“I think I understand stuff when I’m in class. When I try to do homework I get confused and then I try to get help from the book – but none of it makes sense. I get frustrated and give up”  “Julian

Writing and Mathematical Rigor:  
Supporting the Mathematical Practices

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Some Preliminary Thoughts…..

What is rigor?  
Write down a few thoughts.
### Learning experiences that involve rigor ...

- challenge students
- require effort and tenacity by students
- focus on quality (rich tasks)
- include entry points and extensions for all students
- are not always tidy, and can have multiple paths to possible solutions
- provide connections among mathematical ideas
- contain rich mathematics that is relevant to students
- develop strategic and flexible thinking
- encourage reasoning and sense making
- expect students to be actively involved in their own learning

### Experiences that do not involve rigor ...

- are more “difficult,” with no purpose (for example, adding 7ths and 15ths without a real context)
- require minimal effort
- focus on quantity (more pages to do)
- are offered only to gifted students
- are scripted, with a neat path to a solution
- do not connect to other mathematical ideas
- contain routine procedures with little relevance
- follow a rote procedure
- require memorization of rules and procedures without understanding
- often involve teachers doing the work while students watch

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Stories are a natural part of early learning experiences. But how do we promote understanding by emphasizing writing?

[Assessment Practices to Support Mathematics Learning and Understanding for Students (APLUS): Assessing Math Concepts (Richardson, 1998)]
Developing Early Numeracy Must Also Recognize the Importance of Language

The teacher modeling relationships between speaking and recording is essential to developing early skills not only in numeracy but communicating mathematically.

[Assessment Practices to Support Mathematics Learning and Understanding for Students (APLUS): Assessing Math Concepts (Richardson, 1998)]

Goal, Purposes and Types of Writing

- **Exploratory**
  - To personally make sense of a problem, situation, or one's own ideas

- **Informative/Explanatory**
  - To describe
  - To explain

- **Argumentative**
  - To construct an argument
  - To critique an argument

- **Mathematically Creative**
  - To document original ideas, problems, and/or solutions
  - To convey fluency and flexibility in thinking
  - To elaborate on ideas

Exploratory

Some students made mistakes in their drawings. First, write down the names of all the students who made mistakes. Then, choose one student whose drawing you think needs the most improvement and write how you would help them solve the problem correctly.

Alex, Bob, and Cole all make mistakes.

Figure 1: Example of exploratory writing for the purpose of writing mathematics to perceive and describe a problem situation or one's own ideas. This student described a table in Lense before answering the problem addressing the use of the whole as a fraction to a fraction.


Informative/Explanatory

Figure 2: Example of informative explanatory writing for the purpose of writing mathematics to describe. (26) is the transcription of the writing: "How to cut a cup?"

Sample contributed by Edith Lubelli.
Informative/Explanatory

The difference between a cube and a square is that a square has 4 sides and 4 angles, and that you can put something in a cube and close it. A cube has 6 faces and a square has 4 faces. A square is 2D and a cube is 3D. A square is flat and a cube is not. A cube is held in your hands and a square can be passed.

Argumentative

Mia was a discovery. She claims that all squares are rectangles! Do you agree or disagree? Explain your answer.

I agree to Mia's theory. I agree because a square has all the characteristics of a rectangle. It is made up of 4 sides, 4 angles, and lines of symmetry. It is also a quadrilateral. But it also has an additional feature. The sides of the square are equal. All equal sides make a circle. But it's a square, not a circle.
Mathematically Creative

![Image of a mathematical problem and solution]

Figure 11: Create a sample of mathematically creative writing for the purpose of: writing mathematically to accomplish original goals, problems, and/or outcomes and to convey fluency and flexibility in thinking. This student wrote a response to the following problem: How many handshakes will there be if everyone in our class of 25 students shakes each other's hands once? Although students were expected to solve the problem only one way, the sample multiple ways because “I needed to see it differently.” The major two tables check: drew a picture; organized a formula, ensuring the class had not yet learned. This figure includes a portion of the writing, including charts and use of a formula.

Sample contributed by Linda J. Shepherd.
Problem Solving Seven on Angles in a Triangle.

Problem:
Determine the measure of the missing angle. (Triangle is not drawn to scale.) Show all work.

1. \[ \angle 60° \]

Key Words:
- Measure angle →
- Right angle →

Action:
Add and subtract always 90°

Find Problem:
Find missing angle in the triangle

Plan:

\[
\begin{align*}
\text{Add } 3 \text{ angle } & \text{ to get } 180° \\
\text{answer is } & 90° + 60° = 150°
\end{align*}
\]

Examine:

<table>
<thead>
<tr>
<th>Plan</th>
<th>Examine</th>
<th>Verify</th>
</tr>
</thead>
<tbody>
<tr>
<td>Add 3 angles to get 180°</td>
<td>90° + 60° = 150°</td>
<td>Answer is correct! Right triangle</td>
</tr>
</tbody>
</table>

Question 5: In a class three-fifths of the pupils were girls and the rest were boys. If the number of boys were doubled, and 6 more girls joined the class, there will be equal number of boys and girls. How many pupils were in the class at the start?

