Multi-Tiered Systems of Support: What are Effective, Research-informed Interventions and Strategies? (Grades 2–6)

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Your Introductions

- Name
- Teaching role and grade levels
- Go-to source for lesson planning
- Your math super power

Warm up for today's topic

- Using a number family like 9, 6, 15, create an addition or subtraction story problem that you would have students in your classroom solve.
- If you are in middle school, you can substitute a rational number for one of the values.

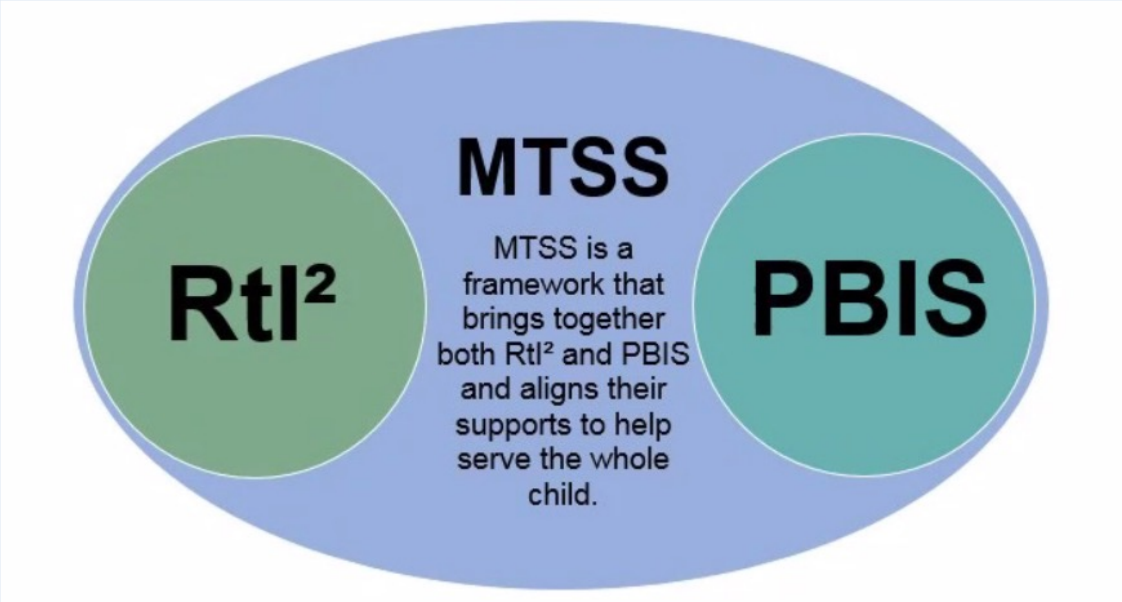
Focus of Session

This session focuses on ways to support students who struggle in Tiers 1 and 2 to become:

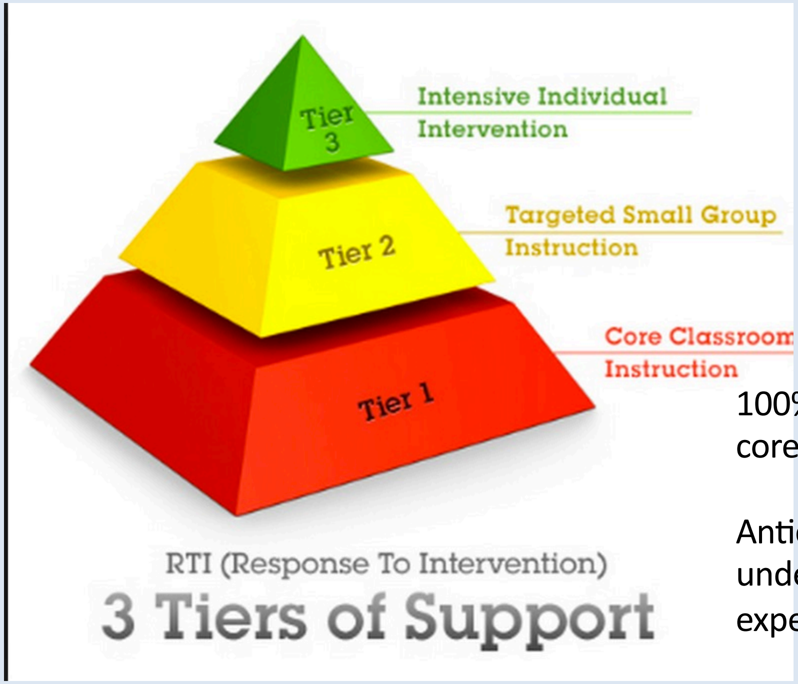
- <u>Confident</u> in mathematics
- Successful with <u>rigorous</u> mathematics
- Capable of working with <u>complex</u> mathematical ideas

Multi-Tiered Systems of Support (MTSS): Response to Intervention (RtI)

How does it all Fit?



3-Tiered Support Model

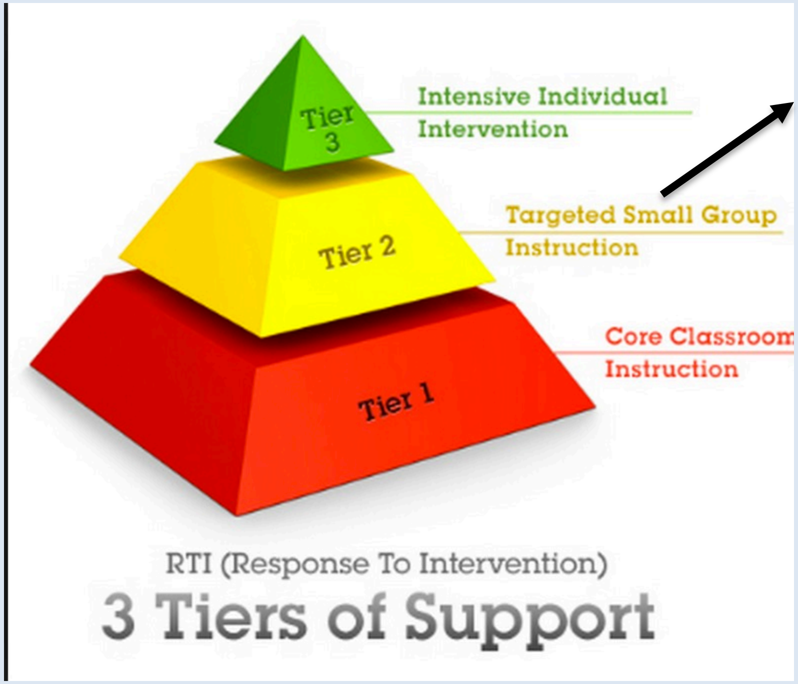


100% of students receive core instruction

Anticipated that 80% understand the content at expected levels

Effective, research-based

3-Tiered Support Model



15% of students receive Tier 2 instruction

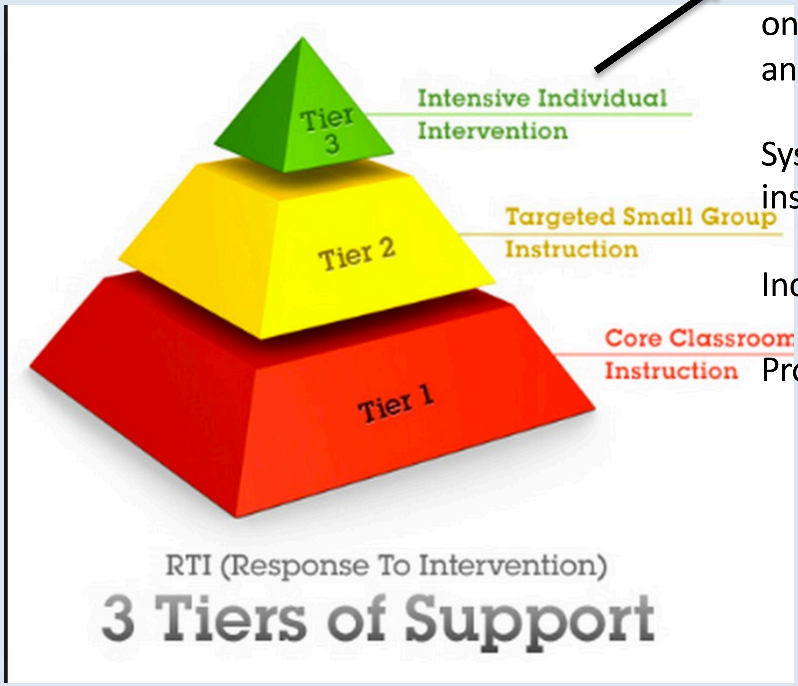
Targeted, focused instruction on topics that 'fill the gaps'

Systematic

Small group or smaller class size

Progress monitoring

3-Tiered Support Model



5% of students receive
Tier 3 instruction

Targeted, focused instruction on topics that 'fill the gaps' and more specific than Tier 2

Systematic with more explicit instruction

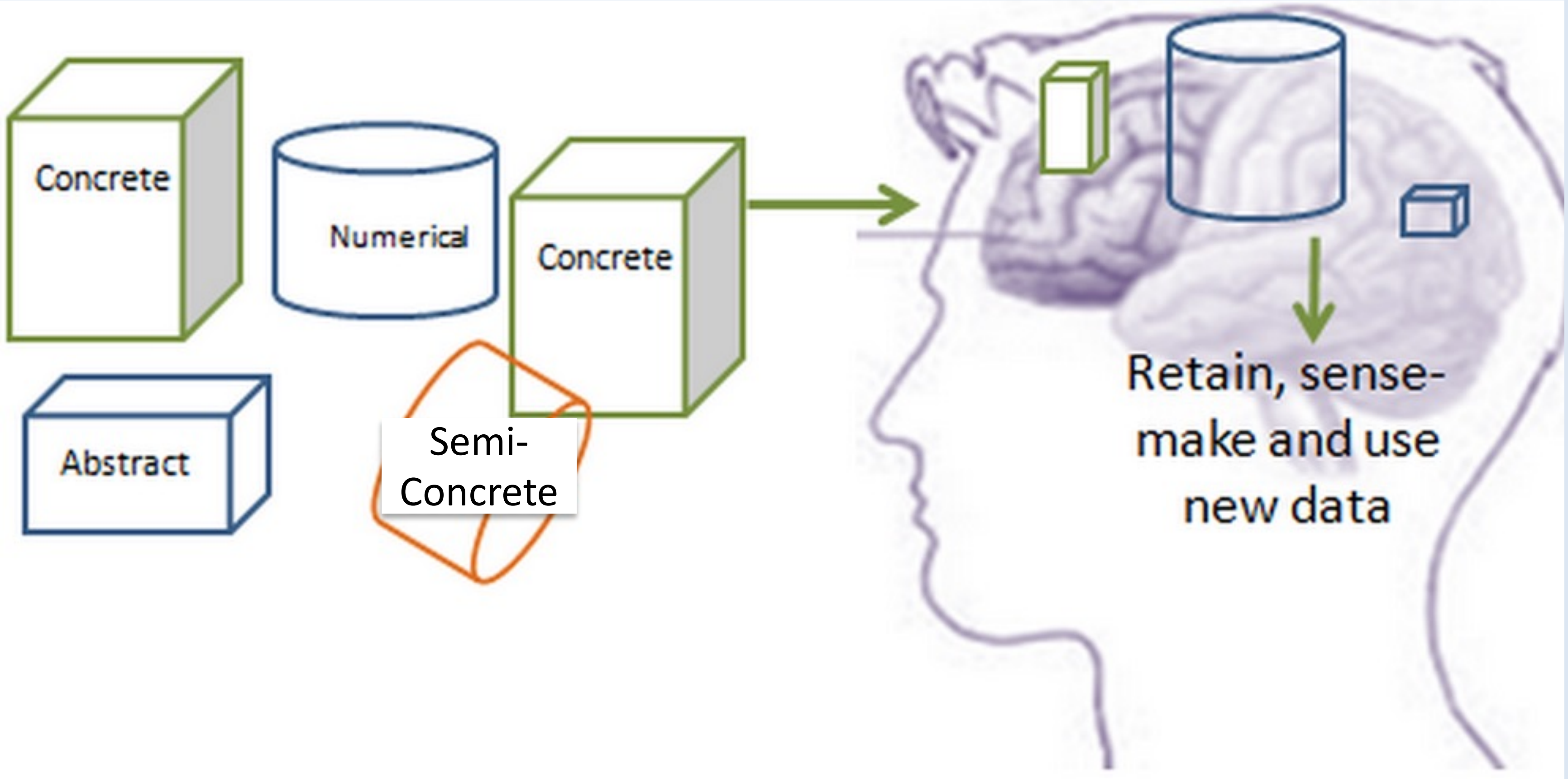
Individual or small group

Progress monitoring

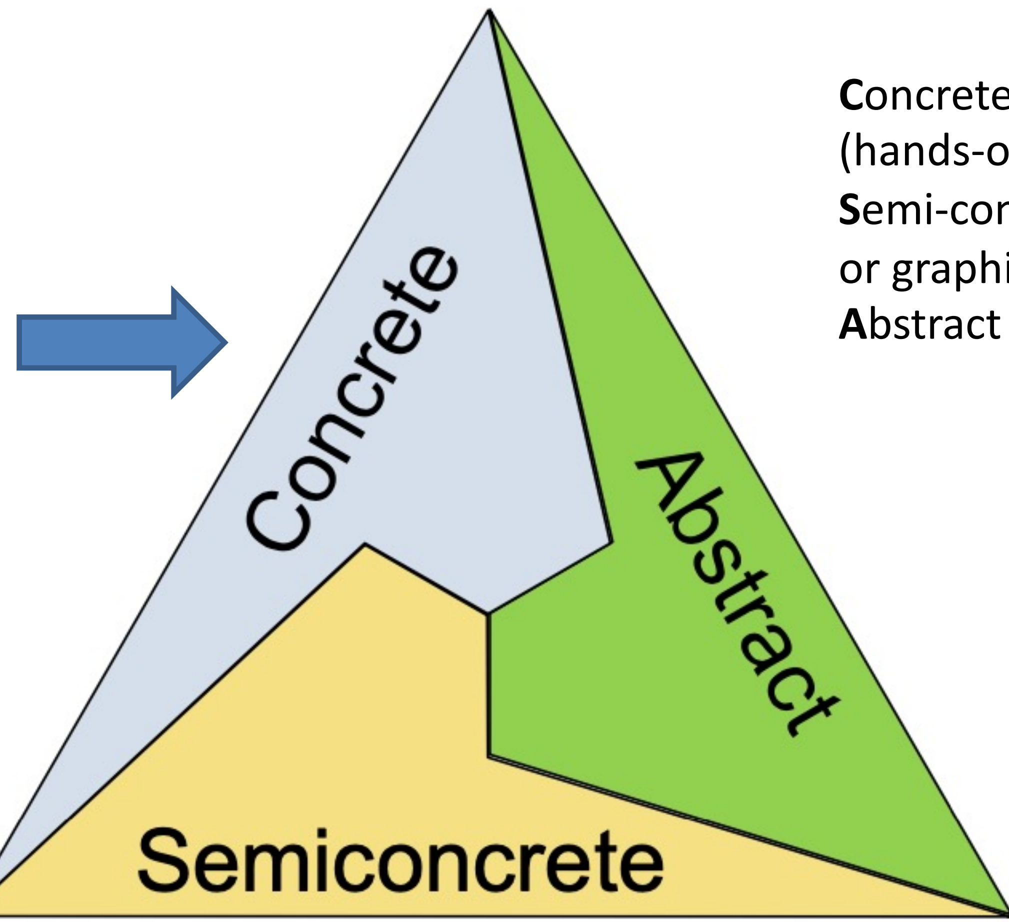
Differentiation versus Intervention

- Differentiation
  - Occurs during Tier 1 instruction
  - Allows for different abilities in the same instructional session with grade-level content – IN the curriculum or specific content within an intervention
  - Offers opportunities for working on different tasks but on the same content as the rest of the class
- Intervention
  - Occurs during Tiers 2 and 3 instruction
  - Develops specific skills and concepts for a targeted group of students to address areas of identified weakness by building on students' strengths
  - Provides intentional foundational support that is most often not at grade level

Structuring Information



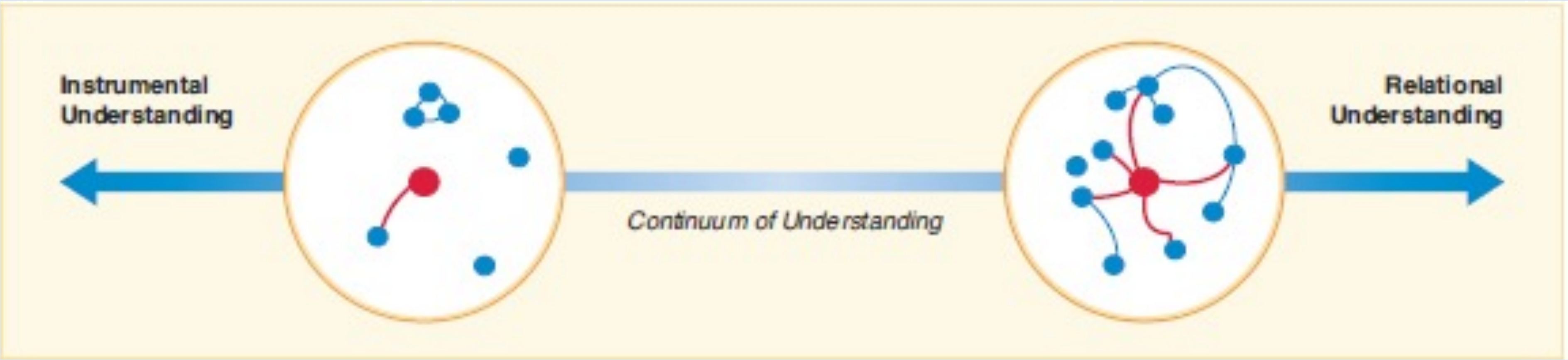
CSA: Concrete—Semi-Concrete—Abstract



Concrete or physical (hands-on)
Semi-concrete (pictorial or graphical)
Abstract or symbolic

Karp, K. S., Dougherty, B. J., & Bush, S. B. (2021). *The math pact: Achieving instructional cohesion within and across grades - Elementary.*

Compared to students who are not struggling, their brains might look very different!



Skemp, R. (1987). *The psychology of learning mathematics,* The Penguin Press.
Van de Walle, J., Karp, K., & Bay Williams, J. (2010). *Elementary and Middle School Mathematics: Teaching developmentally*. New York: Pearson.

What do Blue Lines Look Like?

$$\frac{1}{2} \div \frac{1}{4} =$$

How were you taught to solve this when you were in school?

