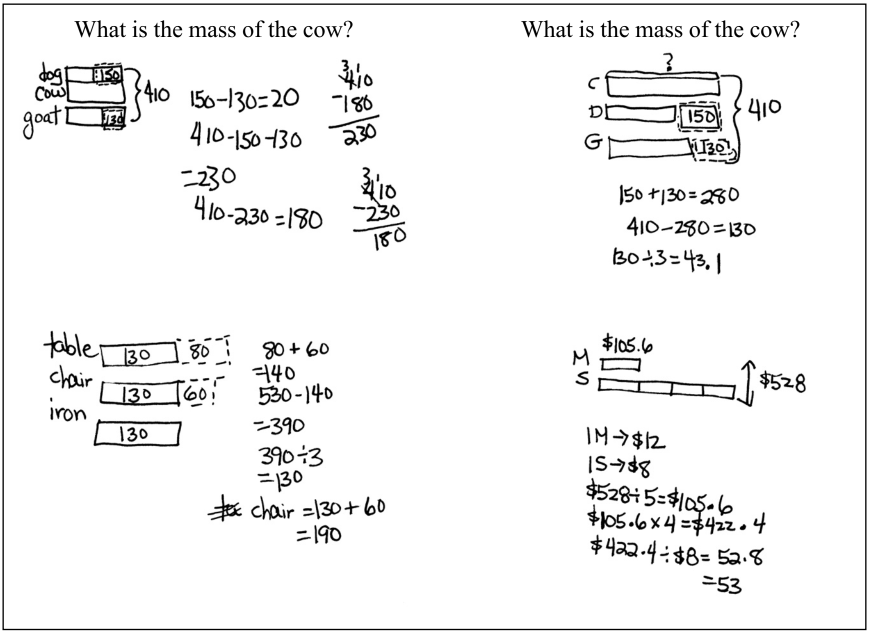

Figure 11. Examples of how representation of a problem is not an all-or-nothing process.

Note. For the Animal Problem, a correct model that would have resulted in a correct solution if the intention were to equalize the mass of the animals to that of the cow (at top, left). For the Animal Problem, a correct model but incorrect set of arithmetic equations (at top, right). For the Furniture Problem, an incorrect base used for comparison but correct numerical answer based on the model (at bottom, left). For the Book Problem, a correct overall representation but incorrect understanding of the role of the rectangle (at bottom, right).

Changing generators midway. This category of error was common to algebraic and arithmetic word problems. In the Enrollment Problem, instead of using the enrollment of Dunearn to construct the representation for Excellent Primary, 25% of the solutions based the enrollment of Excellent Primary on that of Sunshine Primary. In the Furniture Problem, 26% of the solutions made the error of comparing the cost of the table with the cost of the chair rather than with the cost of the iron (see Figure 11, bottom left). Although the SP translation was correct, the answer was incorrect because of an erroneous TS translation.

Correct model drawing, correct arithmetic expressions but failure to keep the goal in mind. Such errors were common to the arithmetic and algebraic word problems but comprised less than 3% of the errors. Children who made such errors constructed a correct model drawing and its related arithmetic equations but failed to answer the question as stated. With the Enrollment Problem, children found the enrollment for each school but did not continue to find the total enrollment of all three schools. For the Furniture Problem, instead of proceeding to find the cost of the chair, these children stopped after finding the cost of the iron, which was used as the generator for the problem. Children who made such errors may not have made the extra effort to check whether they had answered the question. More important, model drawings of correct solutions were more likely to indicate the unknown to be found. For the Enrollment Problem, more correct solutions than incorrect solutions carried a vertical brace and the related question mark to indicate that the total was required. For the Furniture and Animal Problems, a question mark in the selected rectangle identified the unknown to be found. Such details were missing from model drawings with incomplete solutions.

Partly correct model drawings may be the result of poor metacognitive practices. If children check their drawings against the text-based information, they may realize that the model drawing is incomplete or partly correct, or they may realize that they have chosen the wrong generator or failed to answer the question. Correct solutions may be evidence of good practices in problem solving, in which the information captured in the model is checked against the text-based information.

Lack of conceptual knowledge necessary for the solution of the five problems. For the Water Problem, comparison of correct solutions against those that were incorrect highlights how correct solutions offer evidence of sound conceptual understanding of part-whole relationships and fractions. They also illustrate an integrated and well-organized knowledge base of number facts including factors and multiples of numbers as well as skill in operating with fractions.

The Water Problem shows how about 34% of the children were challenged by the concept of fraction. These children constructed the correct model drawings for the volume of containers A and B, but the representation of the relationship between B and C was faulty. This could be because these children were more comfortable working with whole numbers, and the relationship between container A and B was stated in terms of whole numbers, hence the representation was relatively simple. In contrast, the relationship between container B and C was expressed in terms of a fraction. A concept of fraction is necessary to transform the relationships to ones in which operations with whole numbers sufficed. First, the children had to be able to translate the relationship "Container C has 1/4 as much water as container B" to its equivalent "Container B has four times as much water as C." Second, they had to determine the least common multiple of 4 and 3. If they were able to do this, the volume of C could be represented. Erroneous model representations of the volume for container C showed that children were unable to transform the original information to its equivalent. They represented the volume of container C additively by attaching a rectangle representing the value 1/4 to three rectangles representing the volume of container B. They may have misinterpreted the volume of container C as 1/4 more than the volume of B. If these children had a sound knowledge of fractions, multiplicative reasoning, and knowledge of factors and multiples of 3 and 4, they might have been able to solve the Water Problem.

Another example of the importance of conceptual knowledge is that rectangles play different roles. In the first four problems, the rectangles could be used to represent either specific or unknown quantities. In the Book Problem, the children needed to rethink the role of the rectangles because they represent the cost of the books, even though the unknown is not located within any of the rectangles. Instead, the unknown is the question "How many $44 are there in $528?" and this has to be abstracted from the representation. Approximately 21% of the children answered the Book Problem correctly, and another 32% of the children constructed a correct overall representation of the problem—one rectangle for the mathematics books and four rectangles for the science books—but the SP translation was wrong. The SP translation showed how these children continued to treat these rectangles as unknown units and found an answer by dividing the total cost by the total number of rectangles (see Figure 11, bottom right). Perhaps if these children had a clearer understanding of the different roles of the rectangles, they might have solved the problem correctly.