**Algebra**
Let no. of boys be \( x \).
Let no. of girls be \( y \).
\[
\begin{align*}
2x + y &= 12 \\
\frac{2}{5}x &= y \\
\frac{2}{5}x &= 6 \\
x &= 15 \\
\text{There were } 30 \text{ pupils.}
\end{align*}
\]

**Diagram**

<table>
<thead>
<tr>
<th>Diagram</th>
</tr>
</thead>
<tbody>
<tr>
<td>6 girls</td>
</tr>
<tr>
<td>1 boys</td>
</tr>
<tr>
<td>6 girls joined the class, number of boys and girls are equal.</td>
</tr>
<tr>
<td>No. of boys = 2 \times 6 = 12</td>
</tr>
<tr>
<td>No. of girls = 3 \times 6 = 18</td>
</tr>
<tr>
<td>There were 30 pupils in the class.</td>
</tr>
</tbody>
</table>

**Algebra + diagram**

\[
\begin{align*}
\frac{1}{2}x + 6 + \frac{2}{5}x &= 12 \\
\frac{1}{2}x &= 6 \\
\frac{2}{5}x &= 6 \\
\frac{2}{5}x + \frac{1}{2}x &= 6 \\
\text{There were 30 pupils at the start.}
\end{align*}
\]
Fraction Word Problem Composed by Student

I am not computing here, but Ikhine if you add add by going from 5 to 10, you will get the correct answer. For instance, $10 + 5 = 15$ and $15 + 6 = 21$, etc. I think this would be correct by looking at the problem alone. Your first result in the triangle would be the number 60. At first I thought you would multiply by 2, but that didn’t work for 1 because $6 \times 2 = 12$. In the example of the triangle it is 6 less than 10. So I changed my mind!
Place the following fractions in order from smallest to largest: 5/10, 5/3, 5/6.

Describe how you determined the order.
Figure 1. Rubric for Assessing Five Mathematical Facets.

<table>
<thead>
<tr>
<th>Conceptual Understanding</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>Identifies and provides information about major concepts; supplies examples or illustrations with explanations when appropriate.</td>
</tr>
<tr>
<td>3</td>
<td>Identifies and provides information about major concepts but may omit minor details. May use examples or illustrations when appropriate but may not effectively relate them to mathematical concepts.</td>
</tr>
<tr>
<td>2</td>
<td>Identifies and provides support for major concepts but may have minor errors in logic or understanding. Minor details are ignored or supported with incorrect or flawed thinking.</td>
</tr>
<tr>
<td>1</td>
<td>Does not correctly identify major concepts and information contains errors in logic or understanding.</td>
</tr>
<tr>
<td>0</td>
<td>No attempts are made to identify or provide information about major concepts or the information has no mathematical soundness.</td>
</tr>
</tbody>
</table>

- **0** Answer is unresponsive, unrelated, or inappropriate. Nothing is correct.
- **1** Answer addresses item but is only partially correct; something is correct related to the question.
- **2** Answer deals correctly with most aspects of the question, but something is missing. Answer may deal with all aspects but have minor errors.
- **3** All parts of the question are answered accurately and completely. All directions are followed.

Source: Adapted from the 1997–98 North Carolina Open-Ended Assessment, Grade 8
Group Response to Ratio Task

K-W-L on Graphing.

a. Which measuring tool was best to measure the circumference of the eggs?

The best tool to use was the tape measure.

b. Why?

I think that because it can bend, you can place it any where. The ruler has numbers on it, but it can't bend like the pipe cleaner can bend, but it doesn't have numbers on it.

Figure 8. Grade 3 sample of argumentative writing for the purpose of writing mathematically to construct an argument.

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Milan earned 16 more stickers than Rene. She earned 35 stickers. How many stickers did Rene earn?

Georgia earned 47 stickers. How many more does Rene need to have the same amount as Georgia?

Aa. Milan earned 35 stickers.
Rene? 16

Rene earned 19 stickers.
47 - 10 → 37 → 28

Bb. Rene needs 28 more stickers.

Georgia (G) 47
Rene (R) 19
Sentence Cards for Transformations

<table>
<thead>
<tr>
<th>is an example of a ________</th>
<th>________ produces a mirror image.</th>
</tr>
</thead>
<tbody>
<tr>
<td>is an example of a ________</td>
<td>________ turns an object around.</td>
</tr>
<tr>
<td>is an example of a ________</td>
<td>________ has a direction and a distance.</td>
</tr>
</tbody>
</table>
Dear Jack

I hope this letter finds you well. I am currently pursuing a Bachelor of Science degree in Electrical Engineering, which includes extensive studies in differential equations and other higher-level applications. I am currently working on a project involving the use of differential equations to solve real-world problems. The solution to this problem is quite complex, but I believe it is worth a try. I have attached some of my work to this letter.

