The Mathematical Task Analysis Guide (TAG)

Lower-Level Demands	Higher-Level Demands
Memorization Tasks • Involves either producing previously learned facts, rules, formulae, or definitions OR committing facts, rules, formulae, or definitions to memory. • Cannot be solved using procedures because a procedure does not exist or because the time frame in which the task is being completed is too short to use a procedure. • Are not ambiguous – such tasks involve exact reproduction of previously seen material and what is to be reproduced is clearly and directly stated. • Have no connection to the concepts or meaning that underlie the facts, rules, formulae, or definitions being learned or reproduced.	Procedures With Connections Tasks • Focus students' attention on the use of procedures for the purpose of developing deeper levels of understanding of mathematical concepts and ideas. • Suggest pathways to follow (explicitly or implicitly) that are broad general procedures that have close connections to underlying conceptual ideas as opposed to narrow algorithms that are opaque with respect to underlying concepts. • Usually are represented in multiple ways (e.g., visual diagrams, manipulatives, symbols, problem situations). Making connections among multiple representations helps to develop meaning. • Require some degree of cognitive effort. Although general procedures may be followed, they cannot be followed mindlessly. Students need to engage with the conceptual ideas that underlie the procedures in order to successfully complete the task and develop understanding.
Procedures Without Connections Tasks • Are algorithmic. Use of the procedure is either specifically called for or its use is evident based on prior instruction, experience, or placement of the task. • Require limited cognitive demand for successful completion. There is little ambiguity about what needs to be done and how to do it. • Have no connection to the concepts or meaning that underlie the procedure being used. • Are focused on producing correct answers rather than developing mathematical understanding. • Require no explanations, or explanations that focus solely on describing the procedure that was used.	Doing Mathematics Tasks • Requires complex and non-algorithmic thinking (i.e., there is not a predictable, well-rehearsed approach or pathway explicitly suggested by the task, task instructions, or a worked-out example). • Requires students to explore and to understand the nature of mathematical concepts, processes, or relationships. • Demands self-monitoring or self-regulation of one's own cognitive processes. • Requires students to access relevant knowledge and experiences and make appropriate use of them in working through the task. • Requires students to analyze the task and actively examine task constraints that may limit possible solution strategies and solutions. • Requires considerable cognitive effort and may involve some level of anxiety for the student due to the unpredictable nature of the solution process required.

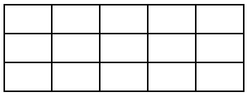
Stein M. K., Smith, M. S., Henningsen, M. A., & Silver, E. A. (2000). Implementing standards-based mathematics instruction: A casebook for professional development, p. 16. New York: Teachers College Press.

Multiplication Tasks A – D

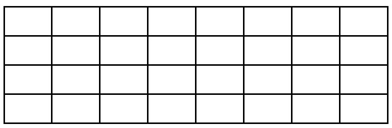
Task A

Manipulatives/Tools Available: Calculator, worksheet showing the figures below.

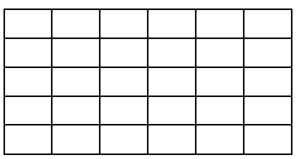
Finding Area



Find the area of the figure.
Count the square units: ______square units



Find the area of the figure.
Count the square units: ______square units

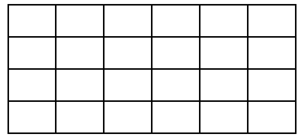


Find the area of the figure.
Count the square units: ______square units

Task B

Manipulatives/Tools Available: Grid paper, interlocking cubes and/or tiles.

Arranging Chairs for a Play



1. If each square is a chair for a student, then write an equation that describes how the number of chairs are arranged for the students.
2. Rearrange the squares to show all the possible different arrangements of chairs. Make sure the chairs are arranged in rows that have the same number of chairs. Write an equation that describes each arrangement.
3. How do you know if you have all possible arrangements?
4. What if one more student came to the play? What happens to the number of different arrangements of chairs if the new arrangements have equal rows, too?

Task C

Manipulatives/Tools Available: Square tiles, grid paper, paper and pencil.

Constructing Area

Step 1: Draw a rectangle on grid paper that has the dimensions of 6 units by 4 units. Write a number sentence that gives the area of the rectangle (the total number of square units inside the rectangle).

Step 2: Draw another rectangle with the same area as the one you drew in Step 1. Write a number sentence that matches the rectangle you drew and give its area.