Children who used the model method also successfully demonstrated a sound conceptual knowledge of comparative phrases such as *more than*, *less than*, and *as many as*, since their model drawings showed that there should be a base for comparison. In contrast, about 7% of the children who did not have the conceptual understanding of such relationships drew models that were devoid of any base for comparison. In the Enrollment Problem, the rectangles represented the differences in enrollment between schools; the children may have treated the difference as the enrollment for each school. Because no base was used in the comparison, the relationships *more than* and *less than* in the Furniture and Animal Problems were irrelevant, and these algebraic word problems were converted into arithmetic word problems. This practice of ignoring the base for comparison may be the result of a local practice in which it is quite common to hear children say "I have more" without referring to a base, and more is treated as a noun. Therefore, they may have focused only on "Sunshine Primary School has 89 more pupils" without attending to Dunearn Primary, the base for comparison. These children, who simply used numbers as they were presented in the problems rather than as comparisons, need more support solving arithmetic problems. Work with algebraic word problems should be deferred until they are confident with arithmetic word problems.

Discussion

Study 2 shows that the model method can be used to solve some, but not all, algebraic word problems. Algebraic word problems involving whole numbers and those that can be solved with the construction and solution of a system of linear equations in one unknown are more amenable to the model method than those that require the construction and solution of a system of linear equations in two unknowns. The children's solutions to the Book Problem show how they have adapted the model method to solve algebraic word problems involving two unknowns.

This study also showed that although most EM1 children were able to use the model method to solve arithmetic and algebraic word problems, many EM2 children could not. We are not sure why this is the case. At the end of Primary 4, the children completed a streaming examination. Based on their proficiency in their mother tongue language (Chinese, Malay, or Tamil), English, and mathematics, these children were streamed into either the EM1 or EM2 class. Could it be that the streaming examination effectively identified those who were more mathematically inclined from those who were less so? Or could it be that the EM2 children faced language difficulties? It could be that the EM2 children had more difficulties than their EM1 peers in understanding the word problems. This would suggest that they had difficulties with the first phase of the three-phase problem-solving process. That about a quarter of the children were unsuccessful with any of the five problems suggests that they do need support with arithmetic word problems and that algebraic word problems should be deferred.

Study 2 showed the challenges that children face when using model representations to solve word problems. Those who were successful with the model method drew detailed models and constructed accurate arithmetic equations to represent the information represented by the model drawing. Such practices are consistent with the behaviors of skilled physics problem solvers (Larkin et al., 1980). In addition, the children's success may be reflective of good monitoring practices in which they countercheck the representation of the model against the text before proceeding to construct the corresponding set of arithmetic equations. Alternatively, the monitoring activity could be the penultimate part of the solution process in which they checked the arithmetic equations against the model method and the text before working out the final answer. Because children were not interviewed, we can only speculate on the sequence of events. Further research is needed to confirm these speculations.

More important, this study shows that using the model method to solve word problems is not an all-or-nothing process. Consistent with the research literature (De Corte et al., 1985), children's errors were due more to erroneous representation of the problem than to computational errors. One third of the partially correct solutions show that children were neither completely incorrect nor completely correct in their problem-solving activity. Although the overall representation may be correct, a single error in the representation can result in an incorrect solution. The error could be a result of misinterpretation of information correctly captured in the model, misrepresentation of a piece of information, or changing the generator midway through the solution.

Successful representation of word problems requires an integrated and well-organized knowledge base of number facts, conceptual understanding of part-whole relationships and fraction, multiplicative reasoning, and knowledge of the four operations (e.g., subtraction is the inverse of addition; division is the inverse of multiplication). This study shows that conceptual understanding of fraction challenged half the children. Albeit fewer in numbers (about 7%), there were children who drew models that illustrated their lack of understanding that comparisons involving the four operations must be made against a base. Without this conceptual understanding, children may continue to draw models that are incorrect.

IMPLICATIONS FOR TEACHING

The visual nature of the model representation means that it is possible for teachers and children to point to specific errors as they appear in the incorrect model drawings. How can teachers help children who drew partially correct models understand the source of their errors? Teachers could offer children a set of correct solutions and ask children to compare and contrast these against those that are partially correct. Children could be asked to discuss from where those errors came and how they could improve upon their model drawings.

Teachers could also use the variations among correct solutions to a particular problem as a stimulus to provoke discussion among children to seek alternate ways to solve the same problem. For example, in the Animal Problem teachers could ask children to compare and contrast solutions in which the mass of the dog was a generator and the mass of the cow was a generator. Children who used the mass of the dog as a generator could be encouraged to see that the mass of the cow might be used as a generator instead, because these different model solutions are within their zone of proximal development (Vygotsky, 1978). Similarly, for the Enrollment Problem, children who found the total enrollment by summing the individual enrollments of the three schools could be encouraged to shift to the alternative solution. Offering children the various Water Problem solutions for discussion may help those who are challenged by the procedural aspects of model drawing to articulate their thoughts. Furthermore, children with errors in the final phase of problem solving could be encouraged to check their arithmetic representations against the model drawing.

CONCLUSIONS

This study contributes to a corpus of research in which the teaching of representation skills supports higher ability children in their work with word problems—arithmetic as well as algebraic. More important, this study offers avenues for average ability children to be supported in their word-problem-solving activity. The partially correct model drawings show that this kind of representation is not an all-or-nothing process. Average ability children's solution of word problems involving whole numbers could be improved if they learn to exercise more care in the construction of related models. Furthermore, by deepening their conceptual knowledge of fraction and its related skills, they may extend their competency to word problems that involve fractions. Their solution of word problems involving two unknowns also could be enhanced if they expanded their appreciation of the different roles that the rectangles can have in model drawings.