Students with Difficulties in Mathematics

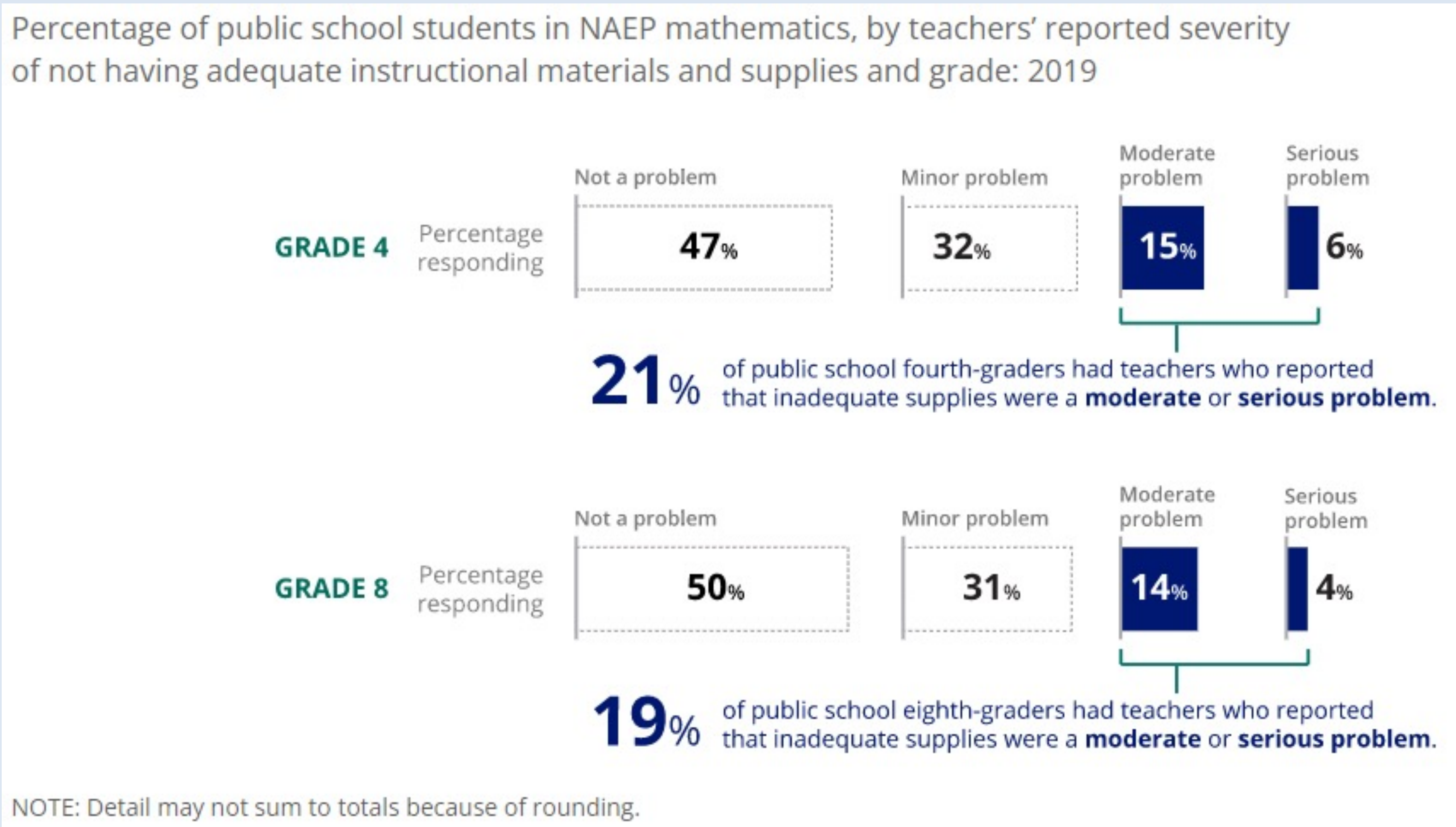
- Students who struggle with mathematics often
  - use procedures that younger, typically achieving students use;
  - make frequent errors when executing procedures; and
  - have a poor understanding of concepts that are foundational to performing procedures (Geary, 2004)
- Additionally, they are often
  - dependent upon the teacher
  - over generalizing
  - quick to give up
  - frustrated by ‘word’ problems (Dougherty & Foegen, 2011)

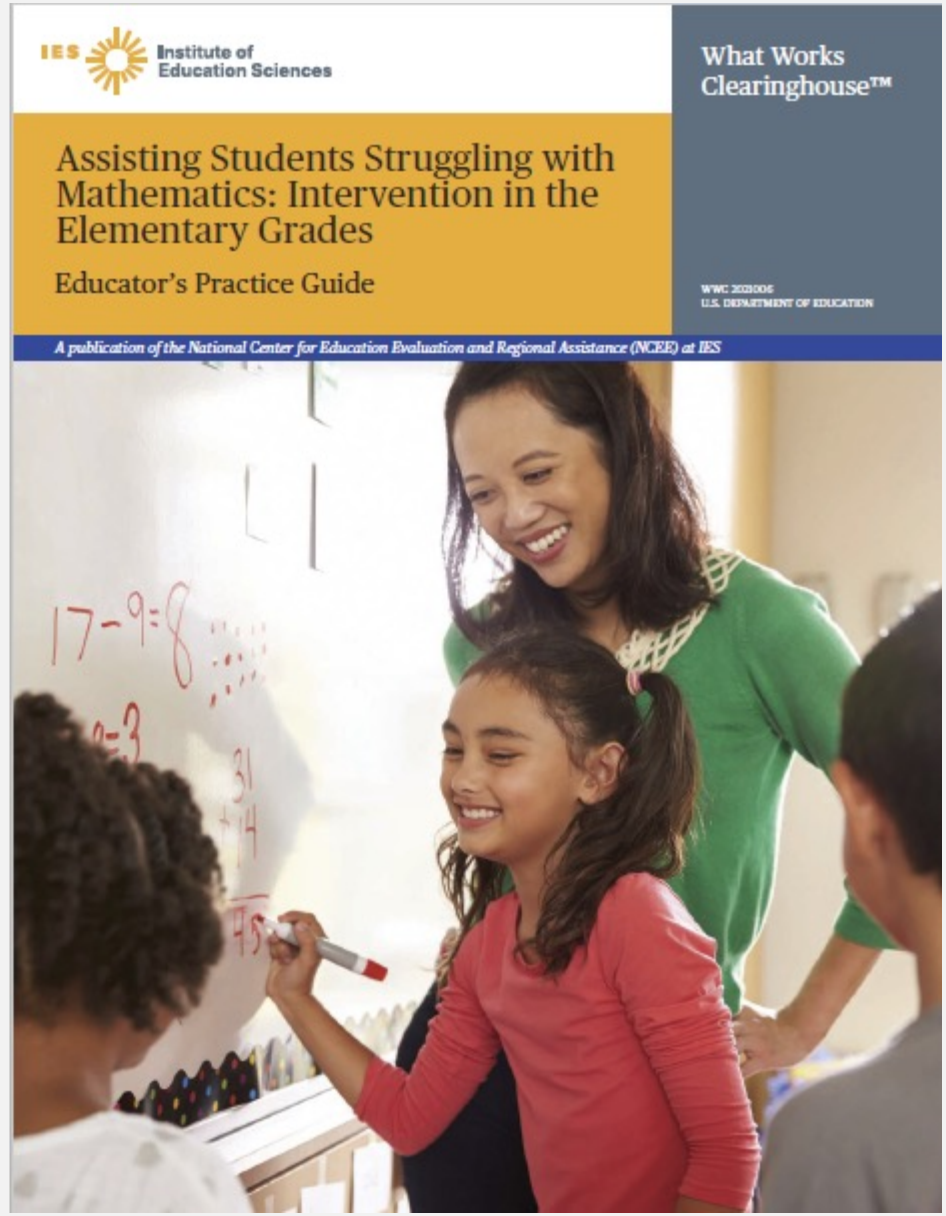
Intervention Materials

Two studies revealed that teachers providing Tier 2 mathematics instruction to elementary and middle grades students largely used worksheets (Foegen & Dougherty, 2011; Swanson, Solis, Ciullo & McKenna, 2012)

One-size-fits-all computer programs

Adequate Instructional Materials a Problem?

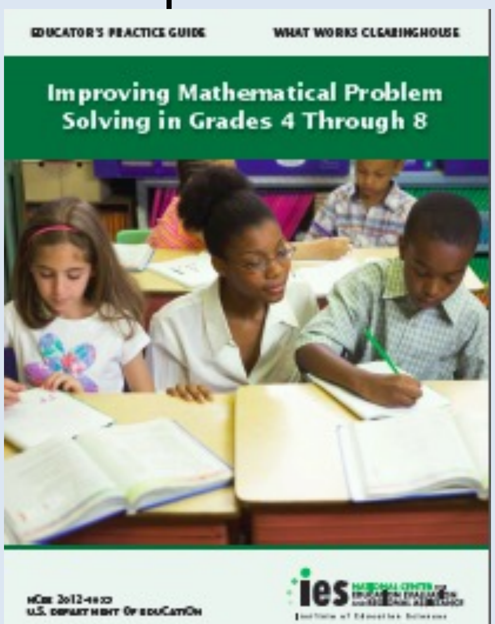


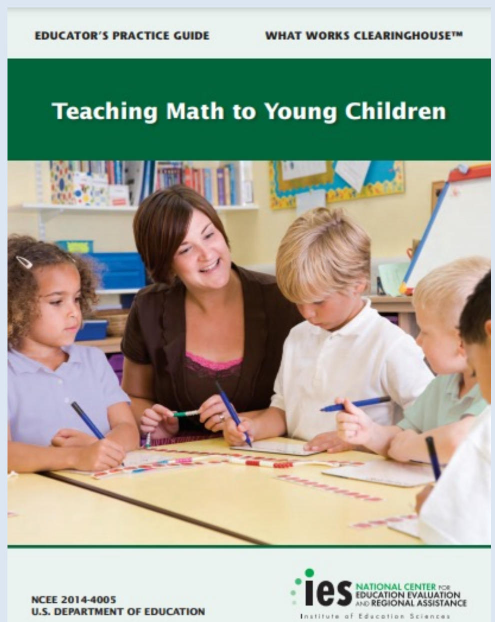


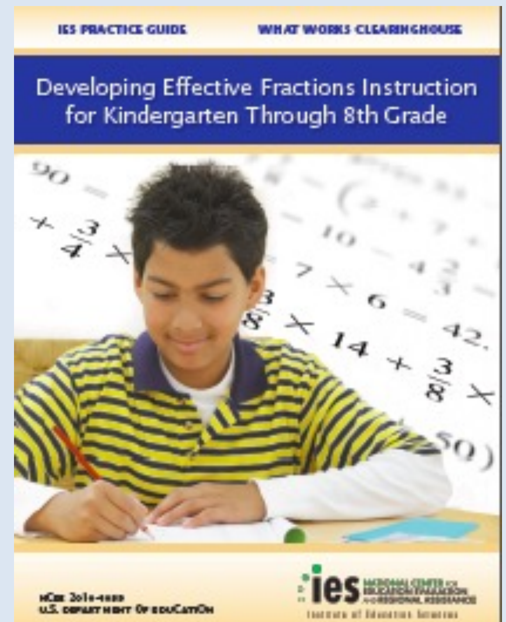
Fuchs, L.S., Newman-Gonchar, R., Schumacher, R., **Dougherty, B**., Bucka, N., **Karp, K.S**., Woodward, J., Clarke, B., Jordan, N. C., Gersten, R., Jayanthi, M., Keating, B., and Morgan, S. (2021). ***Assisting Students Struggling with Mathematics: Intervention in the Elementary Grades*** (WWC 2021006). Washington, DC: National Center for Education Evaluation and Regional Assistance (NCEE), Institute of Education Sciences, U.S. Department of Education. Retrieved from http://whatworks.ed.gov/

Recommendations for identifying and supporting students struggling in mathematics

- Grouped by strong, moderate, and minimal levels of evidence resulting from comprehensive reviews of current research







Practice Guide Recommendations

Practice Recommendation	Level of Evidence: Minimal	Moderate	Strong
1. Systematic Instruction: Provide systematic instruction during intervention to develop student understanding of mathematical ideas.			✓
2. Mathematical Language: Teach clear and concise mathematical language and support students' use of the language to help students effectively communicate their understanding of mathematical concepts.			✓
3. Representations: Use a well-chosen set of concrete and semi-concrete representations to support students' learning of mathematical concepts and procedures.			✓
4. Number Lines: Use the number line to facilitate the learning of mathematical concepts and procedures, build understanding of grade-level material, and prepare students for advanced mathematics.			✓
5. Word Problems: Provide deliberate instruction on word problems to deepen students' mathematical understanding and support their capacity to apply mathematical ideas.			✓
6. Timed Activities: Regularly include timed activities as one way to build fluency in mathematics.			✓

Mathematical Practices

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make sense of structure.
8. Look for and express regularity in repeated reasoning.

NCTM's *Principles to Actions* - Mathematics Teaching Practices for Effective Teaching

1. **Establish mathematics goals to focus learning**.
2. **Implement tasks that promote reasoning and problem solving**.
3. **Use and connect mathematical representations**.
4. **Facilitate meaningful mathematical discourse**.
5. **Pose purposeful questions**.
6. **Build procedural fluency from conceptual understanding.**
7. **Support productive struggle in learning mathematics**.
8. **Elicit and use evidence of student thinking**.

Important Mathematical Understandings

Making Cents

- Take out some coins.
- Multiply the value of the coins in cents by 4.
- Add 10 to the product.
- Multiply your answer by 25.
- Add 115 to your answer.
- Add your age in years.
- Subtract the number of days in a normal year.

Making Cents

- What do you notice about your answers?
- How can you describe what you notice to someone who is not present?

Types of Understandings

Procedural - Student can perform a computation or algorithm by following a series of prescribed steps

Conceptual - Student understands the basis of why a computation or algorithm works. They can apply it later without reteaching. Student can identify, describe, and explain a big idea related to a topic or a class of problems

Problem solving - Student can solve a problem when there is no specific solution pathway or algorithm

Helping Students Find Patterns

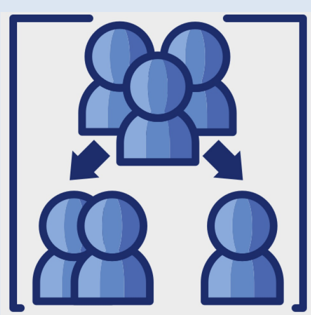
- Questioning techniques
  - Bring attention to mathematical structure
  - Consistent use of ’what patterns do you notice?’
- Problem structures
  - Make the mathematics transparent
  - Use repetition to support identifying patterns (not for practice)

What should students learn first: concepts or skills?

You decide

Rules

- In your breakout room, decide if the rules shown are always true.
- If it is not always true, find a counterexample.



- Addition and multiplication make larger.
- When you multiply by 10, place a 0 at the end of the number.
- The longer the number, the larger the number.
- Use Please Excuse My Dear Aunt Sally.

Addition and multiplication make larger.

32 + 67 = 99

– 3 + (–14) = –17

–17 is not larger than
–3 or –14.

16 + 0 =

15 x 10 = 150

$$\frac{1}{3} \times \frac{2}{7} = \frac{2}{21}$$

0.25 x .16 = 0.04

Neither rational
number product is
larger than the
factors.

When you multiply by 10, place a 0 at the end of the number.

$15 \times 10 = 150$

$4.5 \times 10 = 45.0$

$4.5 \times 10 \neq 4.50$

The longer the number, the larger the number.

$1{,}278{,}931 > 1{,}469$

$1.3 > 1.0118743$

$1.02 < 1.2$

For the Unwavering Fans of PEMDAS

- The “P” in PEMDAS suggests parentheses are first, but there are other grouping symbols such as brackets, braces, square root, absolute value, etc. The order of the MDAS is VERY CONFUSING!!!!!
- Use - GEMA or GEMS
- Grouping, Exponents, Multiplicative, Additive

or

Grouping, Exponents, Multiplication/division, Subtraction/addition

Rules Stick

Mari said, “2*t* is always greater than *t* + 2.” Do you agree with Mari?

A. Yes, because multiplication always gives you a larger answer than addition.

B. Yes, because *t* is a positive number.

C. No, because multiplication is not the inverse of addition.

D. No, because it is possible that 2*t* can be equal to or less than *t* + 2.

Dougherty, B J. & Foegen, A. (2016). *Conceptual progress monitoring for algebra.* Funded by IES, Goal 5, R324A110262

Student Responses

Mari said, "2*t* is always greater than *t* + 2." Do you agree with Mari?

N = 750 students, no IEPs or 504, 2nd semester Algebra I

A. Yes, because multiplication always gives you a larger answer than addition. (41.6%)
B. Yes, because *t* is a positive number. (8.5%)
C. No, because multiplication is not the inverse of addition. (14.3%)
D. No, because it is possible that 2*t* can be equal to or less than *t* + 2. (35.5%)

Dougherty, B J. & Foegen, A. (2016). *Conceptual progress monitoring for algebra.* Funded by IES, Goal 5, R324A110262

Impact of Teaching Rules that Expire

- Students use rules as they have interpreted them.
- They often do not think about the rule beyond its immediate application.
- When even the strongest math students find that a “rule” doesn’t work, it is unnerving and scary.

Do the best you can
until you know better.
Then when you know better,
do better.

-Maya Angelou

Goal – Try to AVOID DEAD ENDS

- "13 Rules that Expire" (Karp, Bush & Dougherty August 2014 in *Teaching Children Mathematics*) (blog too!)

Supporting robust learning means that the focus is on generalizations, not on rules that expire.

Take the Oath!!

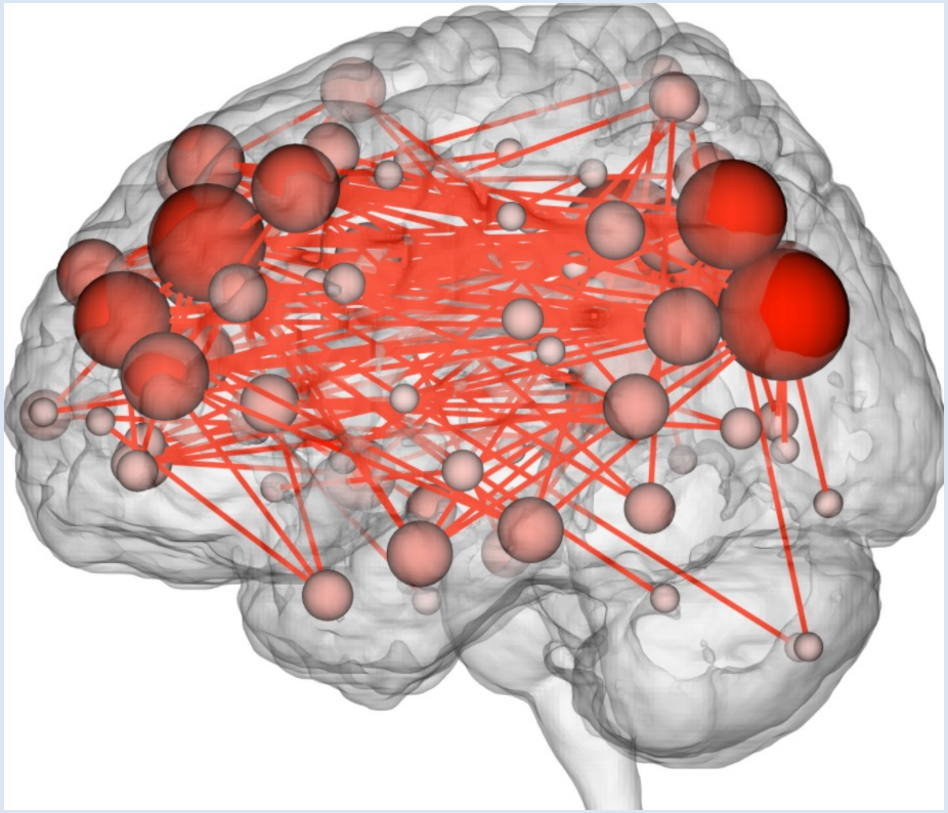
- Borrowing
- Carrying
- “Reducing” fractions
- Talking about Fractions as a “Top Number” and a “Bottom Number”
- Getting “rid” of the decimal
- 2 + 2 “makes” 4

Connections

- Student learning is supported by the connections they make among:
  - Representations
  - Number systems
  - Reasoning and intuition
  - Algorithms and the meanings associated with the operation

Create Mental Residues

- Concurrent representations make connections
  - Physical actions
  - Drawings or diagrammatic representations
  - Symbolic representations



Create Mental Residues

- Establishes foundational understanding
- Models the physical action is the important
- Does not fade away or disappear
- Supports their thinking about the operation

Dougherty, B. J. (2008). Measure up: A quantitative view of early algebra. In Kaput, J. J., Carraher, D. W., & Blanton, M. L. (Eds.), *Algebra in the early grades*, (pp. 389–412). Mahweh, NJ: Erlbaum.

Focus on Skills

$$\frac{1}{12} + \frac{7}{8}$$

Focus on Skill

$$\frac{1}{12} + \frac{7}{8} = \frac{23}{24}$$

Focus on Sense Making

The sum of $\frac{1}{12}$ and $\frac{7}{8}$ is closest to

A. 20

B. 8

$C.$ $\frac{1}{2}$

D. 1

The Two Worlds Collide: Sense Making Meets Skill

The sum of $\frac{1}{12}$ and $\frac{7}{8}$ is closest to

A. 20

B. 8

C. $\frac{1}{2}$

D. 1

Explain your answer.

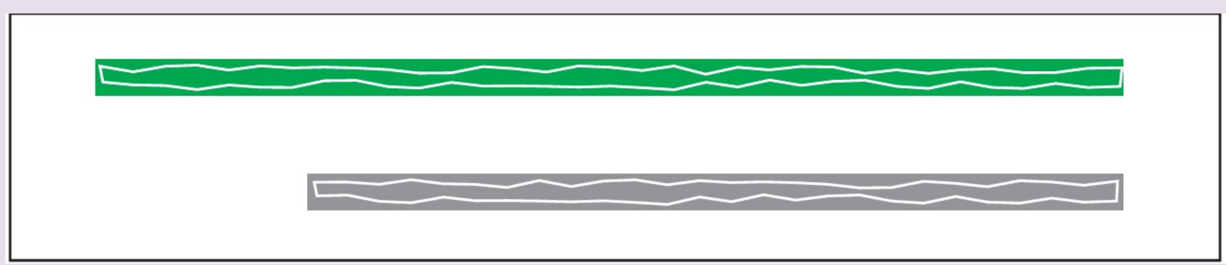
$\frac{1}{12} + \frac{7}{8} = \frac{2}{24} + \frac{21}{24} = \frac{23}{24}$ is closest to 20.