Task D

Manipulatives/Tools Available: Square tiles, grid paper, paper and pencil.

Trays of Candy

Part 1

You work in a candy store. You are in charge of designing trays of candy to show customers. Each square on a tray holds a piece of candy. (*Show tiles to represent pieces of candy*.) How many different trays can you make that hold 12 pieces of candy? The candy has to fill up all the space on the tray and be only one layer high. Explain how you know you found all the different sizes of trays.

Part 2

You are thinking about making trays that hold 48 or 37 candies. Find and show all of the different sizes of trays possible that hold 48 or 37 candies. Which number (12, 37, or 48) gives you the most different-sized trays? Explain your work.

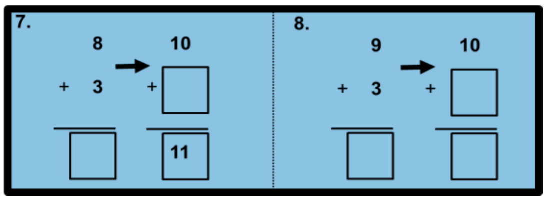
Tell your boss about what you found. She wants to know the number that will give her the most different-sized trays. Draw pictures, write equations and use words to explain your choice.

Addition Tasks   A – C

Task A

Manipulatives/Tools Available: Counters

Solve each problem. Use what you know from first problem to solve the second problem.



If you know 10 + 4 = 14, then how can this help you think about 9 + 4? Explain how the solution to one problem can be used to think of the solution to the other problem.

- 10 + 4
- 9 + 4

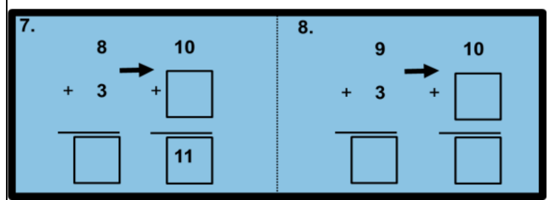
If you know 10 + 6 = 16, then how can this help you think about 8 + 6? Explain how the solution to one problem can be used to think of the solution to the other problem.

- 10 + 6
- 8 + 6

Task B

Manipulatives/Tools Available: Counters

Solve each problem.



Task C

Manipulatives/Tools Available: None

Solve each problem.

8	9	10	6	7
+ 5	+ 6	+ 9	+8	+7

Task D

1. John has 10 pencils. He gets 8 more pencils. How many pencils does John have now?

2. Marisol has 7 pencils. She gets some more. Now she has 18 pencils? How many pencils does Marisol have now?

Marisol claims that she can use what she knows about John's pencils to help her determine the number of pencils that she has. Explain how Marisol can use John's problem to solve her problem?

The Four Levels of Cognitive Demand of Multiplication of Fractions Tasks

Lower-Level Demands

Memorization

What is the rule for multiplying fractions?

Expected Student Response:

You multiply the numerator times the numerator and the denominator times the denominator.

OR

You multiply the two top numbers and then the two bottom numbers.

Procedures Without Connections

Multiply: $\frac{1}{6} \times \frac{1}{2}$

$\frac{2}{3} \times \frac{3}{4}$

$\frac{4}{9} \times \frac{3}{5}$

Expected Student Response:

$$\frac{1}{6} \times \frac{1}{2} = \frac{1 \times 1}{6 \times 2} = \frac{1}{12}$$

$$\frac{2}{3} \times \frac{3}{4} = \frac{2 \times 3}{3 \times 4} = \frac{6}{12}$$

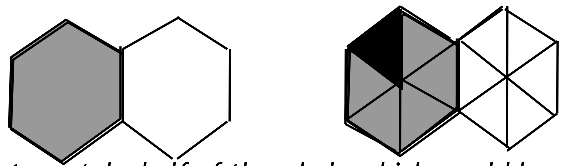
$$\frac{4}{9} \times \frac{3}{5} = \frac{4 \times 3}{9 \times 5} = \frac{12}{45}$$

Higher-Level Demands

Procedures With Connections

Find 1/6 of 1/2. Use pattern blocks. Draw your answer and explain your solution.

Expected Student Response:



First you take half of the whole which would be one hexagon. Then you take one-sixth of the half. So I divided the hexagon into six pieces which would be six triangles. I only needed one-sixth so that would be one triangle. Then I needed to figure out what part of the two hexagons one triangle was and it was 1 out of 12. So 1/6 of 1/2 is 1/12.