This study also shows that although the children may treat the model method as an algorithm, it is not an algorithm learned by rote that is intended to replace other rote algorithms for solving word problems. Instead, it is a problem-solving heuristic that requires children to reflect on how they could accurately represent the information presented in word problems, first in terms of a drawing and then as a series of arithmetic equations. This art of representation first has to be taught, but it is then the children's responsibility how they choose to use this heuristic effectively.

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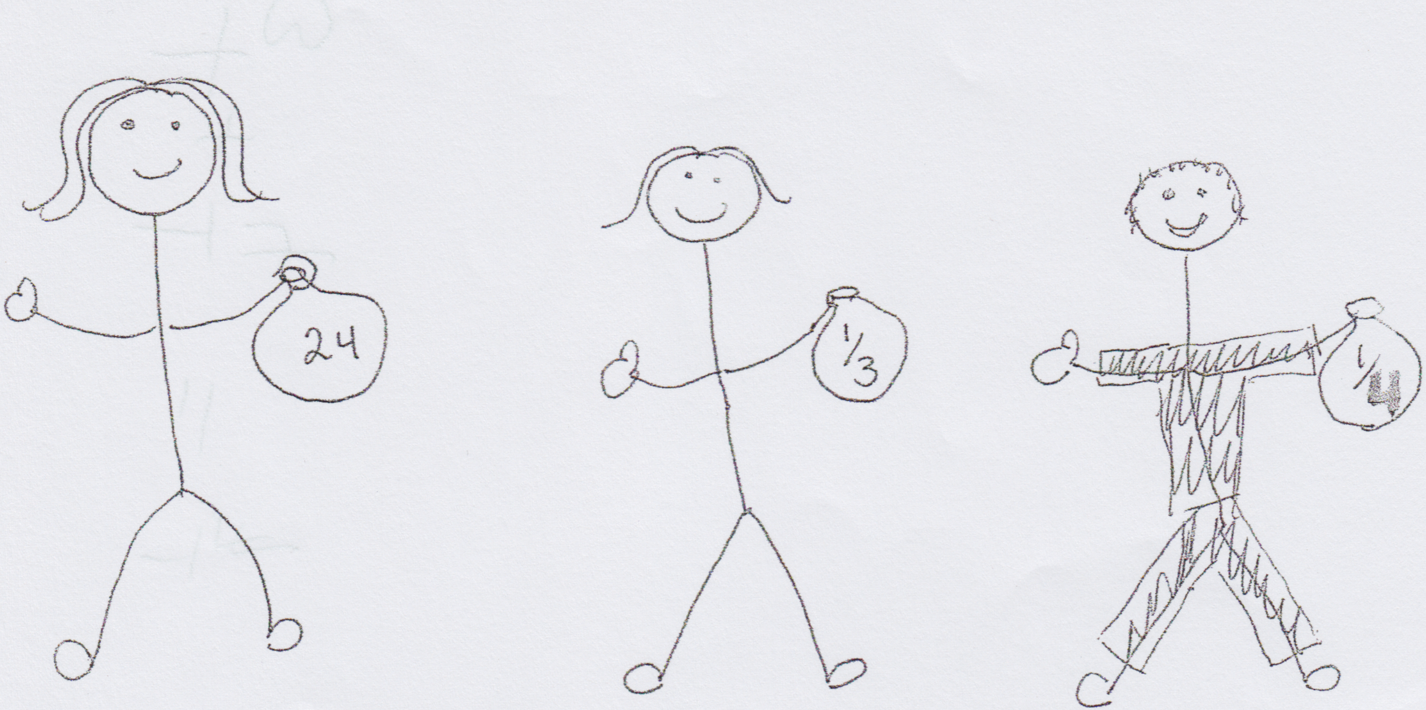
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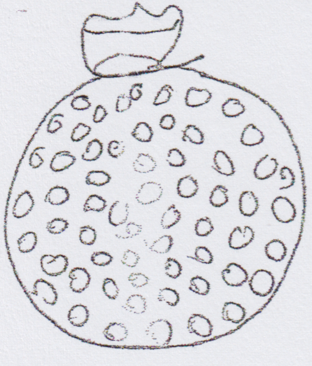

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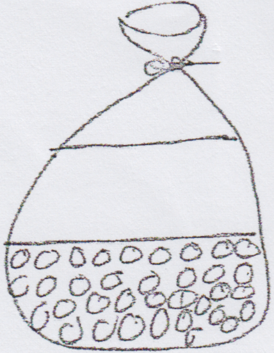

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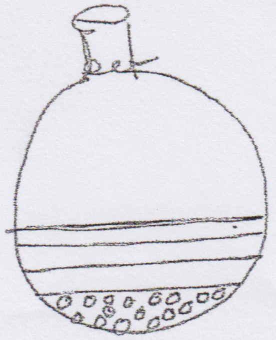

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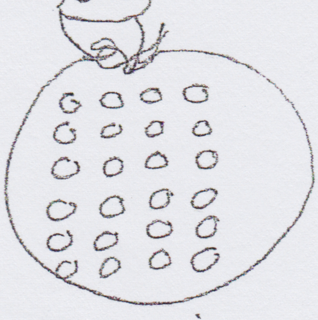