Petit, Laird,& Marsden, 2010

Number Understandings

Number development begins without numbers!

Quantities can be compared without assigning numerical values to them.

Sammi-Jo said, “I counted 4 things.” “No,” said Henna. “There are 8 things.”

Is it possible that Sammi-Jo and Henna counted the same group of objects? Why or why not?

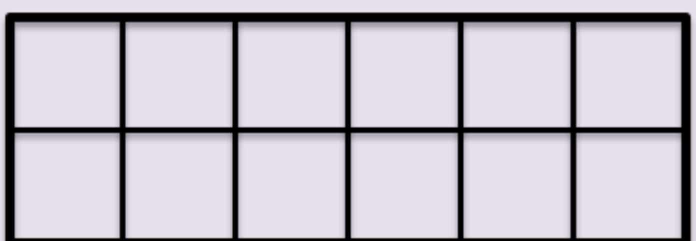
Dougherty, B. J., Flores, A., Sophian, C., & Everett, L. (2010). *Essential understandings: Engaging teachers with mathematics, PreK–2 number concepts*. Reston, VA: National Council of Teachers of Mathematics.

Counting Task

1. How many ways can you find to count the total in the rectangle below?

2. What would the units look like?

3. How would the units compare?



Dougherty, Yagi, Zenigami, & Okazaki (2017). *Measure Up*, grade 1. University of Hawaii.

Units are important!

- Instruction often does not explicitly provide opportunities to discuss the importance of unit.

- Unit is a fundamental concept in mathematics.

What do we assume?

$3 < 8$

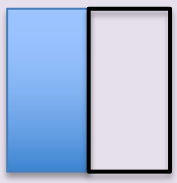
What first graders had to say. . .

$3 < 8$

Richard: You can’t really tell. ‘Cause you could have 3 really really really big units and 8 really really really small units. Then 8 would be less than 3.

Reed: Yeah, but if it’s on a number line then it’s true ‘cause all the units are the same size.

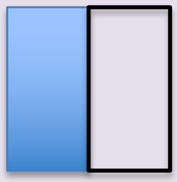
Why it Matters

$\frac{1}{2} \longrightarrow$ 

Does $\frac{1}{2} = \frac{2}{4}$?

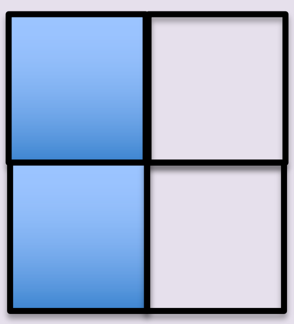
Why it Matters

$\frac{1}{2}$ $\longrightarrow$



Does $\frac{1}{2} = \frac{2}{4}$?

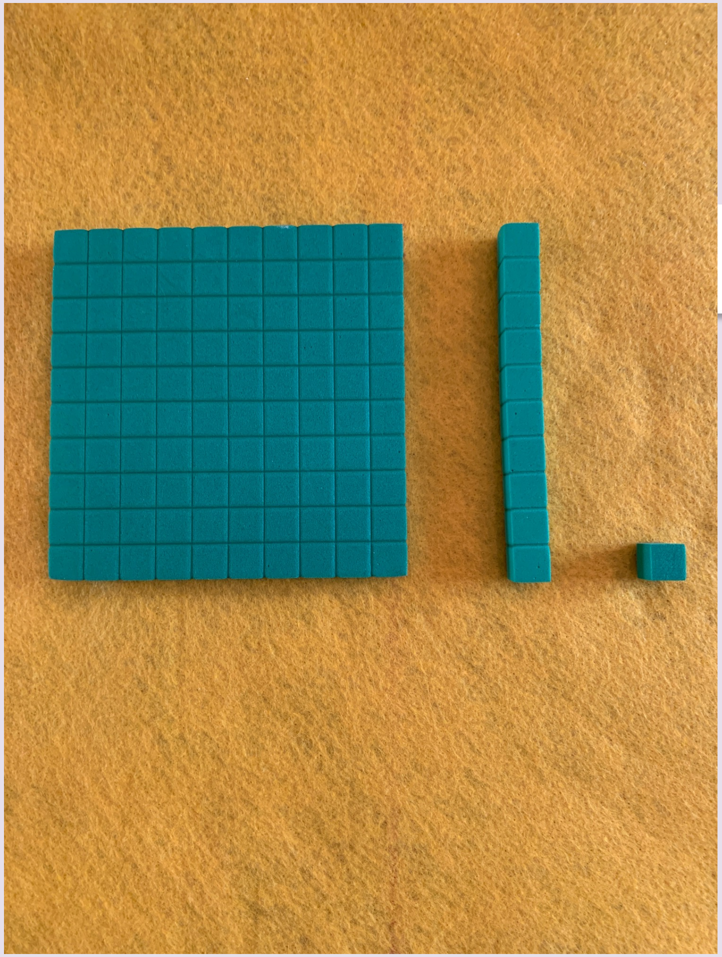
$\frac{2}{4}$ $\longrightarrow$



Whole Numbers

If you wanted to use base ten blocks, how could you model 125?

Work with a partner to think about how many different ways you could model 125.



Whole Numbers

- Decomposition
  - Writing equations to symbolize the representations

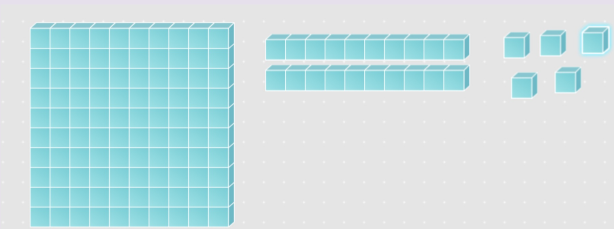
$125 = 100 + 10 + 10 + 5$

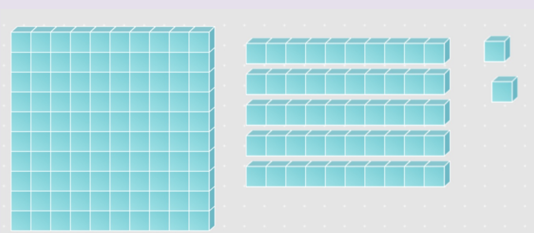
$125 = 10 + 10 + 10 + 10 + 10 + 10 + 10 + 10 + 10 + 10 + 10 + 10 + 1 + 1 + 1 + 1 + 1$

$125 = 60 + 60 + 5$

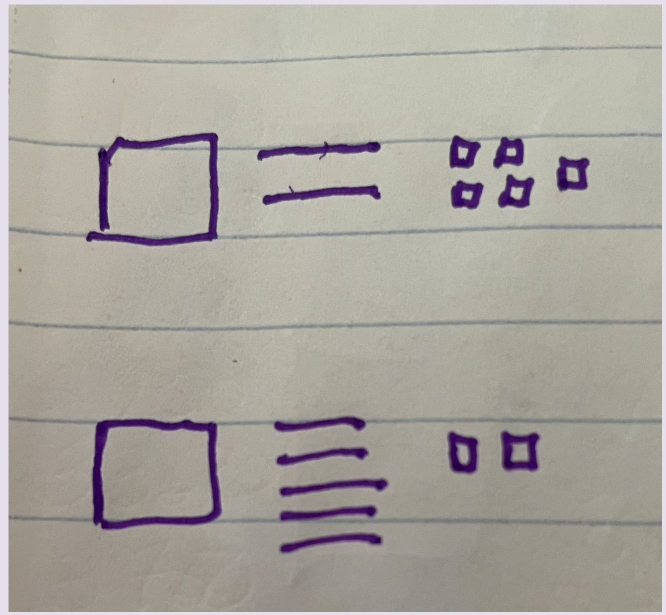
Moving to Comparison: 125 and 152

Concrete





Semiconcrete



Abstract

125 = 100 + 20 + 5

152 = 100 + 50 + 2

Comparing Whole Numbers

- What can you say about the relationship between 125 and 152?

- What statements could you write or say?

Comparing Whole Numbers

- What can you say about the relationship between 125 and 152?
- What statements could you write or say?

$125 \neq 152$

$152 \neq 125$

$125 < 152$

$152 > 125$

Comparing Whole Numbers

$125 \neq 152$

$152 \neq 125$

$125 < 152$

$152 > 125$

$125 < 152$ by 27

$152 > 125$ by 27

Comparing Whole Numbers

$125 \neq 152$

$152 \neq 125$

$125 < 152$

$152 > 125$

$125 < 152$ by 27

$152 > 125$ by 27

$152 = 125 + 27$

$152 = 27 + 125$

$152 - 27 = 125$

$152 - 125 = 27$

Comparing Rational Numbers

$\frac{3}{4} < \frac{7}{8}$ by $\frac{1}{8}$ $\longrightarrow$

$$\frac{3}{4} + \frac{1}{8} = \frac{7}{8}$$

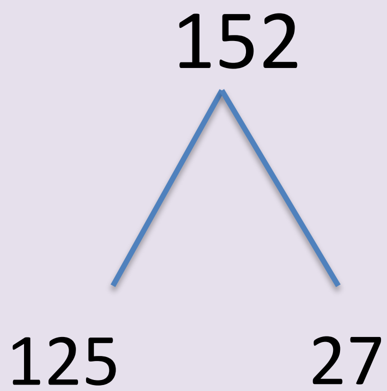
$$\frac{1}{8} + \frac{3}{4} = \frac{7}{8}$$

$$\frac{1}{8} = \frac{7}{8} - \frac{3}{4}$$

$$\frac{3}{4} = \frac{7}{8} - \frac{1}{8}$$

Part-Whole Diagrams

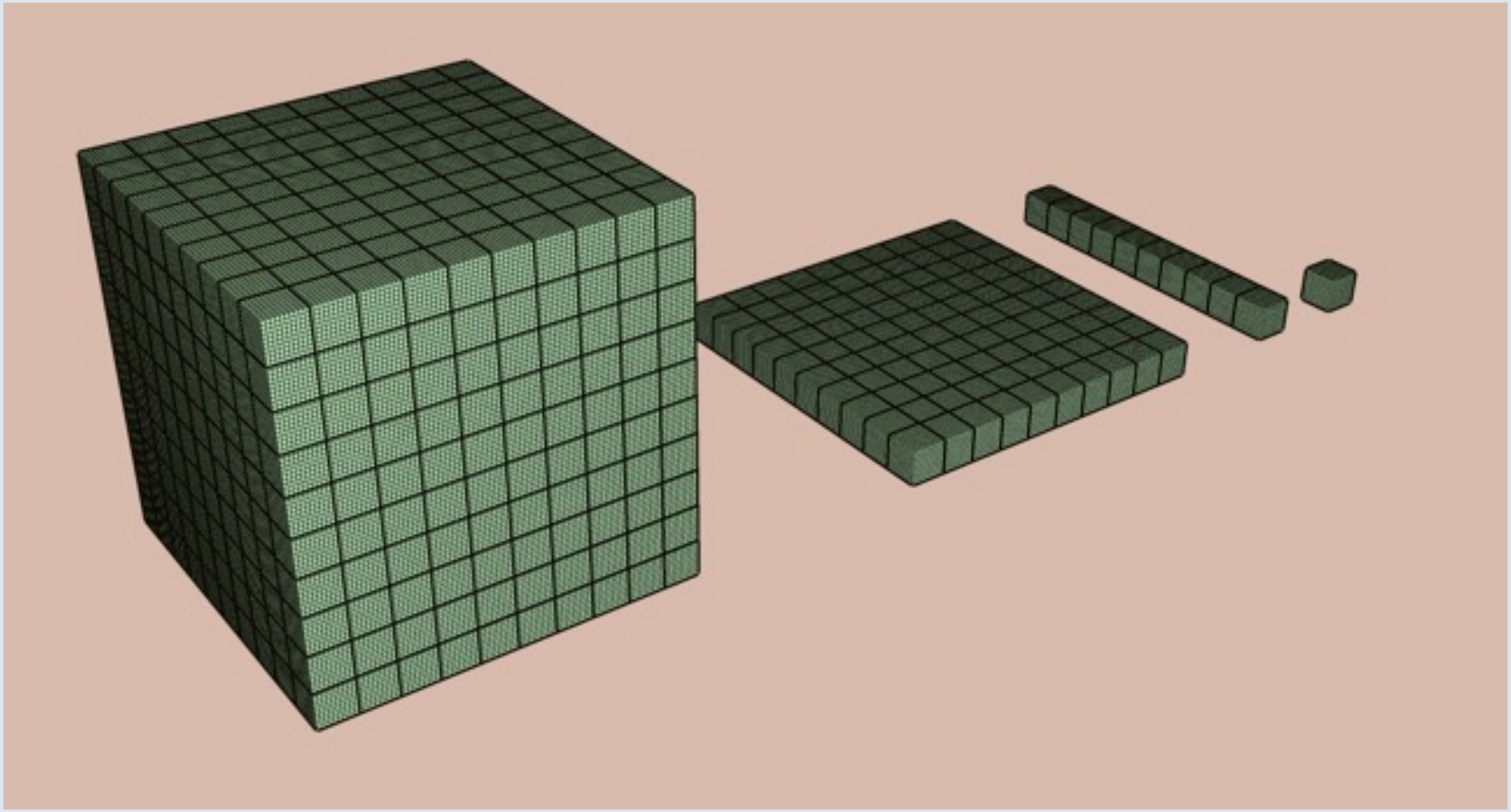
125 < 152 by 27



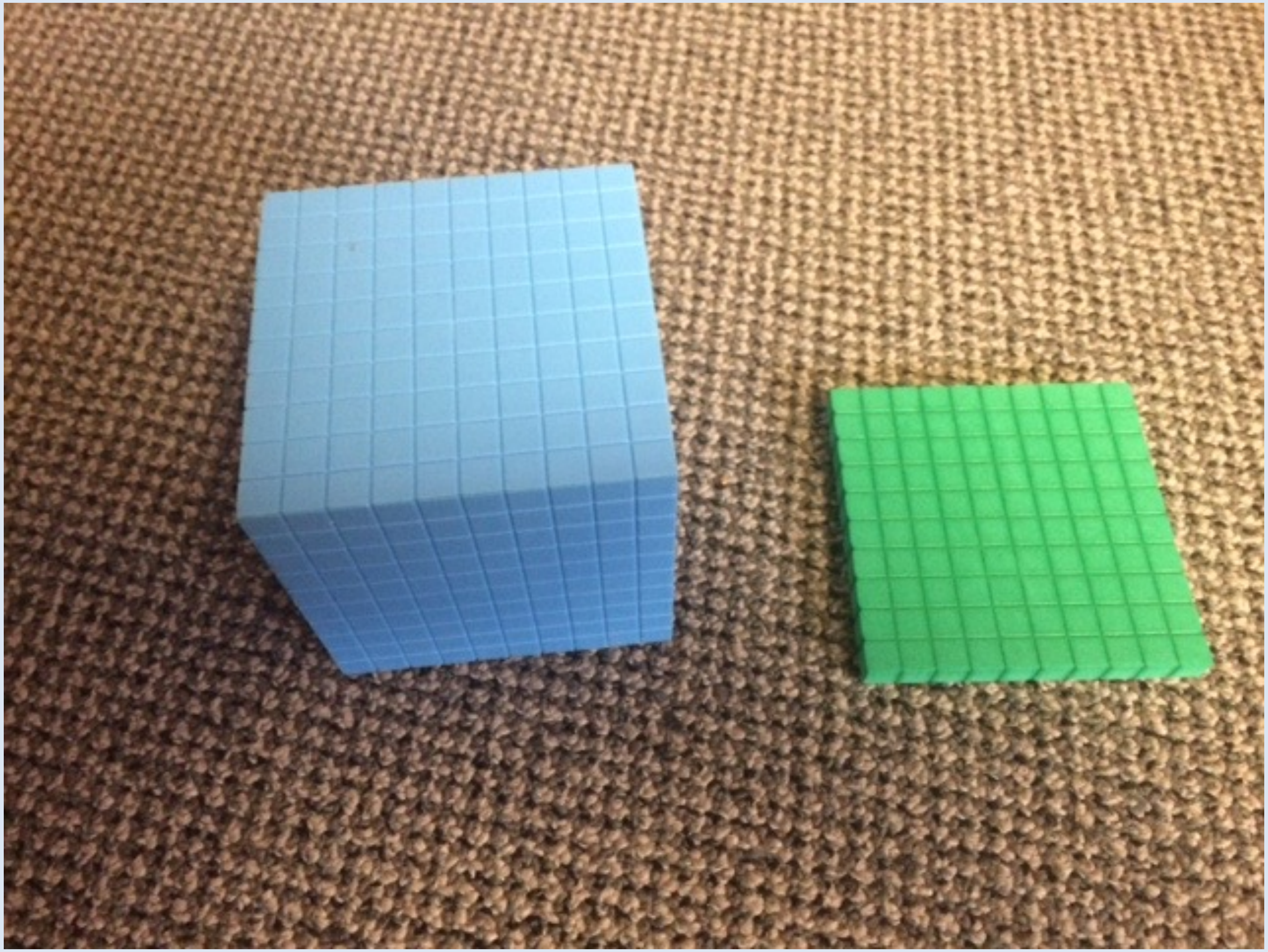
Two Types of Quantities

- Discrete: objects
- Continuous: length, area, volume, and mass
  - Base ten blocks: volume
  - Number line: length

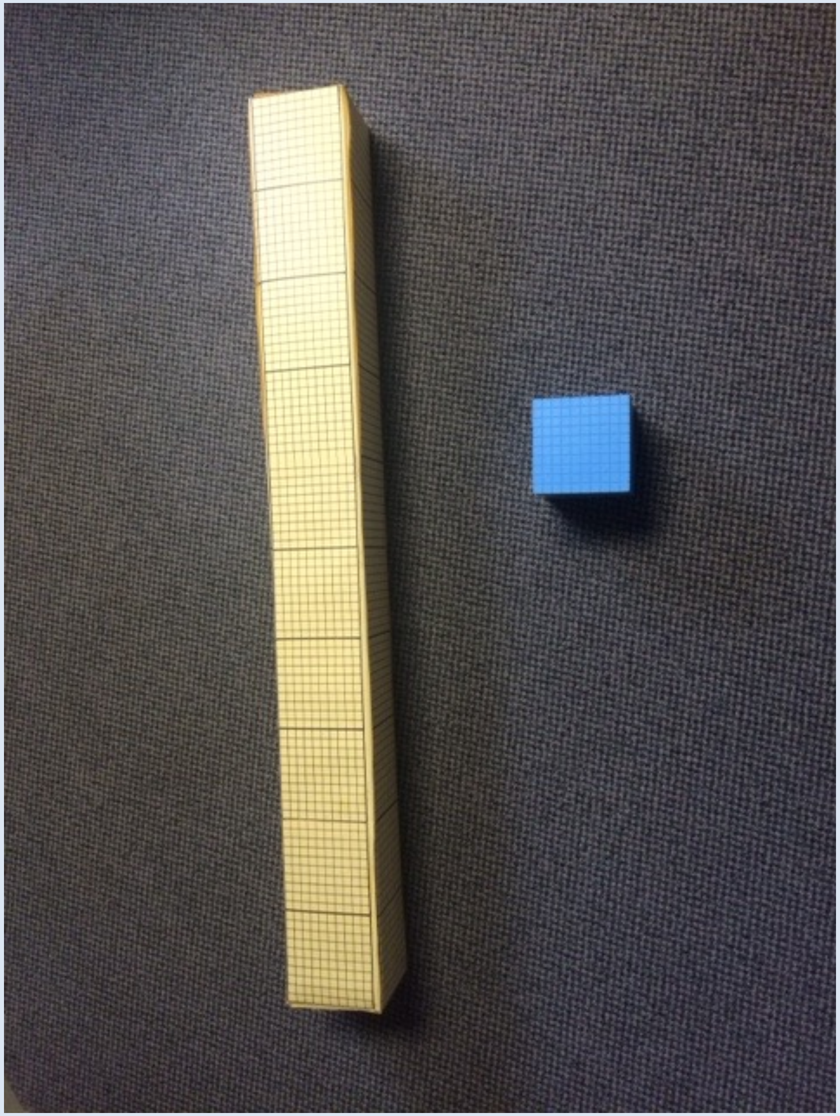
What Patterns do you Notice?