Doing Mathematics

Create a real-world situation for the following problem: $\frac{2}{3} \times \frac{3}{4}$.

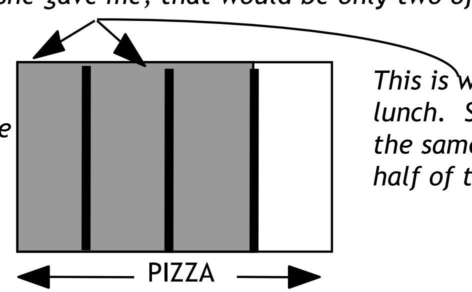
Solve the problem you have created without using the rule and explain your solution.

One Possible Student Response:

For lunch Mom gave me three-fourths of the pizza that we ordered. I could only finish two-thirds of what she gave me. How much of the whole pizza did I eat?

I drew a rectangle to show the whole pizza. Then I cut it into fourths and shaded three of them to show the part mom gave me. Since I only ate two thirds of what she gave me, that would be only two of the shaded sections.

Mom gave me the part I shaded.



This is what I ate for lunch. So 2/3 of 3/4 is the same thing as half of the pizza.

Seth's Birthday Cake

After his birthday party, Seth had one-fifth of his cake leftover. Eight of his friends spent the night. His friends plan to eat the remaining cake as a snack later that night, and they each expect to receive an equal share of the remaining cake. They want to know what fractional part of the original cake will each get to eat for a snack.

A. What fractional part of the cake did each friend eat for a snack? Draw and clearly label a diagram that shows how much of the cake is eaten by one person for the snack. Use words to explain your reasoning and decisions for how you drew the diagram.

B. Seth claims that to find the answer you can use the expression 8 ÷ 1/5. His friend Frankie says that the expression would be 1/5 ÷ 8. Who is correct? Why?

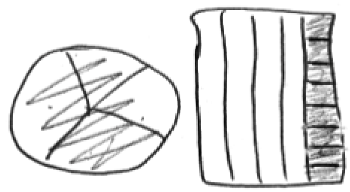
Adapted from *Multiplication of Fractions*, Institute for Learning, University of Pittsburgh, 2015 (ifl.pitt.edu).

Figure 9.2. A task on division contexts involving fractions

Seth's Birthday Party Scenarios

Scenario 1. Nhia

Nhia first drew a circle, and then crossed it out on her paper. She finally seemed to settle on the drawing to the right for the birthday cake task.



Teacher: Tell me about your drawing?

Nhia: I think this one might work. *(Points to* the *diagram on the right.)* This is the cake and I cut it into fifths. Then I cut one of those fifths into eight pieces.

Teacher: So, why did you cut it into eight pieces?

Nhia: Well, there were 8 friends and they shared that leftover piece so I cut it into eighths.

Teacher: So now you have to describe the size of the piece that each friend got.

Nhia: Yeah, but I don't get that. Is the answer one-eighth?

Teacher: Remember when we did a problem like this the other day, and we had to make sure the whole was all cut into equal sized pieces. What is the whole in this problem?

Nhia: The cake.

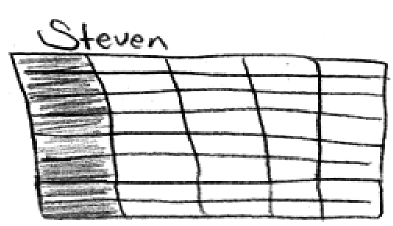
Teacher: So right now you have some big pieces and some small pieces. How many small pieces would you have it you took all five of the big pieces and broke each one into eight smaller pieces just like you did for that one part already?

Nhia: 40?

Teacher: Yes. So now you can figure out how big the pieces are that each friend gets.

Scenario 2. Steven

Steven confidently drew a diagram of the birthday cake determining that each friend got one-fortieth of the cake to eat as a snack. However, he was not as sure which equation modeled the situation.



Teacher: Tell me how you made your diagram.

Steven: I drew a rectangle for the cake and then I cut it into fifths, and shaded one fifth because that's what was left of the cake. Then I cut each of those fifths into 8 pieces because there were 8 friends.

Teacher: So do you know what the answer is yet?

Steven: Yeah. Each friend got one fortieth of the cake.

Teacher: So, which expression do you think matches this situation? Do you agree with Seth or Frankie?

Steven: I'm thinking Frankie (1/5 ÷ 8) because there was a fifth left and it had to be shared among the 8 friends.