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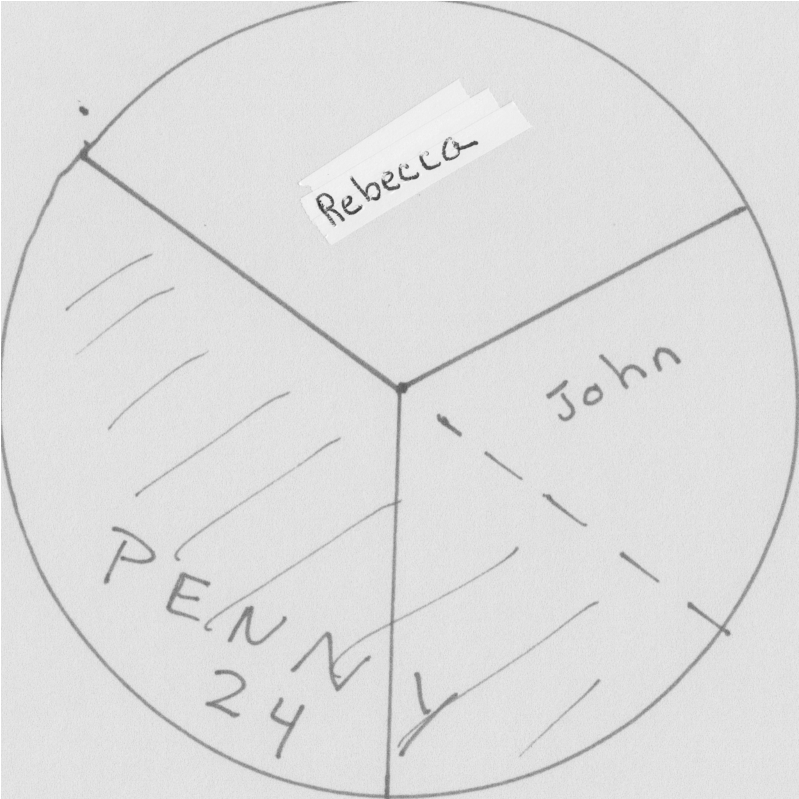
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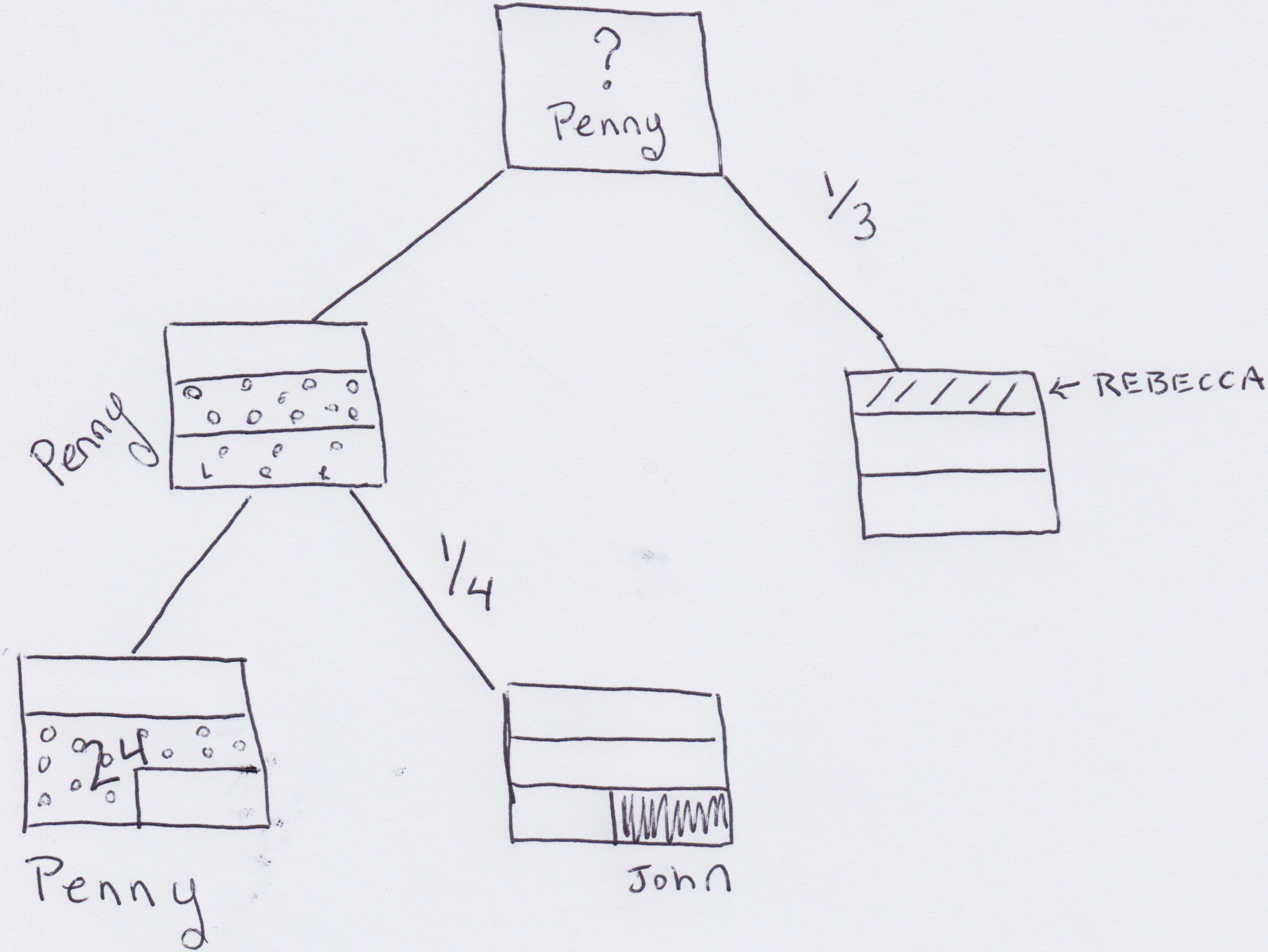
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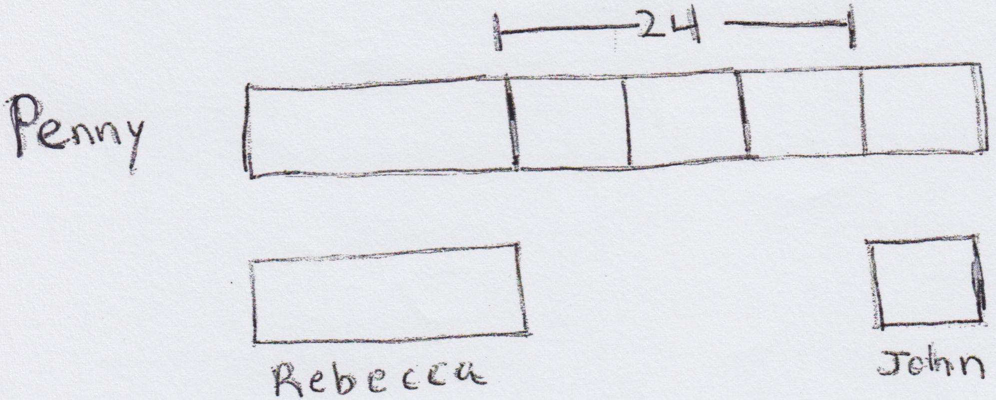
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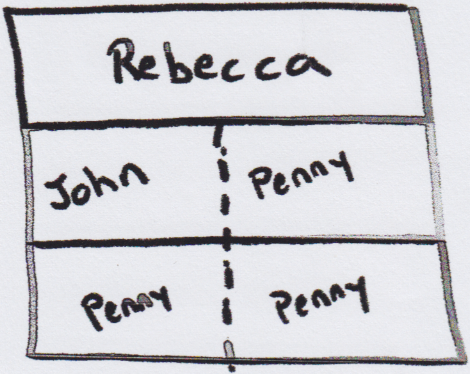