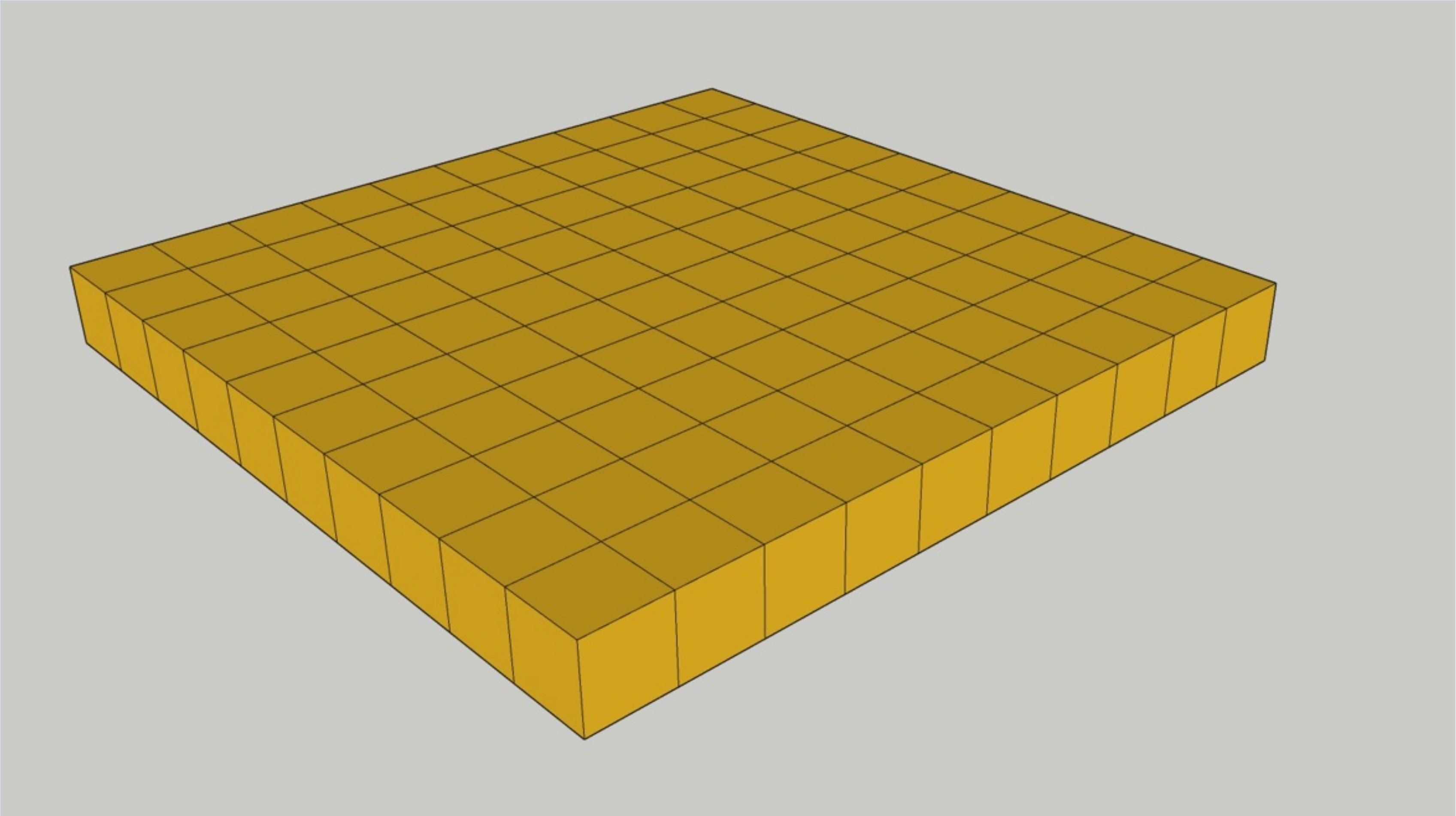
Don't Stop there if you Expect Understanding



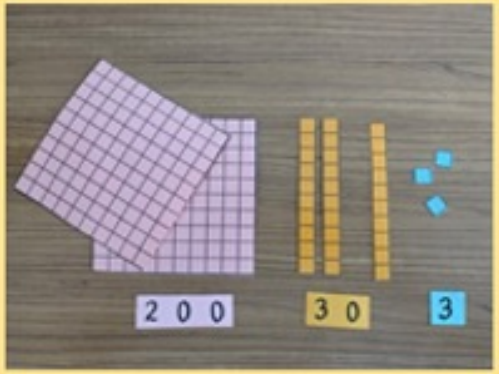
Keep Going



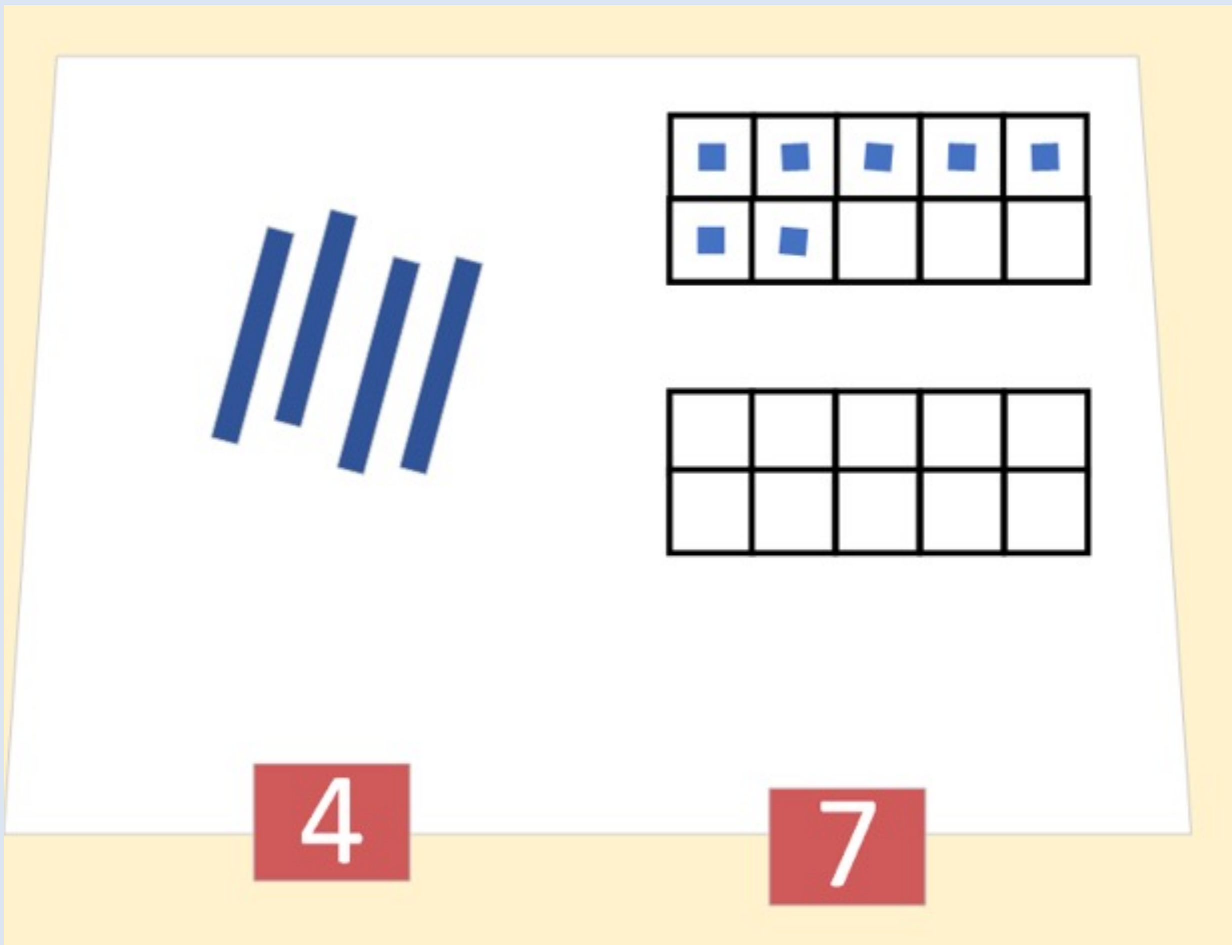
And Going



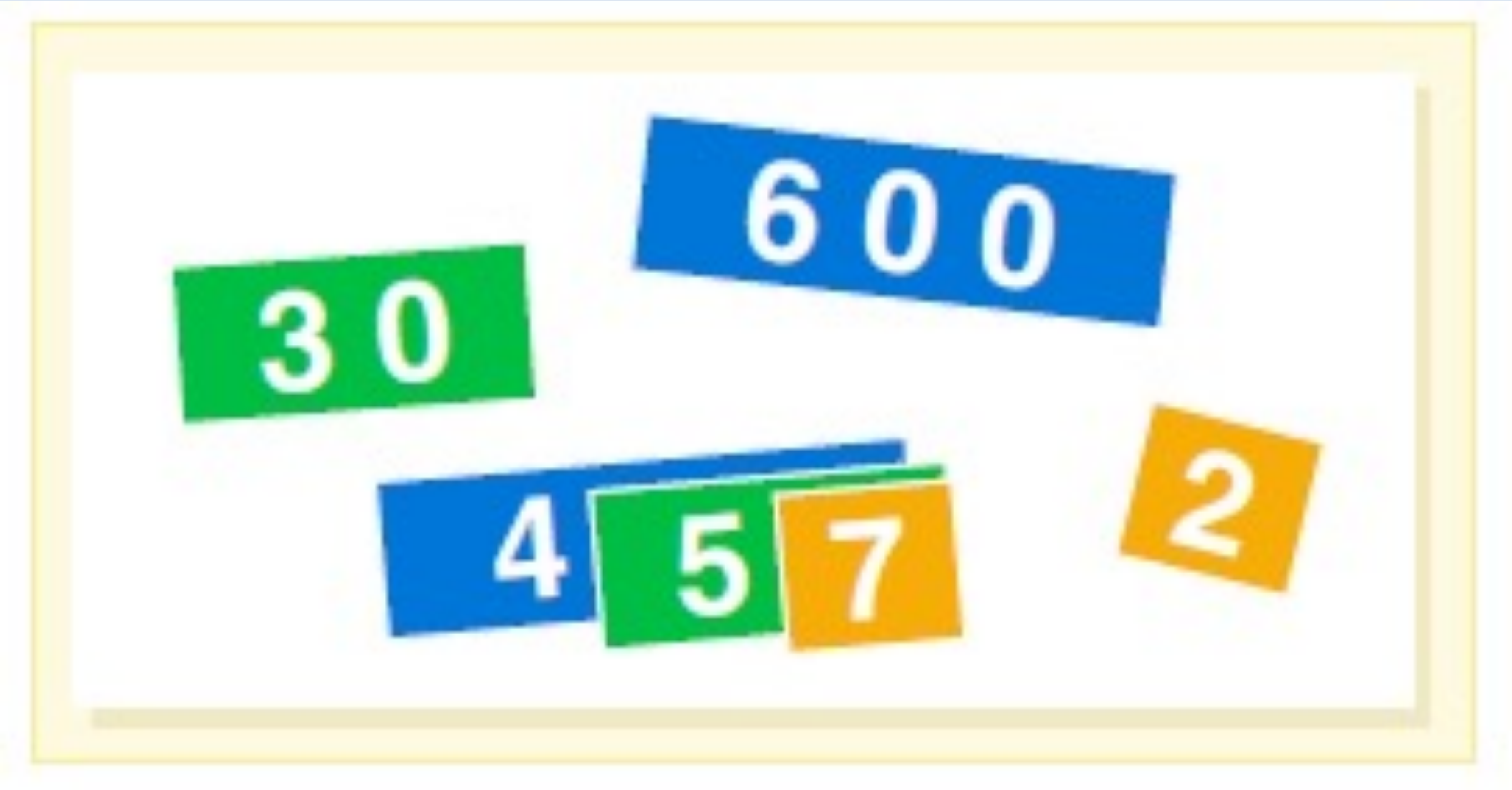
Representations to Avoid – What's better?



Place Value Mat with Double Ten Frame



Place Value Cards



Make Blue Lines
Then Students Say – That Looks Familiar!

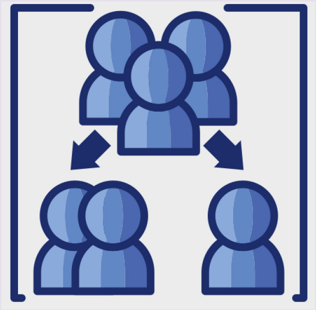
Number Lines

Number lines

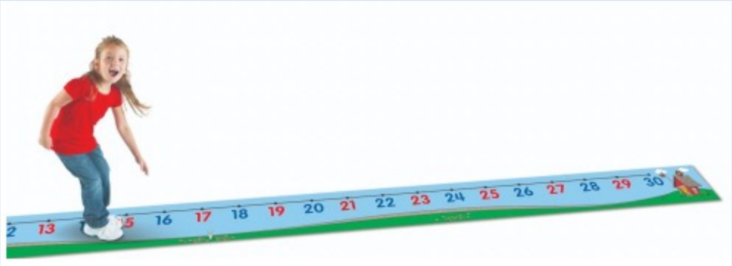
- Are a continuous model, length, extending to the left and right of 0
- Are constructed with consistent units that are placed end-to-end with no gaps and no overlaps
- Allow number representations, comparisons, and computations

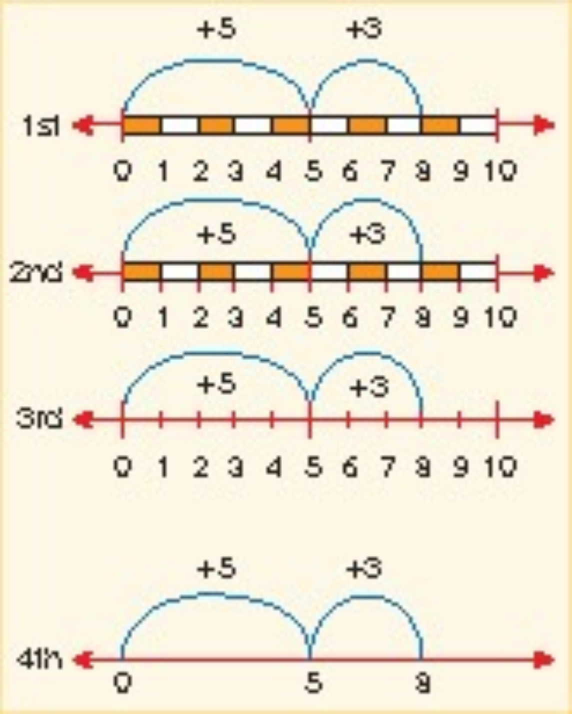
Number Line Development

- Using length units, lay them end-to-end with no gaps and no overlaps
- Construct a length
- Label 0, 1, and so on



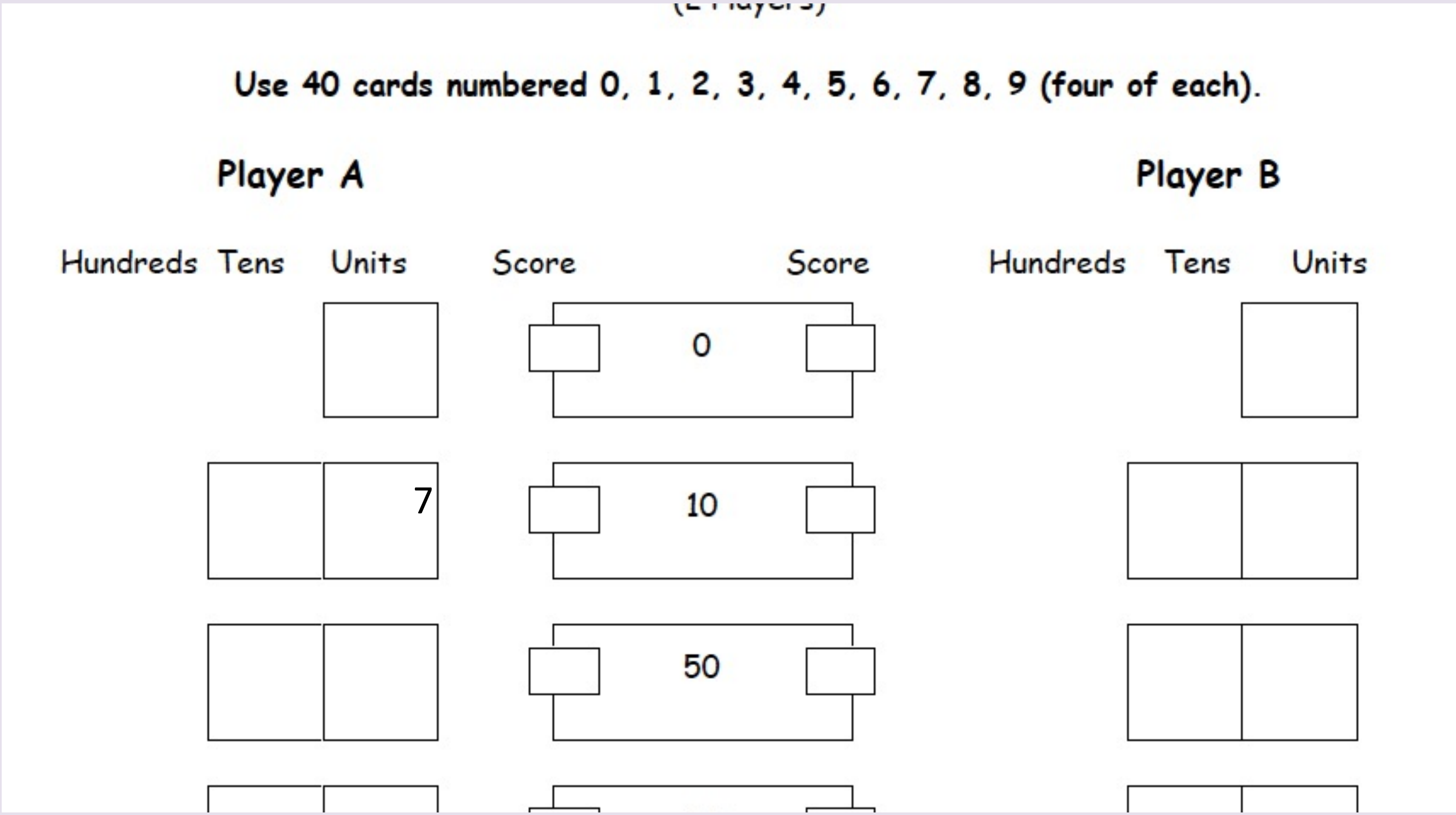
Number Lines
Multiple Representations





Find A Place

Find-A-Place


(2 Players)
Use 40 cards numbered 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 (four of each).
Player A
Player B
Hundreds Tens Units
Score
Score
Hundreds Tens Units
0
7
10
50

Find-A-Place

FIND A PLACE

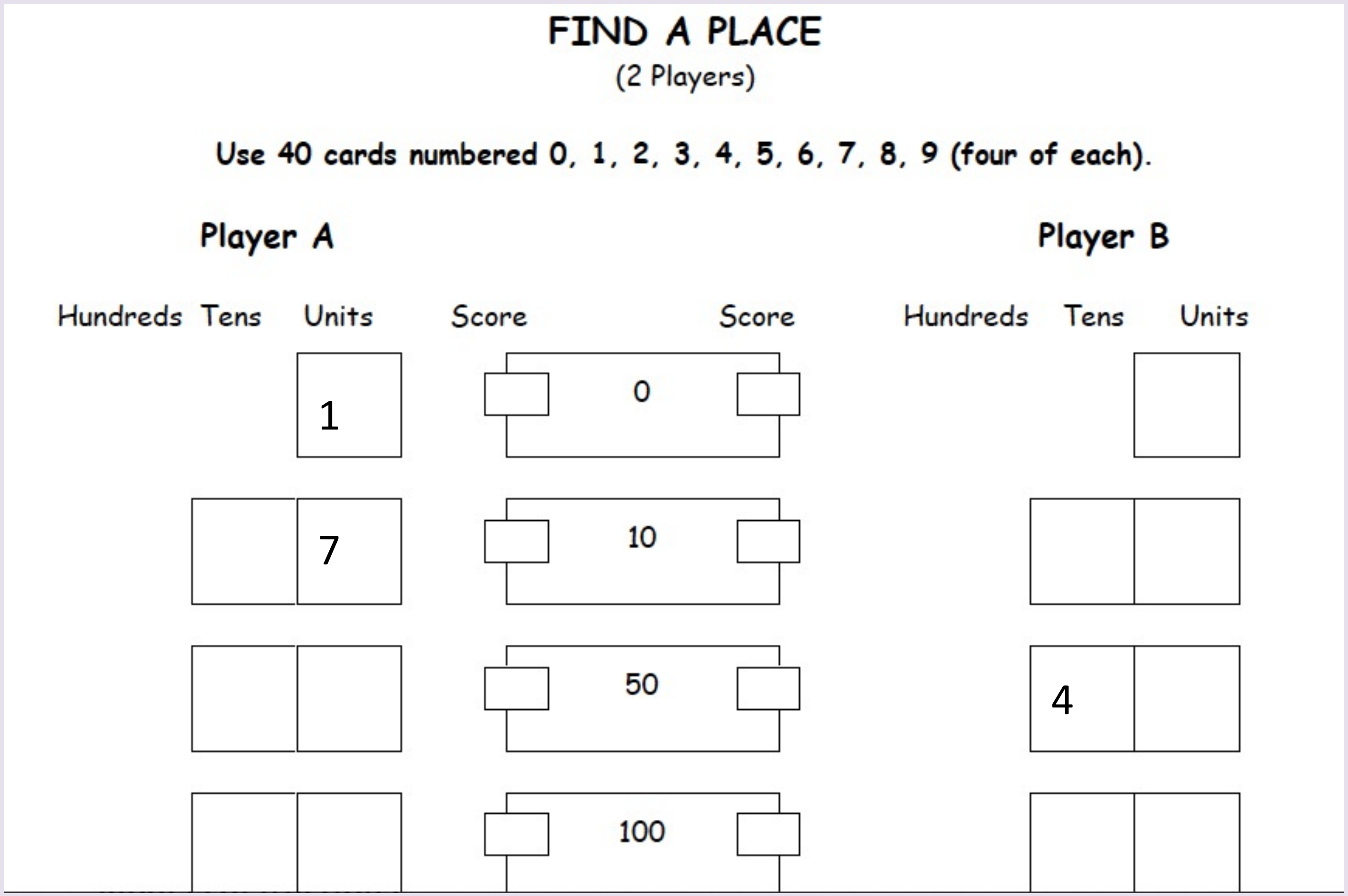
(2 Players)

Use 40 cards numbered 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 (four of each).