Teacher: So you think it is 1/5 ÷ 8. What about the other expression, 8 ÷ 1/5? Is that different?

Steven: Well, we would have to say eight is 8 friends and divide each friend into fifths. That doesn't even make sense.

Teacher: Interesting. So now write an explanation about which expression you think is right and try to explain why it can't be the other expression.

Scenario 3: Zhen

As the teacher approaches Zhen, he is reading and rereading the problem, but has not yet put anything on his paper.

Zhen: I don't get it.

Teacher: What are you suppose to figure out?

Zhen: How to share the cake with 8 friends.

Teacher: Okay, how might you start?

Zhen: I don't know. They only got a fifth.

Teacher: Okay, the first thing you need to do is draw a rectangle for the cake and then cut it into fifths because there was one fifth of the cake left. Then next you need to cut those pieces into 8 smaller pieces each for the 8 friends.

Zhen: Make eighths?

Teacher: No, cut the cake into fifths. Do that much first. Then I'll help you with what to do next.

Zhen: Oh. *(The student draws a rectangle and partitions it into fifths.)*

Teacher: Good, now shade one fifth to show how much cake was left.

Zhen: *(The student shades one fifth.)*

Teacher: Good. Now cut the other way across the cake to get 8 pieces. How many small pieces are in the whole cake?

Zhen: 40.

Teacher: Okay, so if each friend got one of those small pieces, what fraction of the whole cake did each friend get?

Zhen: One-fortieth?

Teacher: Yes, each friend got one-fortieth. Now go on the next part of the task.

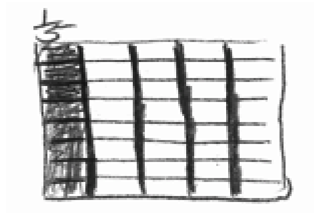
Scenario 4. Diego

Diego determines that each friend gets one-fortieth of the cake and then claims that both equations 8 ÷ 1/5 and 1/5 ÷ 8 work because the order does not matter with the commutative property.

Teacher: Tell me about your thinking, why do you think both could be right?

Diego: They're both right because you can just switch the numbers and you get the same answer.

Teacher: Interesting, so tell me what you did here with your picture and why. *(Teacher points to the student's diagram.)*



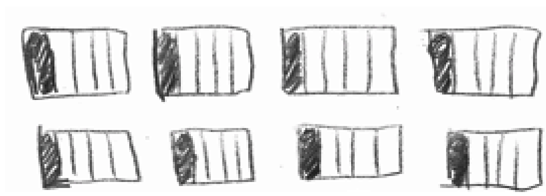
Diego: I made a rectangle for the cake and cut it into fifths cuz that's what they had left. Then I cut it into eighths 'cuz there were 8 friends sharing it. And each friend got one-fortieth.

Teacher: So, did what you just describe sound more like 8 ÷ 1/5 or more like 1/5 ÷ 8?

Diego: Well…um… 8 friends shared that piece, I'm thinking it's more like 1/5 ÷ 8.

Teacher: You said, you thought you would get the same answer with either expression, so tell me about 8 ÷ 1/5 and how that would be the same.

Diego: I could make 8 rectangles (*draws the picture below*) and cut each into fifths (*subdivides each rectangle*). Each person gets one piece (*shades one part of each rectangle*)... mmm ... I got 40 pieces, but…now I'm not sure, is each piece one-fifth or one-fortieth?



Teacher: What do you think?

Diego: I think this is different this would be 8 cakes cut into fifths, and I got 40 fifths, not fortieths.

Teacher: So are the expressions the same or different?

Diego: No, they're different. I guess you can't just switch them. Frankie is right it's 1/5 ÷ 8.

Scenario 5. Sophia

Sophia has not begun to solve the task yet. She is reading the problem as the teacher approaches her.

Teacher: So, tell me about the story. What is happening?

Sophia: There are 8 friends staying at Seth's house for his birthday.

Teacher: That's part of it. Why don't you go back and read the problem again and see what else is happening.

Sophia: Oh, okay.

Teacher: Then start working on Part A where it says to draw a picture of what is happening in the problem.

Fig. 9.3. Impasse scenarios for Seth's birthday cake task.

Huinker, D. and Bill, V. Taking Action: Implementing Effective Teaching Practices. Strategies to support productive struggle. *Mathematics Teaching in the Elementary School*, *20*(7), 390.