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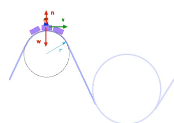
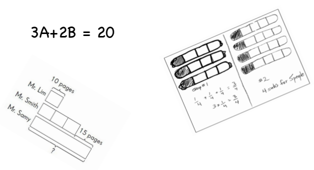
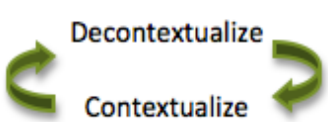
Unpacking Mathematical Practice Standard #2

Reason Abstractly and Quantitatively Mathematically proficient students make sense of the quantities and their relationships in problem situations.			
Key Idea	**Description**	**Learn to ask yourself questions like…**	**Notes**
 Attend to quantities and relationships 	Using a variety of concrete and visual representations to highlight quantities, relationships between quantities, and the underlying mathematical structure of a problem situation	☑ How can I capture important information in a diagram or other representation? ☑ How can I represent the situation so that I can see the important relationships? ☑ Does this representation surface any “hidden” relationships? ☑ What solution path does this diagram or other representation imply?	The focus for this math practice extends beyond simply creating a representation of the problem. It focuses on **using** the representation to help you understand relationships between the important quantities, including quantities not explicitly described in the problem situation.
	*Decontextualize* Abstracting a problem situation <u>and</u> manipulating that abstract representation without attending to referents *Contextualize* Recalling and considering the referents for the abstraction you are manipulating, or assigning a context to the math in order to make sense of it	☑ How can I represent this problem (e.g. using symbols, diagrams, numbers, manipulatives, etc.)? ☑ How will manipulating this, without worrying about what it represents help right now? ☑ What does this (symbol/ diagram/ number/ variable) stand for? ☑ What does this number represent in the problem context? And, does that number make sense given the problem context? ☑ What context will help me understand this math idea/	*“Abstracting” does not just mean using variables.* Diagrams, numbers, manipulatives, invented symbols, etc. can all be used to decontextualize a problem situation.

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Strip Di

Illuminating Proportions

Jessica S. Cohen

A running problem, among others, can help students model proportional thinking.