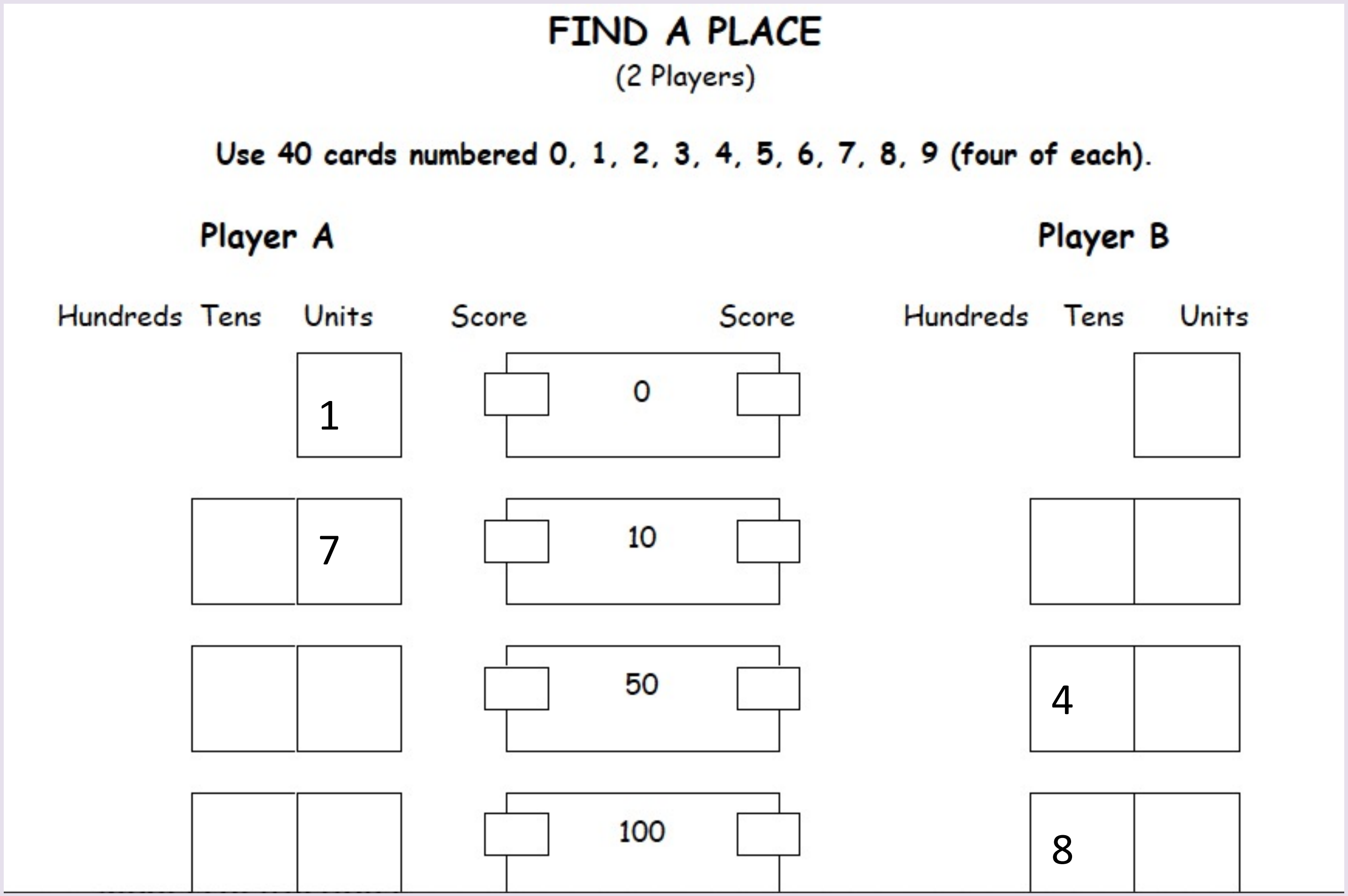
Player A | **Player B**

Hundreds	Tens	Units	Score		Score	Hundreds	Tens	Units
				0				
		7		10				
				50			4	
				100				

Find-A-Place


FIND A PLACE
(2 Players)
Use 40 cards numbered 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 (four of each).
Player A
Player B
Hundreds Tens Units
Score
Score
Hundreds Tens Units
1
0
7
10
50
4
100

Find-A-Place


FIND A PLACE
(2 Players)
Use 40 cards numbered 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 (four of each).
Player A
Player B
Hundreds Tens Units
Score
Score
Hundreds Tens Units
1
0
7
10
50
4
100
8

Find-A-Place

FIND A PLACE

(2 Players)

Use 40 cards numbered 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 (four of each).

Player A						Player B		
Hundreds	Tens	Units	Score		Score	Hundreds	Tens	Units
		1		0				0
	1	7		10			0	2
	5	5		50			4	8
	7	8		100			8	1

Word Problems

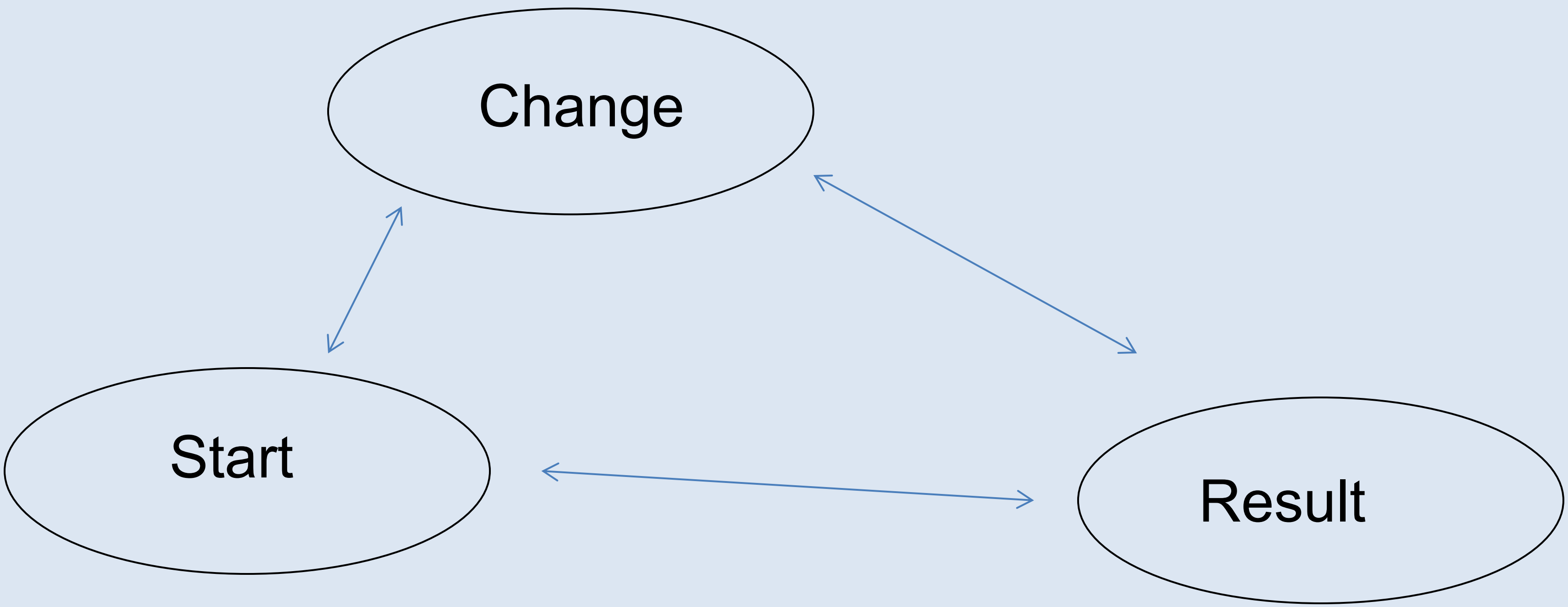
At all grades students who struggle see each word problem as a **separate endeavor**

They **focus on steps** to follow rather than the behavior of the operations

They tend to use **trial and error** – (disconnected thinking – not relational thinking, lack of focus on patterns)

They need to focus on **actions**, **representations** and **general properties of the operations**

Explicit Schema for Additive Structures



Van de Walle, J., Karp, K., & Bay Williams, J. (2016). *Elementary and Middle School Mathematics: Teaching developmentally*. New York: Pearson.
Carpenter, T., Fennema, E., Franke, M. L., Levi, L. & Empson, S.. (1999). *Children's mathematics: Cognitively guided instruction*. Portsmouth, NH: Heinemann.

Distinguish Between the Behavior of the Operations (yes , use the counters!!)

Join Problems: Use two quantities to find the third

Louise has 11 baseball cards. Elliott gave her 6 more. How many baseball cards does Louise have now?

Connect the action to the equation: 11 + 6 = ☐

Louise has 11 baseball cards. Elliott gave her some more. Louise now has 17 cards. How many did Elliott give her?

11 + ☐ = 17 or an equivalent equation 17 – 11 = ☐

Louise has some baseball cards. Elliott gave her 6 more. Now she has 17. How many baseball cards did Louise have to begin with? ☐ + 6 = 17 or 17 – 6 = ☐

Van de Walle, J., Karp, K., & Bay Williams, J. (2016). *Elementary and Middle School Mathematics: Teaching developmentally*. New York: Pearson.
Carpenter, T., Fennema, E., Franke, M. L., Levi, L. & Empson, S.. (1999). *Children's mathematics: Cognitively guided instruction*. Portsmouth, NH: Heinemann.

What's the action in the problem?

Separate Problems: Start is the whole or the largest amount (not the result)

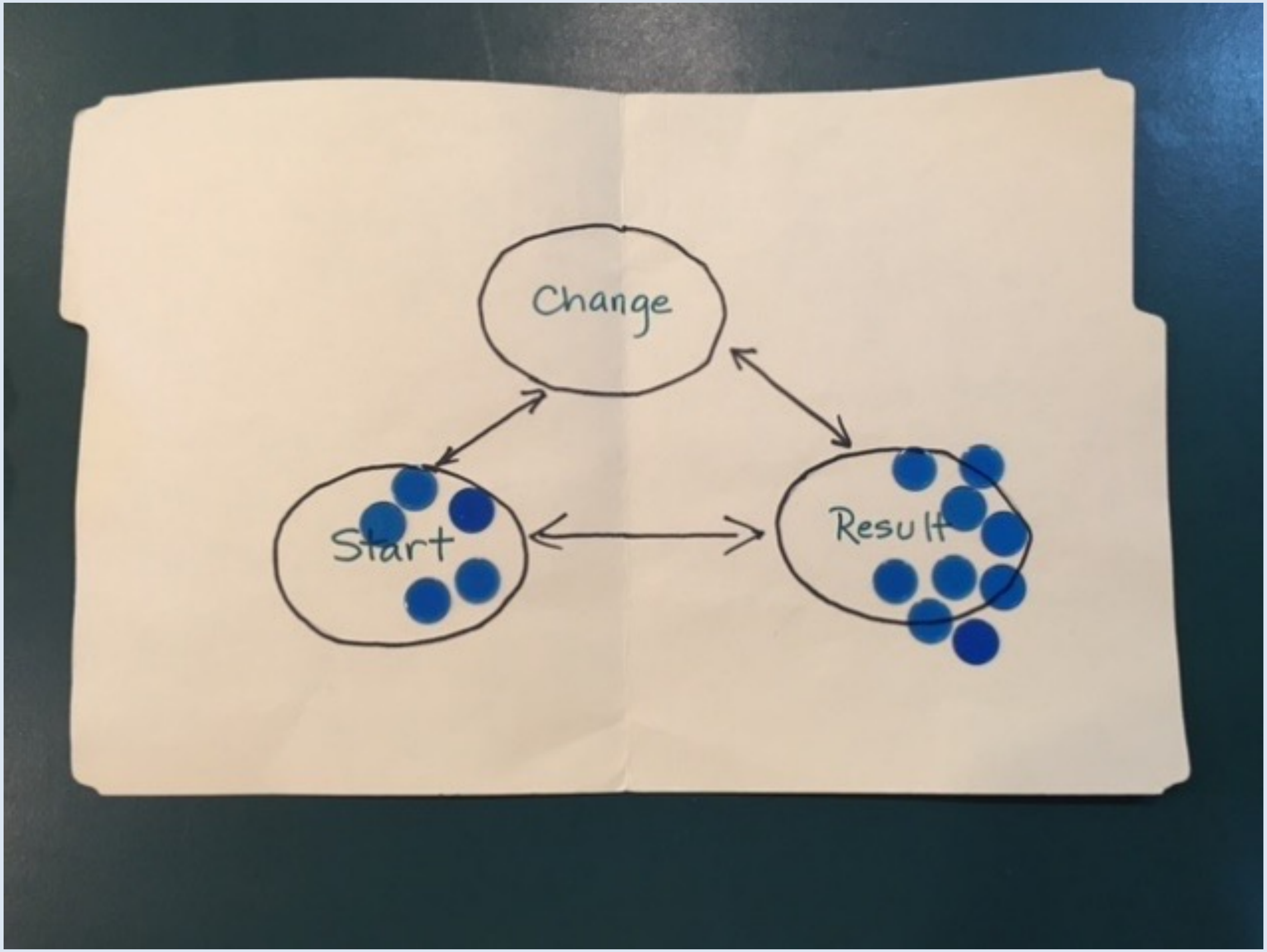
Mikal has 12 T-shirts. He gives 3 shirts to his brother Elron. How many T-shirts does Mikal have now? 12 – 3 = ☐

Mikal has 12 T-shirts. He gives some shirts to his brother Elron. Now he has 9 left. How many T-shirts did Mikal give Elron? 12 – ☐ = 9

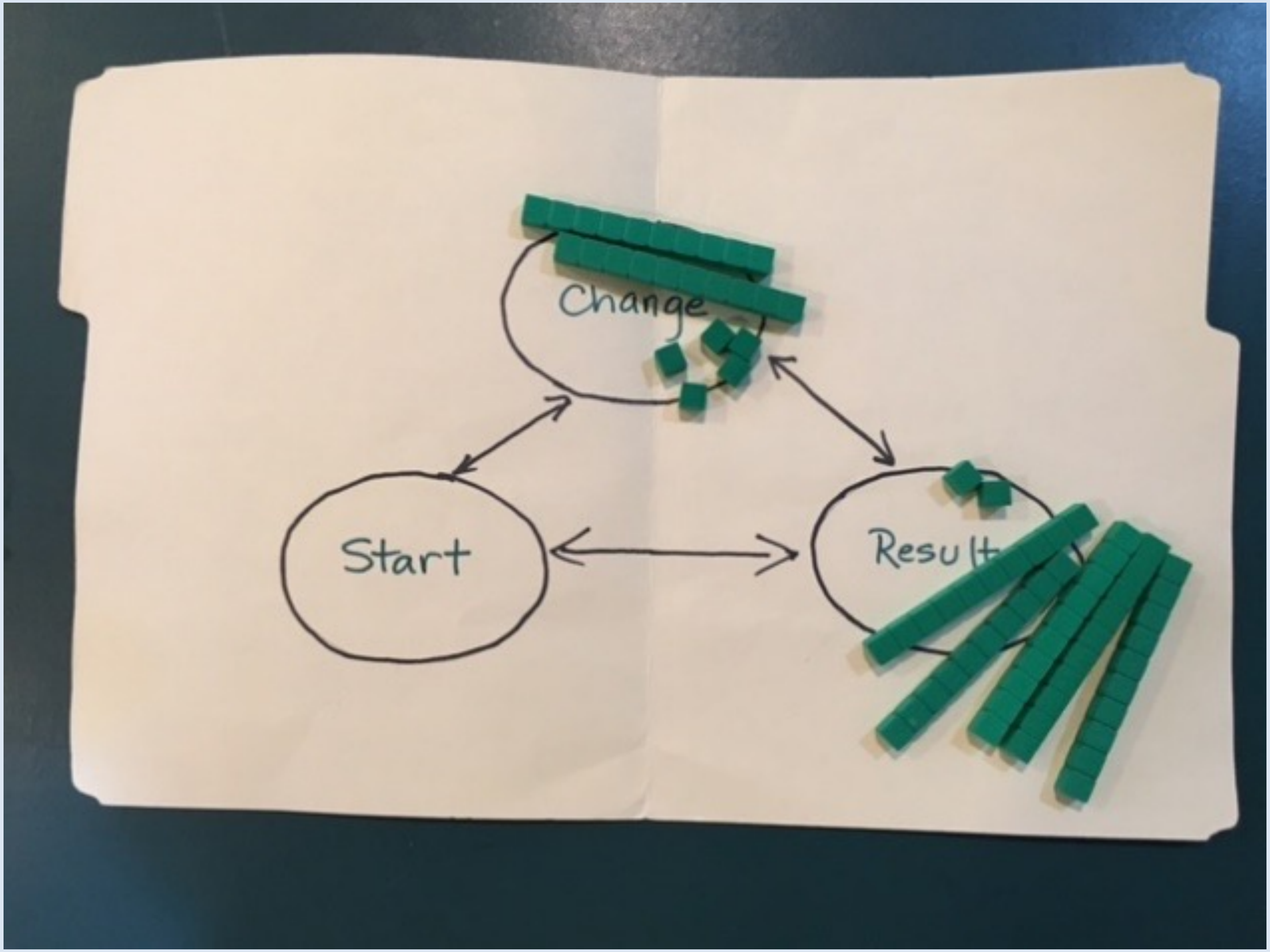
Mikal has some T-shirts. He gives 3 shirts to his brother Elron. Now he has 9. How many T-shirts did Mikal have to begin with? ☐ – 3 = 9

Van de Walle, J., Karp, K., & Bay Williams, J. (2016). *Elementary and Middle School Mathematics: Teaching developmentally*. New York: Pearson.
Carpenter, T., Fennema, E., Franke, M. L., Levi, L. & Empson, S.. (1999). *Children's mathematics: Cognitively guided instruction*. Portsmouth, NH: Heinemann.

Finding a Use for Old Folders



The Graphic Organizer that Keeps on Going!!!