Sincerely,

[Signature]

Date: 7/15/2016

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Look for and make use of structure.

8 + 4 = 12

See the pattern or connection.
For a dinner, you plan to roast a turkey. You saw a sign that turkeys cost $1.25 per pound. You need to buy a big turkey to feed everyone and the biggest turkey in the store weighs 24 pounds. Figure out how much that turkey will cost.
Mark was mixing blue paint and yellow paint in the ratio of 2:3 to make green paint. He wants to make 45 liters of green paint. He began to make a table to help him think about the problem, but is unsure of what to do next.

<table>
<thead>
<tr>
<th>Liters of Blue Paint</th>
<th>Liters of Yellow Paint</th>
<th>Liters of Green Paint</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>3</td>
<td>5</td>
</tr>
<tr>
<td>10</td>
<td>15</td>
<td>25</td>
</tr>
<tr>
<td>20</td>
<td>30</td>
<td>50</td>
</tr>
<tr>
<td>30</td>
<td>40</td>
<td>60</td>
</tr>
<tr>
<td>40</td>
<td>50</td>
<td>70</td>
</tr>
</tbody>
</table>

a. Explain in words how to continue placing values into the table.

b. Write an explanation in words to Mark about how he can use his table to find how many liters of blue paint and how many liters of yellow paint he will need to make 45 liters of green paint.

- Multiply 2, 3, and 5 by 9; then 3 and 9 or big numbers.
- Mark can multiply 2 x 5 so 3 x 7 and 3 x 9 by 9. 10, 15, 25 divided by 5 is 20, 30, 50 so back down by a group of 2, 3, 5 to 45.
“Ben had 5 crackers. He got some more. Now he has 7. How many crackers did Ben get?”

He got 2 crackers.
Model for Creating Expectations for Writing

Key Terms for Clear Writing Tasks

Compare and/or Contrast:
- Write about what is different and/or what is alike.

Describe:
- Provide step-by-step details using key terms, graphs, charts, diagrams, and other illustrations. Make sure the illustrations and diagrams are referred to in the descriptions.

Explain:
- Elaborate on solutions, steps, ideas, concepts, and conjectures using numbers, symbols, illustrations, and examples when it helps to make your explanations clear.

Interpret:
- Provide mathematical reasoning to describe relationships (could involve mathematical information contained in data tables or charts, graphs, illustrations, models, diagrams, symbolic representations, and other ways of representing information and interactions).

Provide Reasons or Justification:
- Give supporting evidence from mathematics to support your thinking including examples, mathematical concepts and definitions, theorems, and other reasons that support what you say.

List:
- Provide ideas in numbered or bulleted format.

Show all work:
- Include all calculations, steps, and ideas that you thought about and used to reach your conclusion: all information that shows your thinking.
# Modifying Text Features to Support Writing

<table>
<thead>
<tr>
<th>Text Feature</th>
<th>Modification for Promoting Writing</th>
</tr>
</thead>
<tbody>
<tr>
<td>Conceptually Dense Information</td>
<td>Restating and/or summarizing important concepts or ideas.</td>
</tr>
<tr>
<td>Diagrams, Cases, Models &amp; Illustrations</td>
<td>Descriptive writing that requires students to incorporate diagrams and illustrations into a problem or explain how a diagram or illustration supports the mathematical concept or idea being emphasized.</td>
</tr>
<tr>
<td>Definitions, Rules, Formulas</td>
<td>Stating definitions, rules, or formulas in one’s own words. Explaining why a rule or formula works.</td>
</tr>
<tr>
<td>Examples Problems &amp; Procedures</td>
<td>Describing how examples are similar and different. Describing how examples demonstrate definitions, rules, formulas, etc.</td>
</tr>
</tbody>
</table>


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# Writing to Support Text Features

<table>
<thead>
<tr>
<th>Text Information</th>
<th>Writing Prompts to Extend Text</th>
</tr>
</thead>
</table>
| Students presented with scenario: 1/3 of students prefer chocolate ice cream and 1/6 prefer vanilla ice cream. What part of the class prefers chocolate and vanilla ice cream? | - What does the fraction 1/3 mean?  
- Use circles or squares to show 1/3 and 1/6. How can you tell which is larger? |
| Worked example using ice cream scenario. | - What does the fraction 3/6 mean? How would this relate to a class of 24 students?  
- Show how 3/6 and ½ represent the same fraction. Write a few sentences explaining your work. |
| Additional worked examples. | - Write the steps for this problem.  
- Why do you not add the denominator when adding fractions? |
| Classwork problems. | - The first problem is 2/5 + 1/5. If someone gets 1/5 as the answer, what did they probably do wrong?  
- Describe how you find a common denominator for ¼ and 1/3. |
| Practice Problems. | - Tell how you worked problem 3.  
- Draw a diagram and tell how you know your answer for number 3 is correct. |