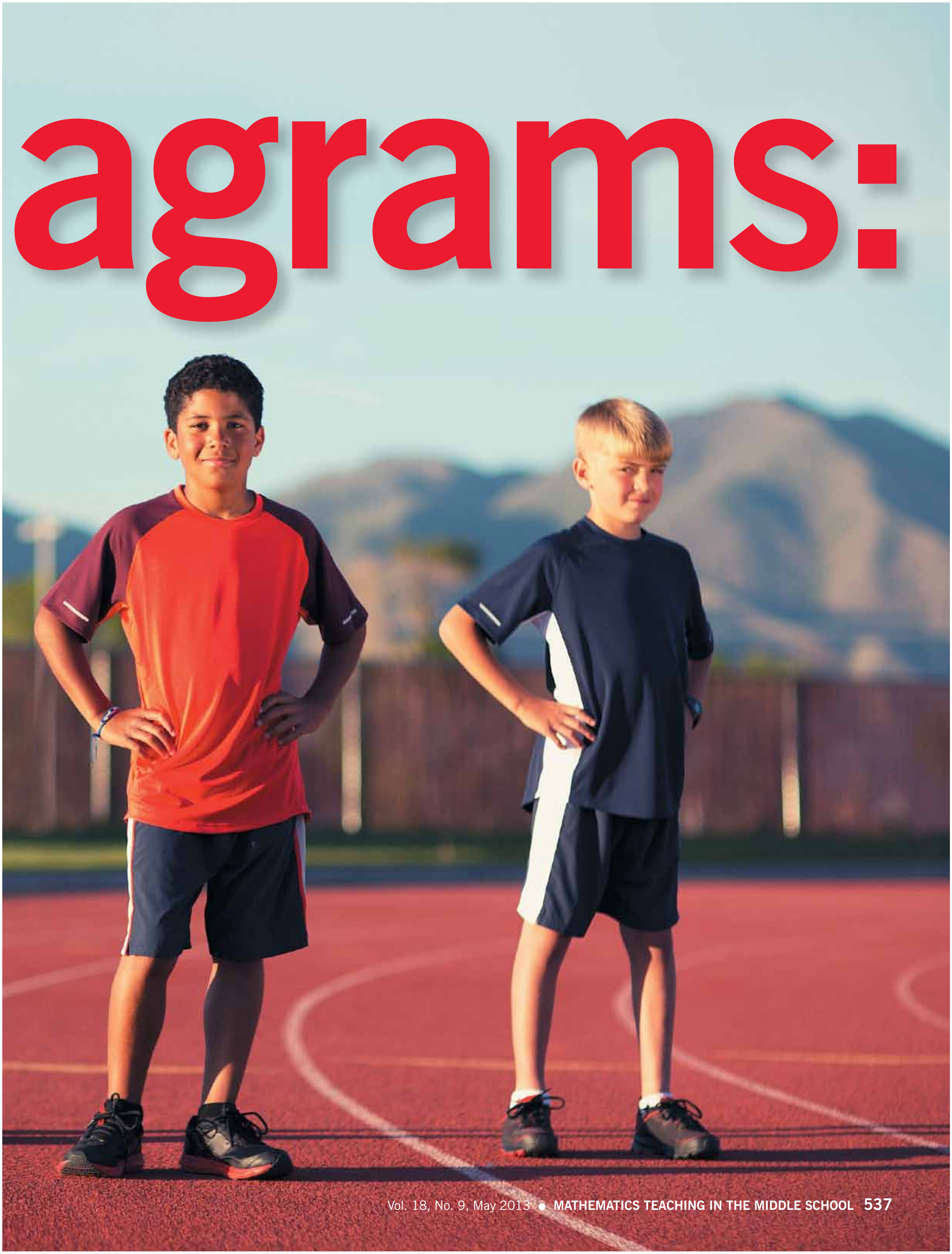
RICHVINTAGE/ISTOCKPHOTO

Proportional reasoning is both complex and layered, making it challenging to define. Lamon (1999) identified characteristics of proportional thinkers, such as being able to understand covariance of quantities; distinguish between proportional and nonproportional relationships; use a variety of strategies flexibly, most of which are nonalgorithmic, to solve proportion problems and compare ratios; and understand ratios as being separate from the quantities they compare. Proportional reasoning is multifaceted and is not an all-or-nothing ability. Rather, it is a collection of understandings and abilities that must be developed through a range of activities and experiences (Van de Walle, Karp, and Bay-Williams 2010).

It is well documented that proportionality is a complex and time-consuming concept to master (Hoffer and Hoffer 1988; Lesh, Post, and Behr 1988). This is compounded by the tendency to teach proportional reasoning through the cross-multiplication algorithm (Lesh, Post, and Behr 1988). Although this algorithm is a useful tool for solving proportion problems, students can successfully implement the algorithm without using or demonstrating proportional thinking.

It is common for the preservice teachers I work with to be able to properly set up the equivalent ratios needed to solve a proportion problem and efficiently use the cross-product

agrams:



Teachers may wish to delay presenting the cross-multiplication algorithm to students until after they have acquired the algebraic reasoning skills developed in algebra 1.

algorithm to find a solution. However, it is also common for these same preservice teachers to be unable to articulate the covariation in the problem or to be able to identify an alternate solution strategy. These skills are two components of Lamon's hallmarks of proportional thinkers. Including alternate methods, particularly those that provide deeper insight into the proportionality at the heart of the problem, can help students develop their proportional reasoning abilities.

Using Singapore strip diagrams (Beckmann 2004) can be a particularly effective strategy for solving proportion problems, building proportional reasoning skills, and connecting proportionality to other mathematical topics. These simple, visual models use strips to represent known and unknown quantities in problems in a meaningful way by displaying relationships between those quantities.

Strip diagrams were presented to students in a mixed mathematics content and methods course for preservice elementary and middle-grades teachers. This strategy was one of several given for solving ratio and proportion problems. These students began a proportional reasoning unit by considering proportion problems that did not involve numbers (see **fig. 1**). These problems had been adapted from the NCTM 2002 Yearbook companion (Litwiller and Bright 2002). Proportion problems without numbers were included to ensure that students used nonalgorithmic thinking and to help students distinguish between proportional and nonproportional relationships.

Students then encountered proportion problems *with* numbers and were allowed to use their own strategies to solve other proportion problems, provided they could explain why the strategy worked. This part of the unit was included to help students develop a range of nonalgorithmic strategies for solving proportions. It revealed students' developing understanding of covariance and of ratios and quantities as different mathematical entities. Students also learned how to use ratio tables and were finally shown strip diagrams.

The goal of this part of the unit was to help students continue to build an understanding of covariance. The approach was to connect the strips model to the relationship between covarying quantities and to use this model to deepen their understanding of the cross-multiplication algorithm. Developing a meaningful understanding of the algorithm, with particular focus on *why* it can be used to solve proportion problems, is important for preservice teachers who will then be teaching the algorithm to their own students.

A description of preservice teachers' use of strip diagrams follows. It highlights their evolving perception of the usefulness of the strategy and

Fig. 1 This introductory proportion problem does not involve numbers.

An orange juice drink is made by combining powdered juice mix and water. The pictures below show two glasses of juice. If glass A has a more orangey taste, and one scoop of powdered juice mix is added to both glasses, which would have a more orangey taste?

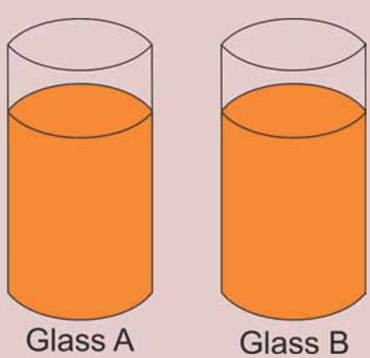
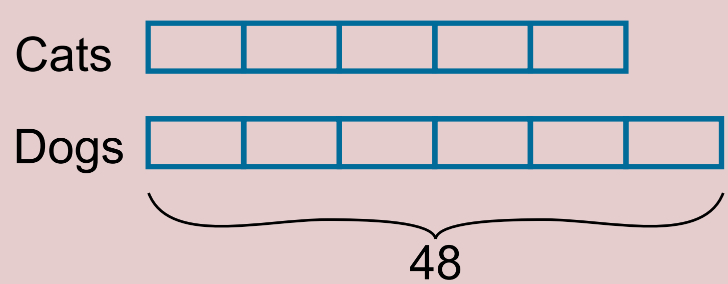


Fig. 2 Elena's strip diagram solution to the Cats and Dogs problem was typical of her classmates' work.



The 48 dogs represent 6 parts, so each part must have 8 dogs in it (48/6). For the ratio to work, the size of the parts has to be the same, so each of the cat parts must have 8 cats in it. That means there are $5 \times 8 = 40$ cats. The ratio is seen in the number of boxes in each strip. The strip makes it very easy to see the ratio because the number of boxes is the same as the number of parts.

the use of strip diagrams to build insight about the cross-multiplication algorithm. Although the problems described were used with preservice teachers, many tasks in the unit were adapted from materials intended for middle school students.