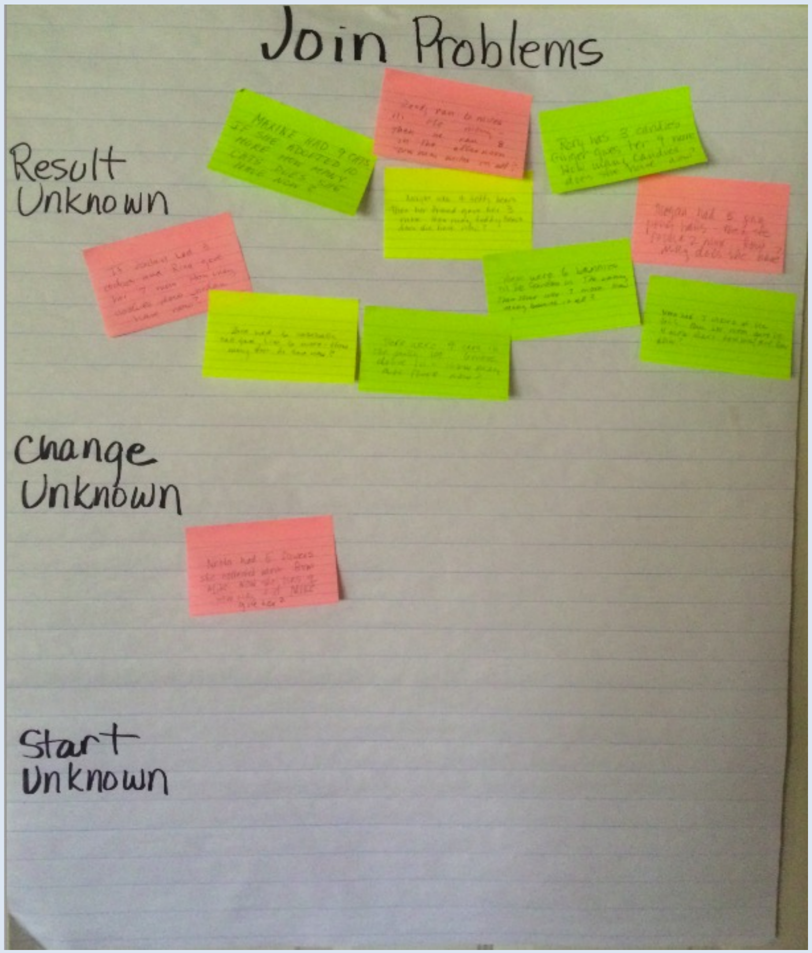


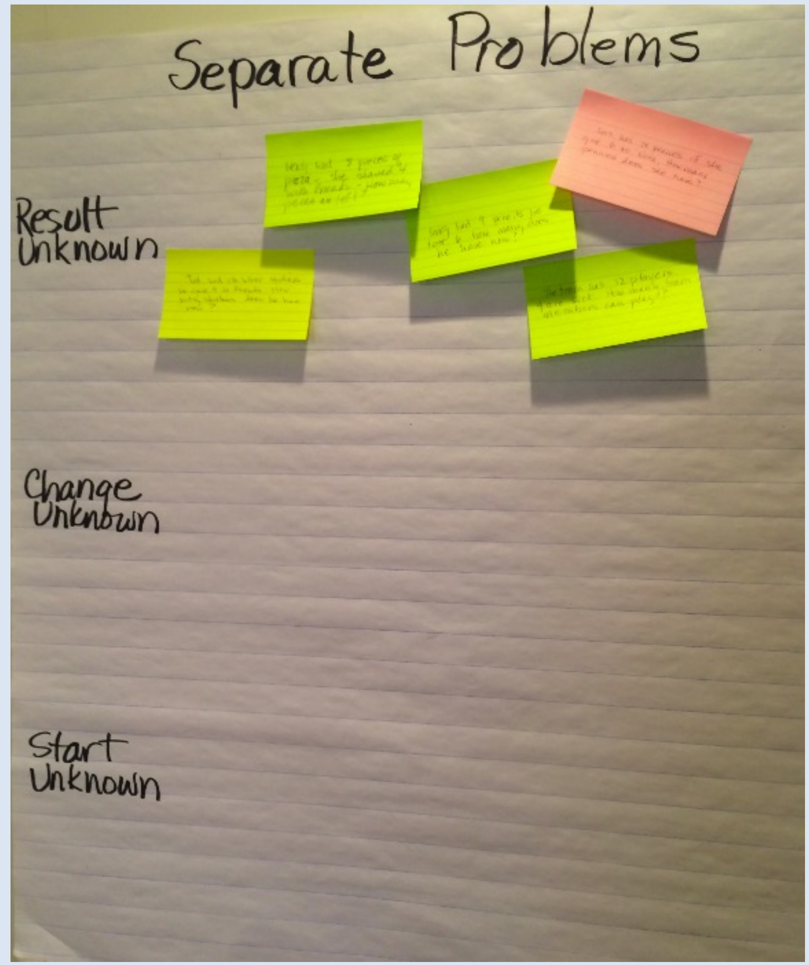
Warm up for today's topic

- Using a number family like 9, 6, 15, create an addition or subtraction story problem that you would have students in your classroom solve.

- If you are in middle school, you can substitute a variable for one of the values.

Teachers Responses to this Task



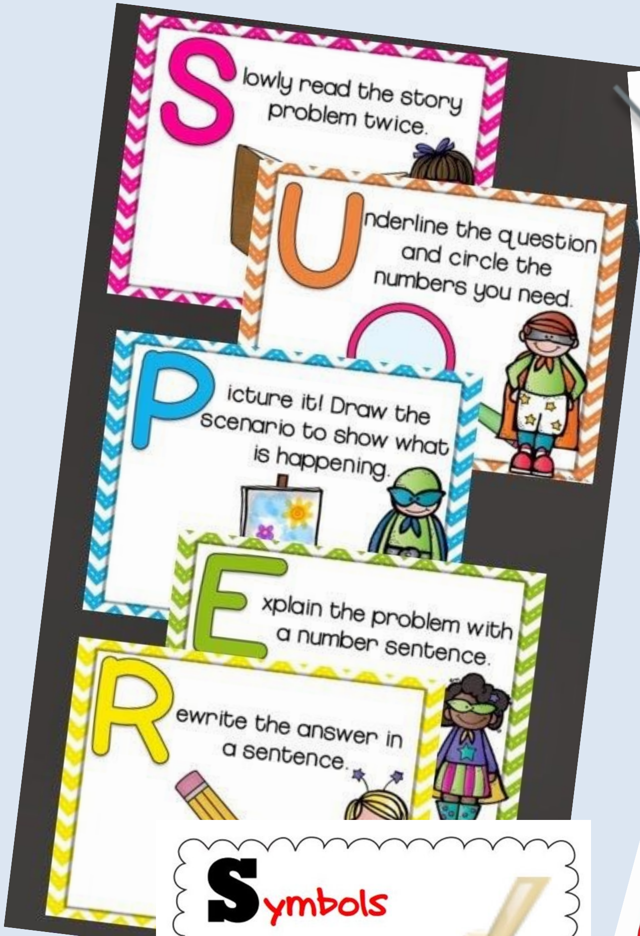


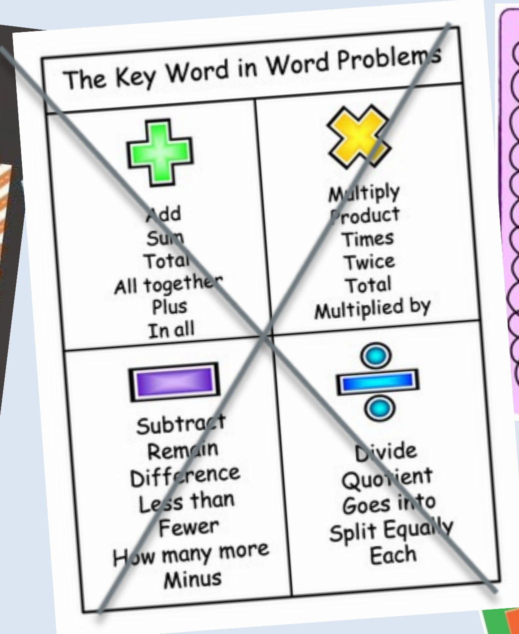
Common Addition and Subtraction Situations

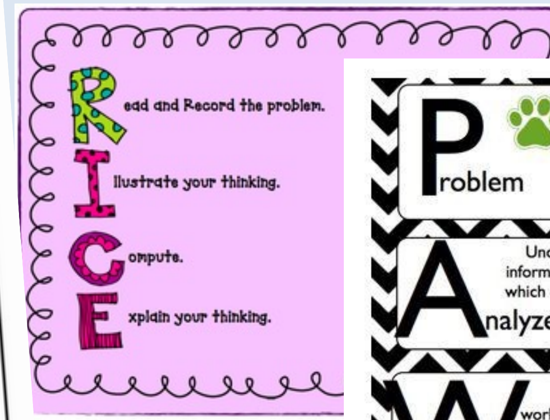
	Result Unknown	Change Unknown	Start Unknown
Add to	Two bunnies sat on the grass. Three more bunnies hopped there. How many bunnies are on the grass now? 2 + 3 = ?	Two bunnies were sitting on the grass. Some more bunnies hopped there. Then there were five bunnies. How many bunnies hopped over to the first two? 2 + ? = 5	Some bunnies were sitting on the grass. Three more bunnies hopped there. Then there were five bunnies. How many bunnies were on the grass before? ? + 3 = 5
Take from	Five apples were on the table. I ate two apples. How many apples are on the table now? 5 – 2 = ?	Five apples were on the table. I ate some apples. Then there were three apples. How many apples did I eat? 5 – ? = 3	Some apples were on the table. I ate two apples. Then there were three apples. How many apples were on the table before? ? – 2 = 3
	Total Unknown	**Addend Unknown**	**Both Addends Unknown**[1]
Put Together/ Take Apart[2]	Three red apples and two green apples are on the table. How many apples are on the table? 3 + 2 = ?	Five apples are on the table. Three are red and the rest are green. How many apples are green? 3 + ? = 5, 5 – 3 = ?	Grandma has five flowers. How many can she put in her red vase and how many in her blue vase? 5 = 0 + 5, 5 = 5 + 0 5 = 1 + 4, 5 = 4 + 1 5 = 2 + 3, 5 = 3 + 2

What about the problem-solving posters or acronyms in classrooms?

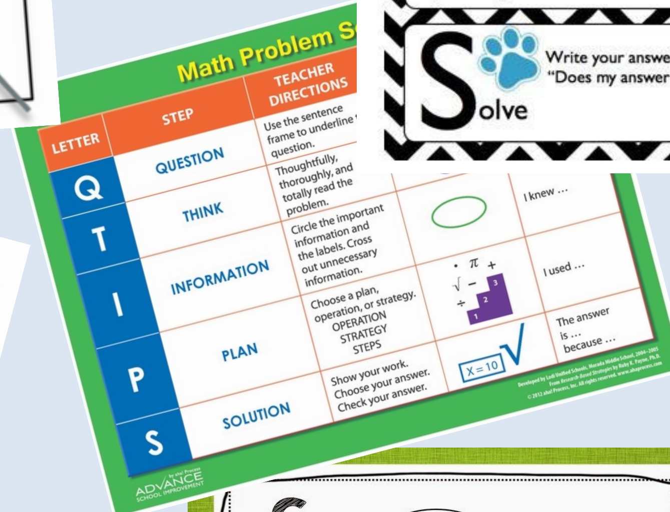
- What posters do you use to support problem solving?
- What acronyms to you use to support problem solving?
- Is it what others are using?

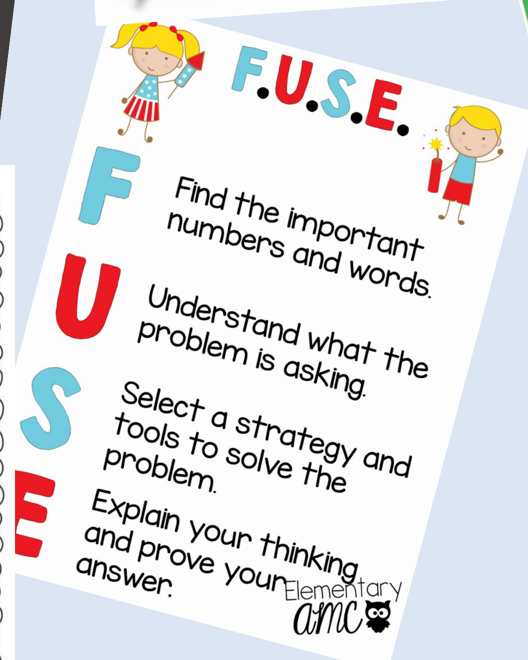

S lowly read the story problem twice.
U nderline the question and circle the numbers you need.
P icture it! Draw the scenario to show what is happening.
E xplain the problem with a number sentence.
R ewrite the answer in a sentence.


The Key Word in Word Problems
Add
Sum
Total
All together
Plus
In all
Multiply
Product
Times
Twice
Total
Multiplied by
Subtract
Remain
Difference
Less than
Fewer
How many more
Minus
Divide
Quotient
Goes into
Split Equally
Each

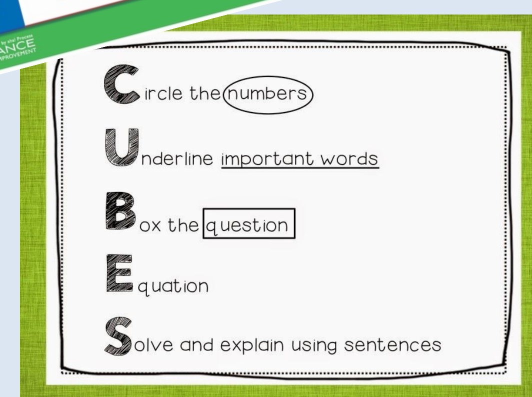

R ead and Record the problem.
I llustrate your thinking.
C ompute.
E xplain your thinking.


P roblem
Read your problem.
A nalyze
Underline key words. Cross out information you do not need. Decide which operation you will need to use.
W ork it out
Use the operation to work the problem out. Show your work.
S olve
Write your answer. Ask yourself, "Does my answer make sense?"


Math Problem S
LETTER
STEP
TEACHER DIRECTIONS
Q
QUESTION
Use the sentence frame to underline question.
T
THINK
Thoughtfully, thoroughly, and totally read the problem.
I knew ...
I
INFORMATION
Circle the important information and the labels. Cross out unnecessary information.
I used ...
P
PLAN
Choose a plan, operation, or strategy. OPERATION STRATEGY STEPS
The answer is ... because ...
S
SOLUTION
Show your work. Choose your answer. Check your answer.
X = 10
ADVANCE SCHOOL IMPROVEMENT


F.U.S.E.
F Find the important numbers and words.
U Understand what the problem is asking.
S Select a strategy and tools to solve the problem.
E Explain your thinking and prove your answer.
Elementary AMC


S ymbols
W ords or Labels
E quations
E xplain
P ictures or diagrams


C ircle the numbers
U nderline important words
B ox the question
E quation
S olve and explain using sentences

KEY WORDS - Not So Helpful…

Grade	Process	E means	P means	S means	U means
Key Word Posters					
1	CUBES	equation	--	solve	underline important words
2	SUPER	explain with a number sentence	picture	slowly read the problem	underline the question
3	FUSE	explain your thinking	--	select a strategy	understand
4	QTIPS	--	plan	solution	--
5	SWEEP	equation explain	pictures	symbols	--

Andrews, D. & Kobett, B. M. (2017, July). *Connection to discourse: Word problems.* Presentation for the NCTM Discourse Institute, Baltimore, MD.

The Myth of Keywords

- Keywords do not—
  - Develop of sense making or support making meaning
  - Build structures for more advanced learning
  - Appear in many problems

- Students use key words inappropriately

- Multi-step problems are impossible to solve with key words

Van de Walle, J., Karp, K., & Bay Williams, J. (2010). *Elementary and Middle School Mathematics: Teaching developmentally*. New York: Pearson.

Danger: Key Words Ahead

Mark has 33 packages of pencils. There are 6 pencils in each package. How many pencils does he have in all?

39 because it says in all.

The Infamous Shepherd Problem

:

There are 25 sheep and 5 dogs in a flock. How old is the shepherd?

How do you think students solved the problem?

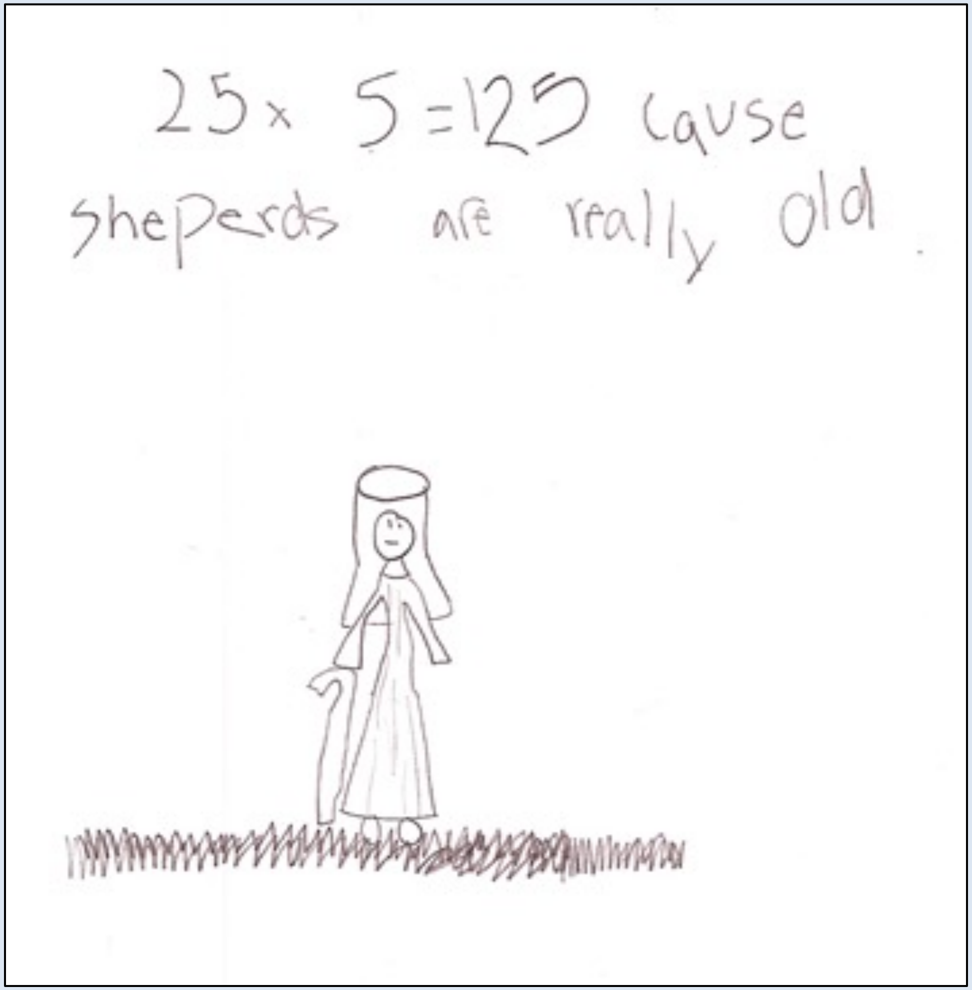
Caldwell, Kobett & Karp (2014) Putting Essential Understandings into Practice: Addition and Subtraction , Grade PreK–2. NCTM.

Results from 214 Students

	Added the numbers	Subtracted the numbers	Multiplied the numbers	Created a ratio	Other Incorrect procedure	Suggested no solution is possible
Third grade (n = 58)	76%	8%	0%	0%	14%	2 %
Sixth grade (n = 71)	48%	9%	21%	8%	6%	8 %
Seventh grade (n = 85)	48%	2%	17%	14%	9%	10 %

Caldwell, Kobett & Karp (2014) *Putting essential understandings into practice in grades preK–2: Addition and subtraction.* A volume in the series edited by B. J. Dougherty, National Council of Teachers of Mathematics..

Really?



Caldwell, Kobett & Karp (forthcoming) *Putting essential understandings into practice in grades preK–2: Addition and subtraction.* A volume in the series edited by B. J. Dougherty, National Council of Teachers of Mathematics..