Writing Prompts Aligned to Learning Targets

**Developing Students' Knowledge of Mathematics**
- Write personal definitions of terms, rules, theorems, etc.
- Write explanations of mathematical concepts and ideas
- Write a summary of a lesson or task
- Write explanations of errors (What went wrong? How can the error be addressed?)
- Offer examples and justify selection
- Describe rules, their application, and mathematical importance

Writing Prompts Aligned to Learning Targets

**Developing Problem Solving Methods**
- Write problems, applications, and provide solutions
- Describe how to solve a problem
- Compare and contrast alternative approaches to a problem
- Describe how technology helped in finding a solution (Describe the mathematical processes required for the output.)
- Write a formal report for approaching a problem situation.

Writing Prompts Aligned to Learning Targets

*Developing Self-Monitoring and Reflective Behaviors*

- Describe what made a problem or task easy and/or difficult
- Explain why an answer or solution is reasonable
- Identify and react to questions one may raise about your work (or respond to a question someone raised about your work)
- Analyze the quality of one’s work (process, methods, mathematical soundness, communication)
- Describe how different decisions might impact an answer
- Describe how problems are similar and/or different


Writing Prompts Aligned to Learning Targets

*Promoting Affective Issues*

- Write an autobiography about a mathematics experience
- Write about the role mathematics plays in your life or might play in the future.
- Describe how mathematics changes or changed one’s life.
- Explain what helps or hinders you in understanding mathematics
- Describe how you feel about your performance on a task or problem
Writing Prompts Aligned to Learning Targets

Promoting Discourse

- Write a note or notes to the teacher for additional information
- Specify lesson components which were not understood or components which you understood well
- Write a journal entry about some aspect of the day’s class
- Summarize and interview with a peer or other individual about a topic, problem, or other mathematically related idea
- Write a response for a group or team to a problem or task
| 1. Summarize a part of class discussion or lecture. |
| 2. Summarize a part of the text. |
| 4. Use key words in an explanation or description. |
| 5. Construct test or quiz questions. |
| 7. Develop a decision or action. |
| 8. Create a graph or table. |
| 9. List characteristics or steps. |
| 10. Compare understanding of a concept to what was known before a lesson or exercise. |
| 11. Write a proposal for a project. |
| 12. Write a memo dealing with a math topic or problem. |
| 13. Create and defend a statement about a math topic. |
| 14. Write a biographical sketch of a mathematician. |
| 15. Paraphrase a section of text. |
| 16. Write a summary or analysis of an interview. |
| 17. Write a report on a math topic or problem. |
| 18. Create an outline of a lesson. |
| 19. Write a description of a graph or table. |
| 20. Write a definition of a term given in class or the text. |
| 21. Write a letter explaining a mathematical idea, problem, or process. |
| 22. Write a letter explaining how a mathematician used a particular concept (such as what mathematicians have used in sports, travel, a hobby...). |
| 23. Write a dialogue between student and teacher. |
| 24. Write a dialogue between student and another person. |
| 25. Describe a graph or table. |
| 26. List characteristics or steps. |
| 27. Identify personal goals for mathematics learning. |
| 28. Write a list of unfamiliar terms or concepts. |
| 29. Write a list of important points. |
| 30. Write a list of unfamiliar terms to be learned. |
| 31. Write a list of unfamiliar terms to be learned. |
| 32. Write a list of unfamiliar terms to be learned. |
| 33. Write a list of unfamiliar terms to be learned. |
| 34. Write a list of unfamiliar terms to be learned. |
| 35. Write a list of unfamiliar terms to be learned. |
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| 39. Write a list of unfamiliar terms to be learned. |
| 40. Write a list of unfamiliar terms to be learned. |
| 41. Write a list of unfamiliar terms to be learned. |
| 42. Write a list of unfamiliar terms to be learned. |
| 43. Write a list of unfamiliar terms to be learned. |
| 44. Write a list of unfamiliar terms to be learned. |
| 45. Write a list of unfamiliar terms to be learned. |
| 46. Write a list of unfamiliar terms to be learned. |
| 47. Write a list of unfamiliar terms to be learned. |
| 48. Write a list of unfamiliar terms to be learned. |
| 49. Write a list of unfamiliar terms to be learned. |
| 50. Write a list of unfamiliar terms to be learned. |

Fifty Activities for Writing in Mathematics