STRIP DIAGRAMS IN PRACTICE

Students were initially presented with the following Cats and Dogs problem:

> An animal shelter has 5 cats for every 6 dogs. If the shelter has 48 dogs, how many cats must it have?

Students were asked to find a solution using a strip diagram and consider how the strip diagram emphasized the ratio of cats to dogs. The most common student response is shown in **figure 2**. The consensus was that the problem was fairly straightforward; students had virtually no issues while using the strip diagram. Moreover, most students identified the ratio in the problem, and many students explicitly stated that the strip diagram emphasized the ratio.

Students were next given the Gus and Ike Running problem:

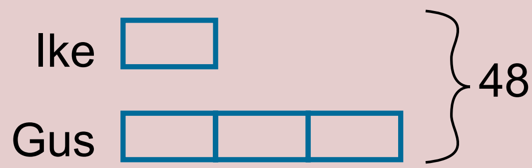
> Gus's and Ike's combined running distance this week was 48 miles. If Gus ran three times as far as Ike, how many miles did Ike run?

In addition to solving the problem, students were asked to consider the proportion used and how to solve it using the cross-multiplication algorithm. This question was selected to extend students' understanding of the use of the strip diagram because the ratio in the problem compares parts of the whole, whereas the quantity in the problem relates to the combined parts.

Identifying the appropriate representation for the ratio was initially challenging for some students. However, most were able to successfully resolve those challenges within their small groups by drawing on previous understandings. They recalled ideas from the portion of the unit in which students used their own strategies to solve proportion problems. In particular, considering possible distances that Gus and Ike could have run helped students identify the 3:1 ratio in the problem. Finally, students also had to recognize that the 48 total miles covered represented the sum of the two boys' distances, or the total quantity for all four boxes in the strip diagram.

The most common strip diagram representation is shown in **figure 3**. As students attempted to connect to cross multiplication, they met some challenges. **Figure 4** is the transcript of a small-group discussion typifying those challenges.

Fig. 3 Martin's solution to the Ike and Gus Running problem, using strips, was typical of those produced by his classmates.



Together the 4 boxes represent 48 miles, so 1 box must be 48/4 = 12 miles. So Ike ran 12 miles, and Gus ran 12 × 3 = 36 miles.

Fig. 4 The conversation among Tanya, Sam, and Elena typified the challenges that students faced when solving the Ike and Gus Running problem.

Tanya: I can't figure out what to do with this. I know the ratio is, like, 1:3, so that should be part of the cross multiplication, so I thought that $1/3 = x/48$. But this gives $x = 16$, and I know neither one of them [Gus and Ike] ran 16 miles. So I know I must be setting it up wrong, but I can't see how it should be.

Sam: I see in the strip diagrams that all four boxes represent the 48 miles, so I think this means the 48 miles goes with 4 parts in the equation, so I think this should be $x/4 = y/48$. But then I don't know what x and y should be.

Elena: I think this depends on what we want to find first. What goes with the 4 should be the number of parts we're thinking of and what goes with the 48 should be the number of miles in those parts. So to find, um, Ike, that's one part, so $1/4 = x/48$, and so $48 = 4x$, divide by 4, $x = 12$.

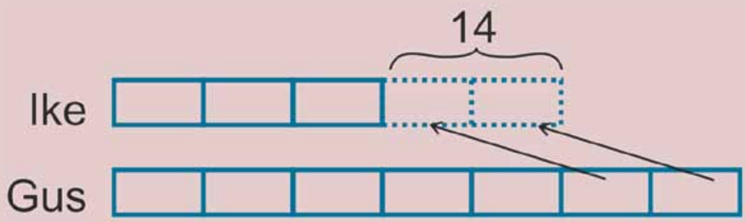
Tanya: Oh, right! And the dividing by 4 is the same thing we did to find how much went in one part when we were using the diagram!

The next task, the Gus and Ike Car problem, was designed to help students observe that setting up a proportion for the cross-multiplication algorithm can be more complicated.

> Gus and Ike are playing with toy cars. The ratio of Gus's cars to Ike's cars is 7 to 3. Gus gives Ike 14 cars, so now they each have the same number of cars. How many cars do they each have now?

This problem was selected next because the ratio of Gus's cars to Ike's cars changes in the problem. This is difficult to represent in the cross-multiplication algorithm without using some algebraic reasoning in the process. Students were asked to

Fig. 5 Belle's solution to the Gus and Ike Car problem was similar to that experienced by her classmates using a strip diagram.



For Gus and Ike to have the same number of cars, they have to have the same number of parts. So Gus has to give 2 of his boxes to Ike so they both have 5 boxes. Those 2 boxes have 14 cars in them, so every box has to have 7 cars. This means that each of the 5 boxes has $7 \times 5 = 35$ cars in them.

write a brief explanation of how the strip diagram helped them solve the problem. **Figure 5** shows the most common strategy used by the class.

A class discussion revealed that students were unsure how to set up equivalent ratios so that cross multiplication could be used to solve the problem. By carefully parsing the steps used to solve the problem with a strip diagram, students were able to connect the process to a cross-product algorithm based on the ratio of 14 cars to 2 parts. This proportion is based on the idea that to make the ratio of Gus's cars to Ike's cars equivalent to 1:1, the difference between the two ratios (four parts) must be evenly distributed between them. This means that Gus gives two parts' worth of cars to Ike, making the new ratio 5 parts to 5 parts. Thus, the equivalent ratios for cross multiplication would be 14:2 and x:5.

However, this problem can be solved using the ratio in the problem stem and employing cross multiplication, but it requires algebraic reasoning. **Figure 6** shows the process, which involves using two variables and then developing two equations and using substitution.

Several students noted the advantages to the strip diagram strategy for such a problem. They indicated that it helped them organize their thoughts

Fig. 6 This is an algebraic solution to the Gus and Ike Car problem.