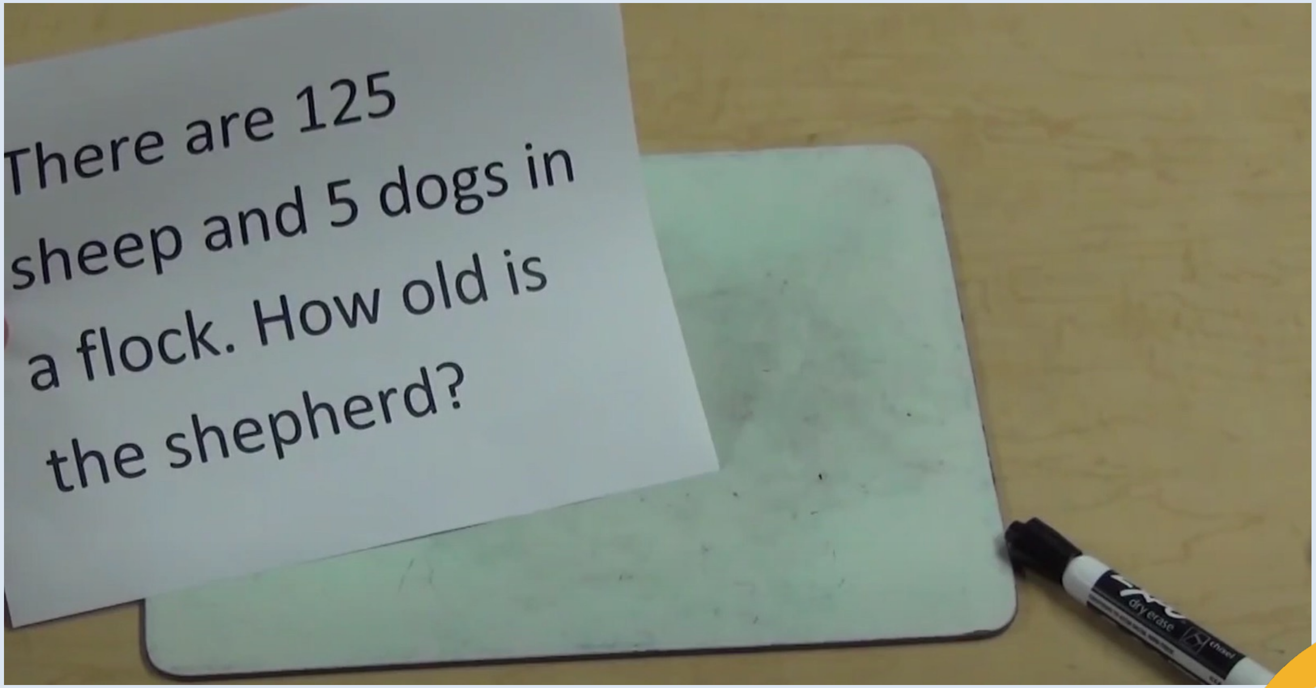
Another Option

Would your students be able to discern which of the following three options would be the correct answer?

- The shepherd is 30 years old
- The shepherd is 125 years old; and
- It is not possible to tell the shepherd’s age from the information in the problem.

Caldwell, Kobett & Karp (2014) *Essential understanding of addition and subtraction in practice, grades K-2*. NCTM.

Can 8^{th} Graders Make Sense of This??



Systematic Instruction with Word Problems

Students need structure through systematic instruction

The structure should be based on the meaning of the operations – not on key words or guesswork

Have students try problems they don't have to solve they just need to sort into addition or subtraction

They need to focus on **actions**, **representations, reasoning strategies,** and **general properties of the operations**

Increase Readability

- Quality of ease of reading can be influenced by
  - Length of the problem
  - Difficulty of the words used
  - Using second person pronouns such as you or yours to put the student “into the problem”

Ensure Topic Relevance

- Use problems that are relevant to students - it’s common sense!
  - Elementary students are not interested in buying a car! Middle schoolers mixing paint?
  - Identify their interests and use them in a problem

Support Reading and Understanding of Mathematics Words

- Emphasize interpreting words that are spelled the same but have different meanings (e.g., table, face, degree, similar, mean, formula, right, etc.)
- Vocabulary is at the core of content literacy!
- Vocabulary should not be pretaught!

Make Problems Easier with Concrete Materials

- Use manipulatives to represent the situation in the context of the word problem!
- Use word problems that easily link to concrete representations

Have Students Imagine the Situation

- Have students think about and articulate the situation
- Focus on quantitative relationships within the context

Reading and Problem Solving

Strategies

Highlighting important information

Caryn baked 4 dozen cookies. She gave half of them to Jacklyn. She gave half of what remained to Corbin. How many cookies did she have left?

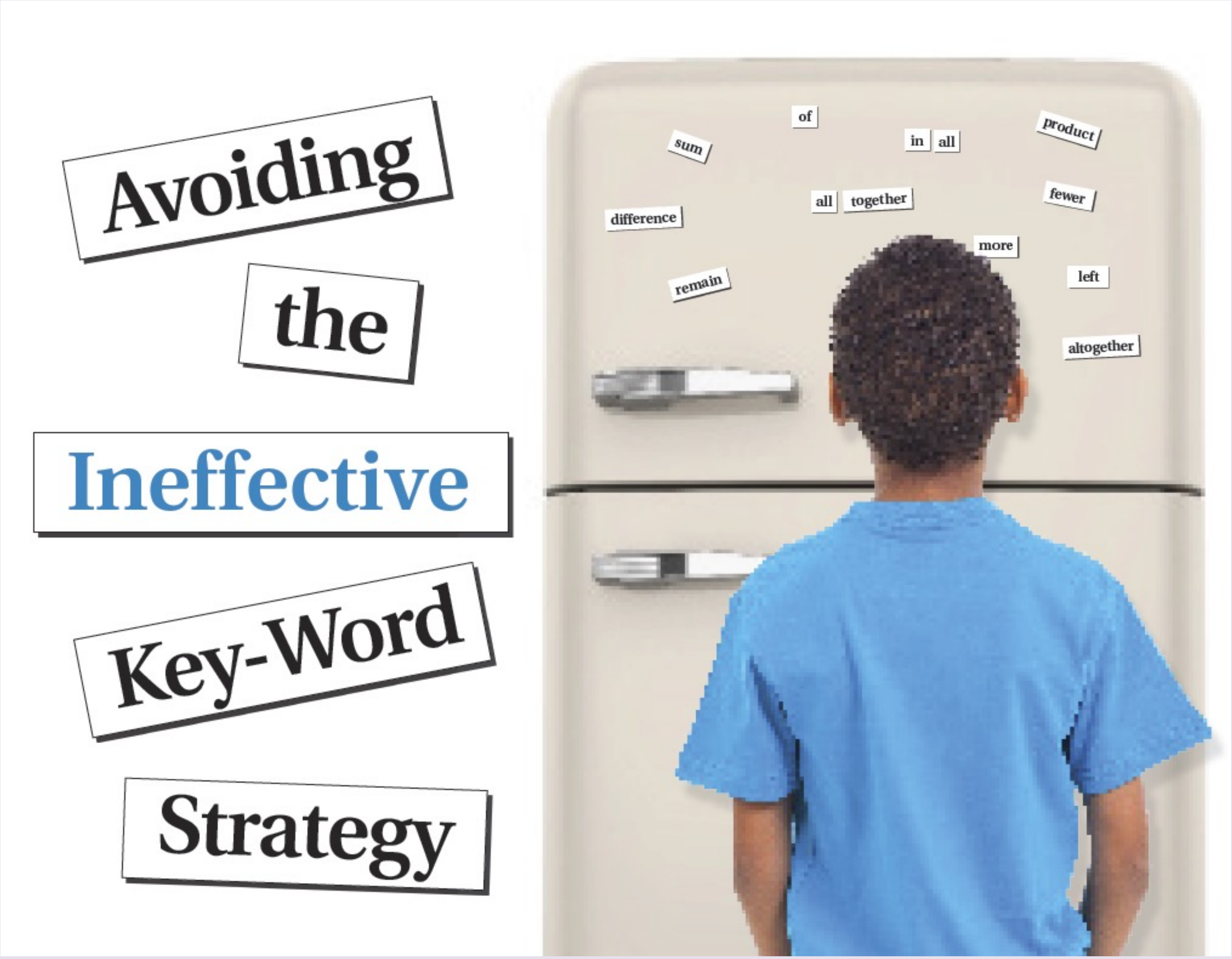
Reading and Problem Solving

Strategies

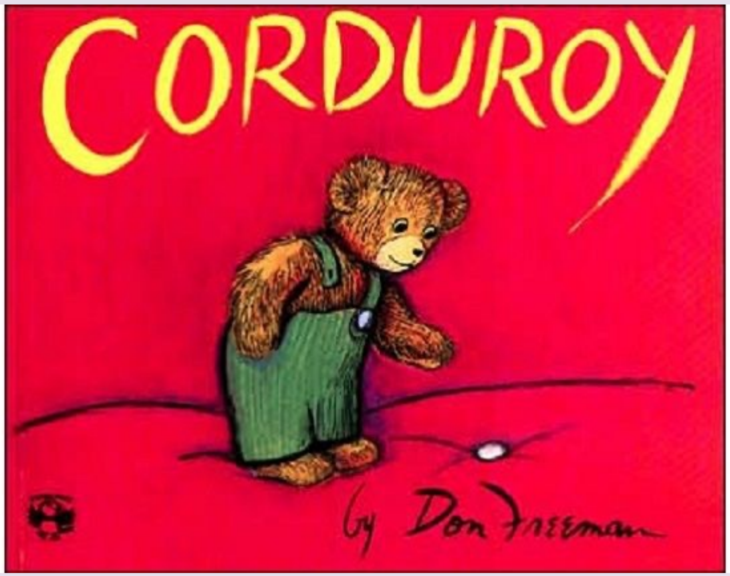
Highlighting important information

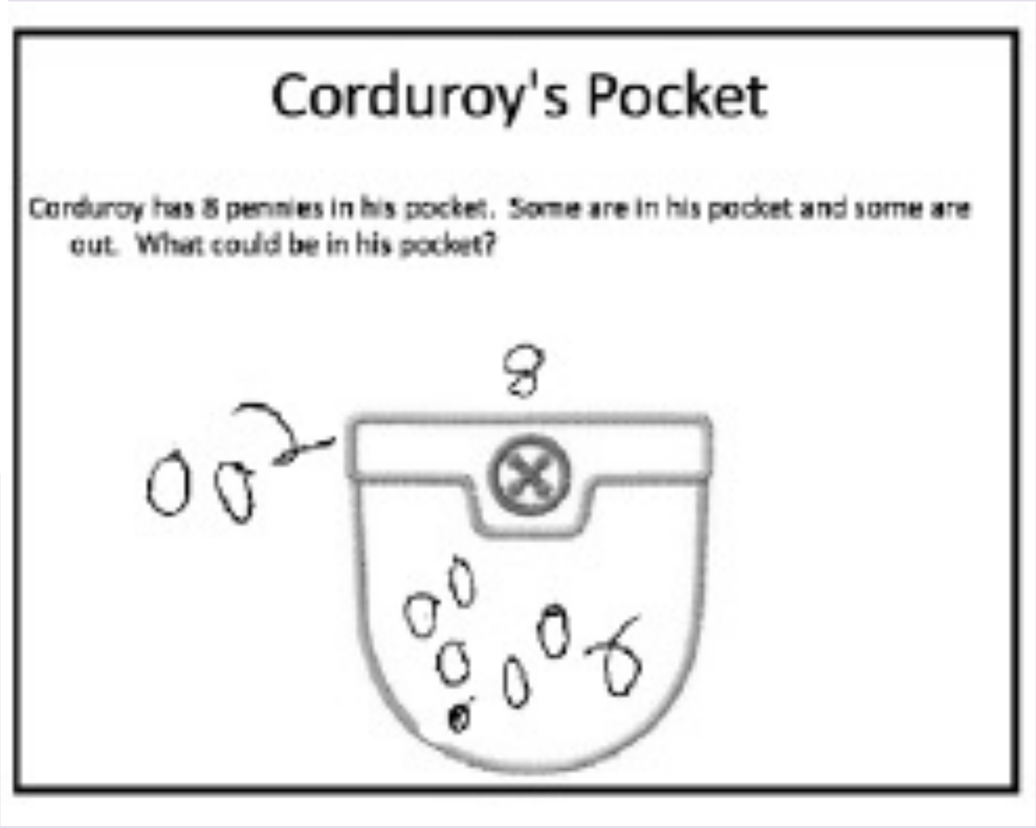
Caryn baked 4 dozen cookies. She gave half of them to Jacklyn. She gave half of what remained to Corbin. How many cookies did she have left?

Teaching Children Mathematics – May 2019



Two parts (addends) unknown





Caldwell, Kobett & Karp (2014) *Essential understanding of addition and subtraction in practice, grades K-2*. NCTM.

Part-Part-Whole Problem Structure

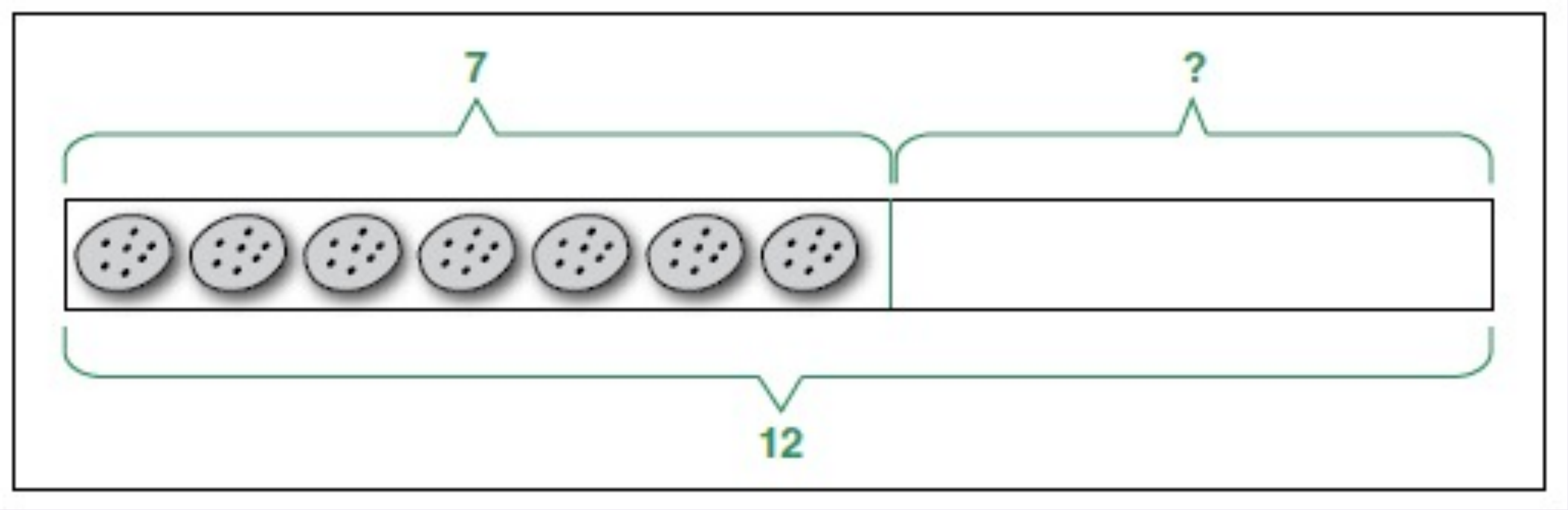
Lynnette has 14 fiction and 23 nonfiction books. How many books does she have?

Lynnette and her friend Victoria put 37 books into a backpack. Lynnette put in 14 books. How many books did Victoria put in the backpack?

If there are 37 books in the backpack. What are the different combinations that both girls could have placed in the backpack?

Van de Walle, J., Karp, K., & Bay Williams, J. (2010). *Elementary and Middle School Mathematics: Teaching developmentally*. New York: Pearson.

Part-Part-Whole Diagrams – Where does this lead?



Caldwell, Kobett & Karp (2014) *Essential understanding of addition and subtraction in practice, grades K-2*. NCTM.

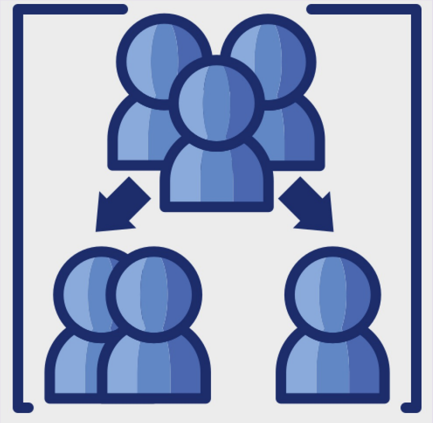
Make Mine 100

Make Mine 100 Score Sheet

Addition Equation	Score
____ ____ + ____ ____ = ________	______
____ ____ + ____ ____ = ________	______
____ ____ + ____ ____ = ________	______
____ ____ + ____ ____ = ________	______
____ ____ + ____ ____ = ________	______
Total Score	______

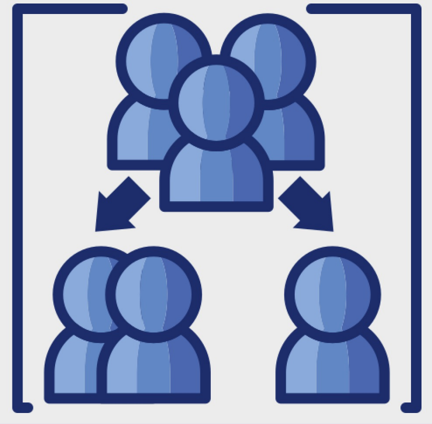
Patterns in Addition and Subtraction

- 100s charts
- Patterns in sums and differences to support generalizations



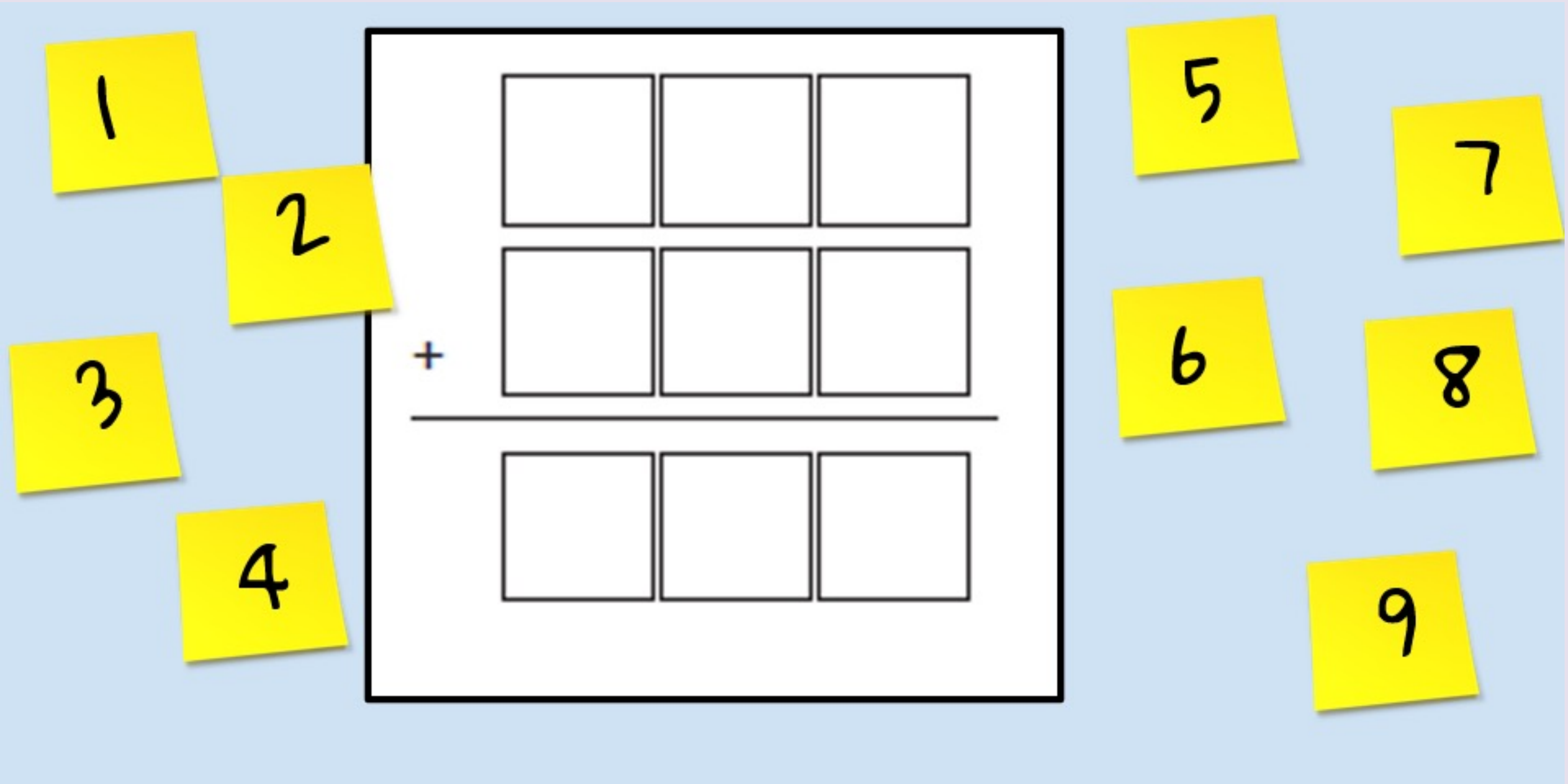
Hundred Chart

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100



Problem of Nine

Number Sticky Notes from 1-9



Extension questions

- What pattern do you notice with the sums?
- What is the smallest digit that can be in the hundreds place of the sum?
- What is the smallest digit that can be in the hundreds place of the addends?
- Is it possible to find two addends that do not require regrouping? Why or why not?