Let g be the number of cars that Gus has originally. Let i be the number of cars Ike has initially. Then

$$\frac{i}{g} = \frac{3}{7}.$$

Furthermore, we also know that when Gus gives away 14 cars, Ike gains 14 cars. At this point, each boy has the same number of cars. We can represent this scenario as $g - 14 = i + 14$, so $i = g - 28$. This gives:

$$\begin{aligned} \frac{g-28}{g} &= \frac{3}{7} \\ 7(g-28) &= 3g \\ 7g - 196 &= 3g \\ -196 &= -4g \\ 49 &= g \end{aligned}$$

Gus had 49 cars to begin with, and Ike had 49 – 28 – 21 cars.

and allowed for easy modeling of the change in the proportion of Gus's cars to Ike's cars. Although the foundation of the problem is a ratio, students must consider an abstract rate (7 cars per part). Furthermore, the strip diagram allows for a more intuitive, meaningful solution process as compared with an algebraic one that is required for the cross-multiplication algorithm.

RECONSIDERING THE CROSS-MULTIPLICATION ALGORITHM

After using strip diagrams to represent part-to-part ratios and solve problems, students had sufficient experience with this strategy to reconsider the cross-multiplication algorithm. The following task was assigned to elicit reasoning:

> Why does the cross-multiplication algorithm work? Use your understanding of strip diagrams and, if you desire, the Cats and Dogs problem to aid your argument.

The majority of students constructed arguments that, at a minimum, connected the steps of cross multiplication to the steps followed when using the strip diagram. **Figure 7** shows Tanya's solution that is representative of the successful responses generated by students. It is worth noting that the strip diagrams seem to help illuminate the construction of the initial equivalent ratios in the cross product, in part, by connecting parts to quantities. Moreover, the use of strip diagrams appears to connect to the computations followed using the algorithm. Students seem to better understand the often seemingly arbitrary process of multiplying across the equal sign.

Tanya's experience with number operations and algebra allowed her to connect the operations used with the strip diagram to those in the cross-multiplication algorithm. She was also able to conclude that the only difference between the two has to do with the order of multiplication and division. However, the process used with the strip diagram is the exact opposite of the process used with the cross-multiplication algorithm.

For middle-grades students with minimal algebraic understanding and experience, this ordering of steps is a significant difference. The order in which the operations are performed when using the strip diagram connects the process to the covariance of quantities and confirms students' understanding of proportions. The order in which the operations are performed with the cross-multiplication algorithm does not connect well to understanding of proportions and may be a barrier to students' understanding the algorithm. The justification for the steps in the algorithm requires algebraic understanding that middle-grades students likely will not have before algebra 1. For this reason, teachers may wish to delay presenting the cross-multiplication algorithm to students until after they have acquired algebraic reasoning skills in algebra 1.

Fig. 7 Tanya's solution connects cross multiplication and strip diagrams and was typical of those who completed the task correctly.

The cross-multiplication algorithm is just a shortcut with rearranged steps from how we solve problems with a strip diagram. For this problem it would be:

$$\frac{5}{6} = \frac{x}{48}$$

$$\frac{5 \times 48}{6} = x$$

$$x = 40$$

The equation is set up this way because the 6 parts for the dogs are 48 dogs, and the 5 parts of the cats are some unknown number of cats, so we want the same things on the same side of the division. This is also because the ratios have to be the same, so we are making equivalent fractions.

With the strip diagram, we start by showing 5 boxes for the cats and 6 boxes for the dogs, and we know those 6 boxes are 48 dogs, so we divide 48/6 to get the number in each box, and then we know each of the 5 boxes have 8 in them, so we times 5 and 8 to get 40 cats. The only way this is different from the cross-multiplication is we're dividing first and then multiplying instead of multiplying and then dividing, but it still comes out the same.

The order in which the operations are performed with the cross-multiplication algorithm does not connect well to an understanding of proportions and may be a barrier to students' understanding the algorithm.

THE POWER OF THE STRIP DIAGRAMS

For problems requiring proportional reasoning, using strip diagrams to model ratios and solutions provides an excellent sense-making context among preservice teachers. The meaningful understanding of the relationships between parts and quantities would be difficult to accomplish with a traditional algorithmic method. The activities described in this article can build understanding of a variety of complex and multistep proportion problems. Building in opportunities for reflection and discussion about the use of strip diagrams can promote conceptual understanding of a topic that is often challenging. Reflection and discussion can also provide opportunities for preservice teachers to revisit and solidify their understandings of challenging ideas.

The visual nature of the strip diagram and the clear connection to the ratios in a proportion task may be a useful tool for middle-grades students. The procedure of using a strip diagram is more conceptually connected to covariance and the relationships between ratios and quantities than the procedure of using the cross-multiplication algorithm. This may allow middle-grades teachers to delay presentation of the algorithm until students have sufficient algebraic understanding to digest the process while providing a tool for facilitating problem solving.

If strip diagrams are presented as a strategy after students have had opportunities to develop their own strategies for solving proportion problems, they become tools for developing relationships between covariance and ratios and quantities. The proportion problems presented here are similar to those found in materials for use in the middle grades (Van de Walle, Karp and Bay-Williams 2010; Litwiller and Bright 2002), yet they are rich enough for use with preservice teachers, particularly when students are directed to focus on connections to their understanding of proportions and their own experiences with algorithmic solutions.

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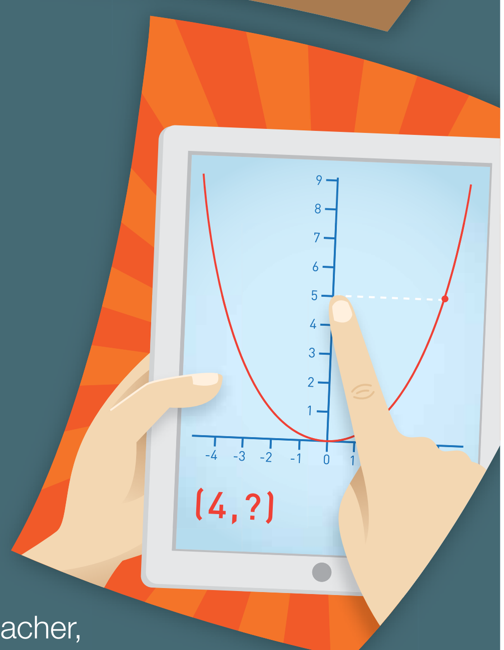

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