Compare the Tasks

- Problem of Nine

3-Digit Plus 3-Digit Addition (A)

Name: ____________ Date: ____________ Score: ___ /25

Calculate each sum.

593 + 954	217 + 229	783 + 708	736 + 304	230 + 846
406 + 447	577 + 553	582 + 557	832 + 717	428 + 526
332 + 146	399 + 455	557 + 653	675 + 459	942 + 225
456 + 573	957 + 462	971 + 772	456 + 286	328 + 424
115 + 316	870 + 107	425 + 306	436 + 306	311 + 484

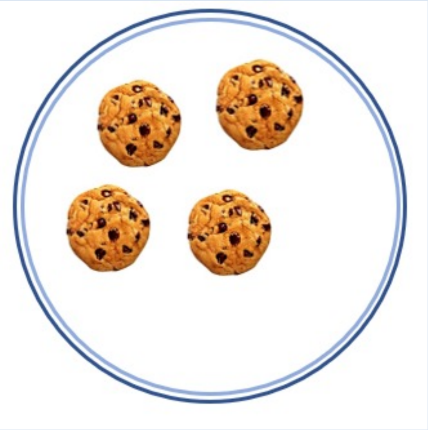
Math-Drills.com

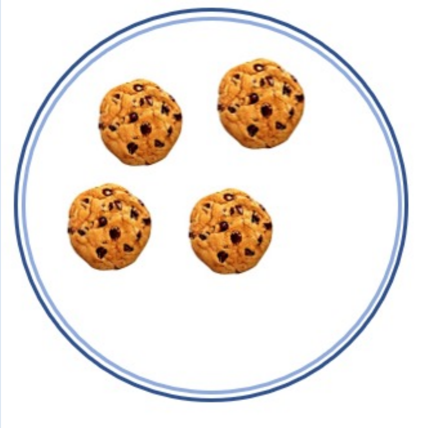
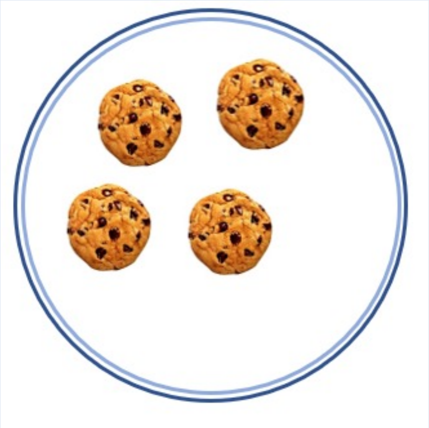
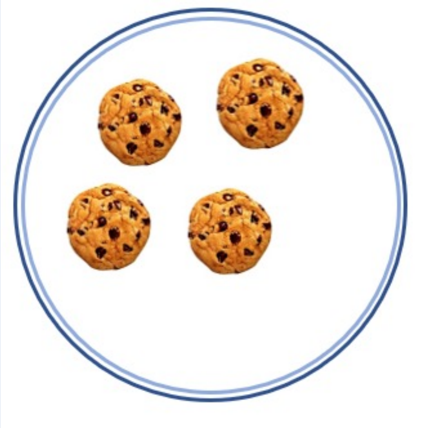
Multiplication and Division Problem Structures

- Equal-groups
- Area
- Array
- Comparison
- Combination

Explicit Schema for Multiplicative Problems – Equal-size Groups

Lena has 3 plates of cookies. There are 4 cookies on each plate. How many cookies does Lena have all together?

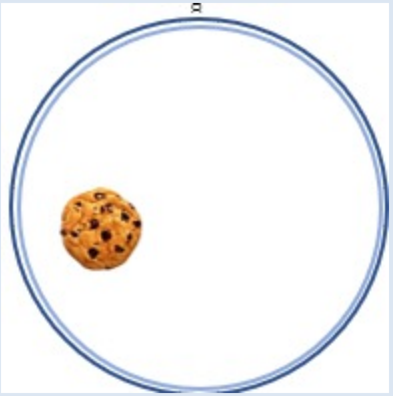


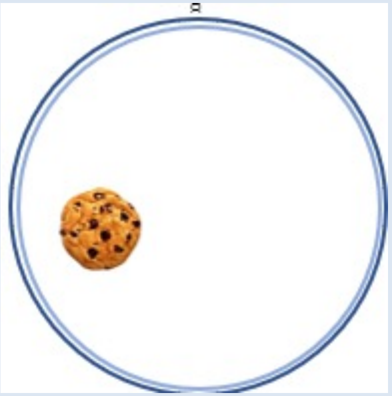




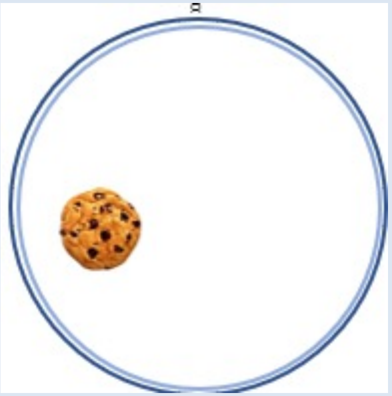
Lena has 3 plates of cookies. There are 4 cookies on each plate. How many cookies does Lena have all together?

$3 \times 4 = ?$ product unknown

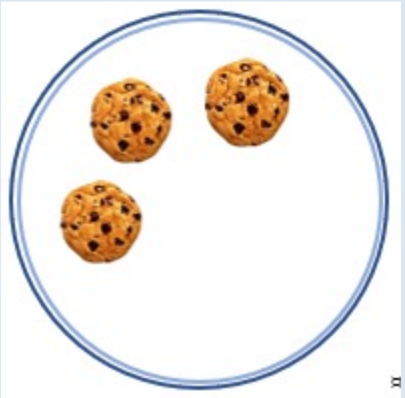
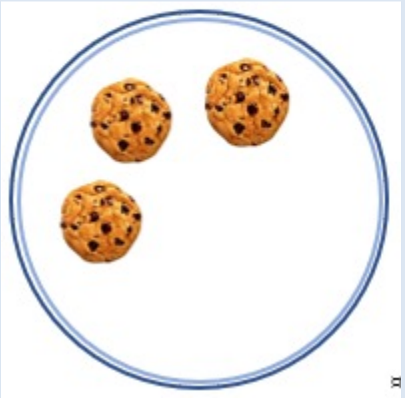
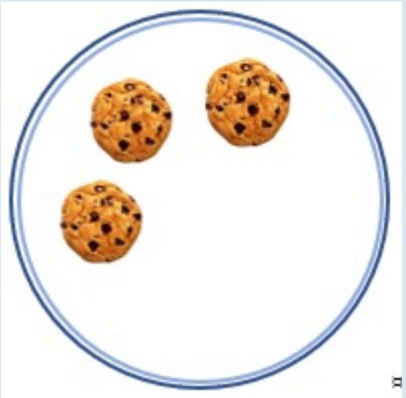
Wynn has 9 cookies. She wants to give these cookies in equal amounts to 3 friends. How many cookies will each friend receive?









Wynn has 9 cookies. She wants to give these cookies in equal amounts to 3 friends. How many cookies will each friend receive?

$9 \div 3 = ?$ Group size unknown

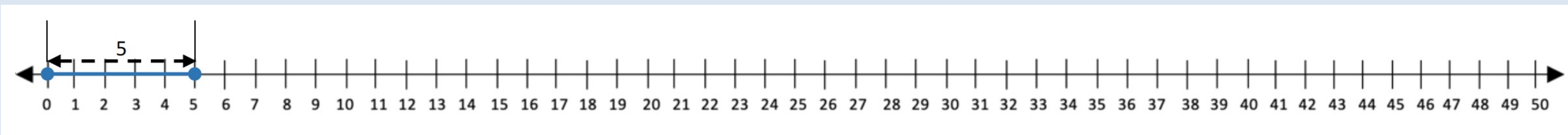
Sam wants to put 15 cookies on plates with 5 cookies on each. How many plates will he need?



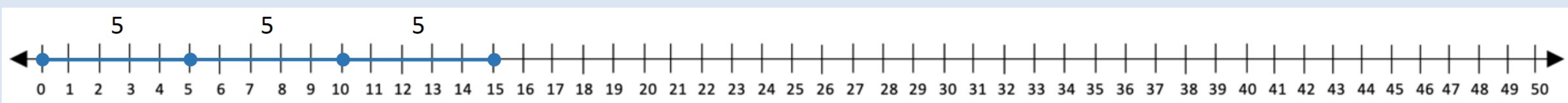

Sam wants to put 15 cookies on plates with 5 on each. How many plates will he need?

$15 \div 5 = ?$ Number of groups unknown

Continuous Model



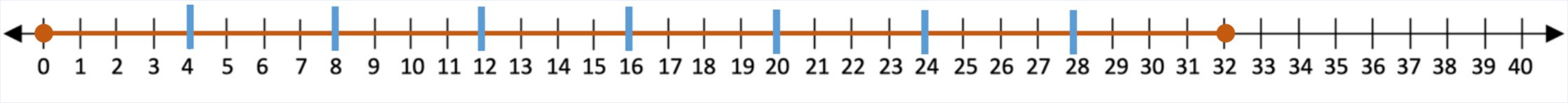
1 x 5 = 5



3 x 5 = 5

Dougherty, B., Jitendra, A., & Sanchez, V. (2021). Multiplicative reasoning for struggling learners.
Funded by US Dept. of Education, IES, R324A190101

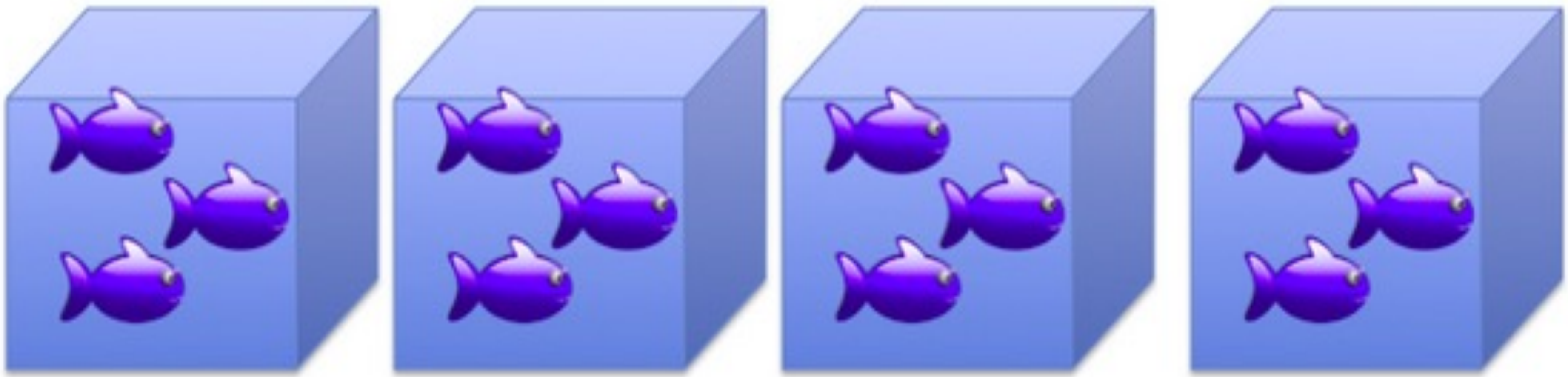
Continuous Model



32 ÷ 4 = 8

Dougherty, B., Jitendra, A., & Sanchez, V. (2021). Multiplicative reasoning for struggling learners.
Funded by US Dept. of Education, IES, R324A190101

So, we are not sure our students can handle this...



Suppose there are 4 tanks and 3 fish in each tank. The total number of fish in this situation can be expressed as $4 \times 3 = 12$.

a. Describe what is meant in this situation by $12 \div 3 = 4$

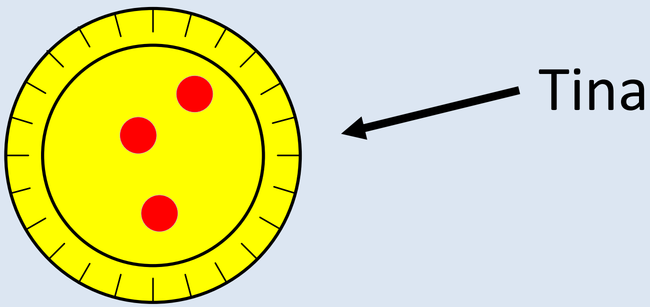
b. Describe what is meant in this situation by $12 \div 4 = 3$

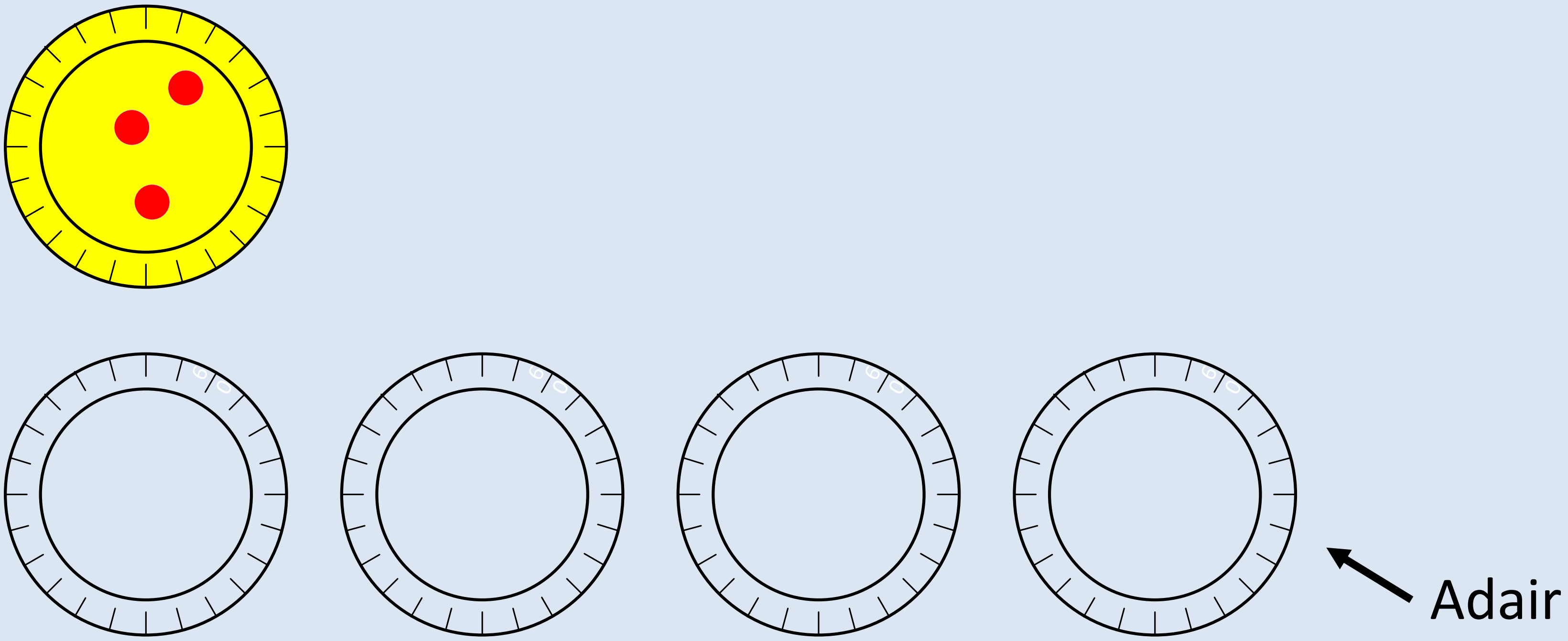
Explicit Schema Multiplicative Comparison Problems

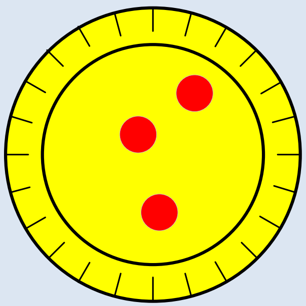
Tina picked 3 apples. Adair picked 4 times as many as Tina. How many apples did Adair pick?

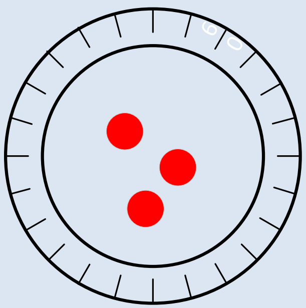
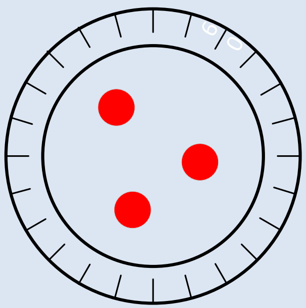
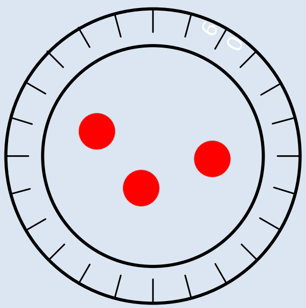
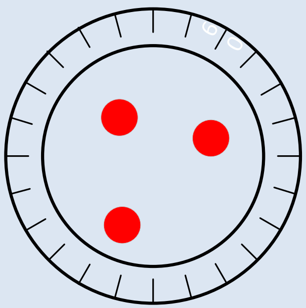
Think about how you might use plates and counters (CSA) to help students understand this problem.

Act this out – draw a sketch if you need. How can you show what you know?


Tina


Adair



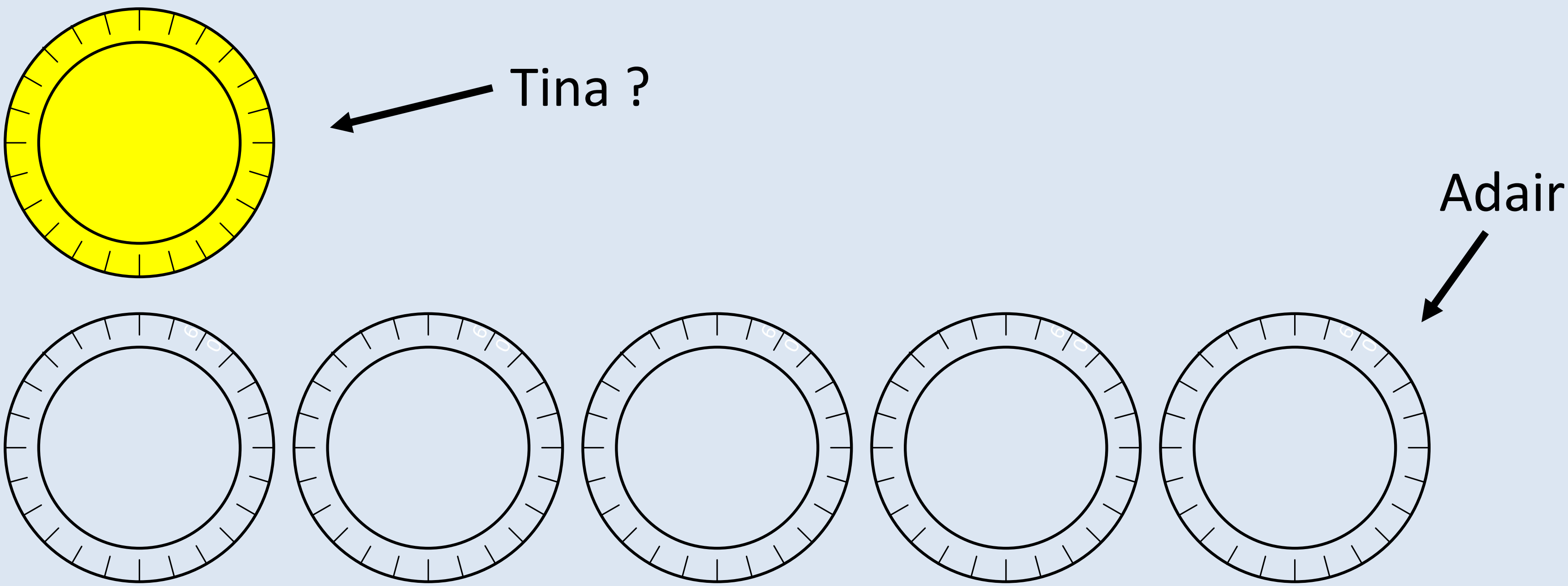
Include Symbolic Representation

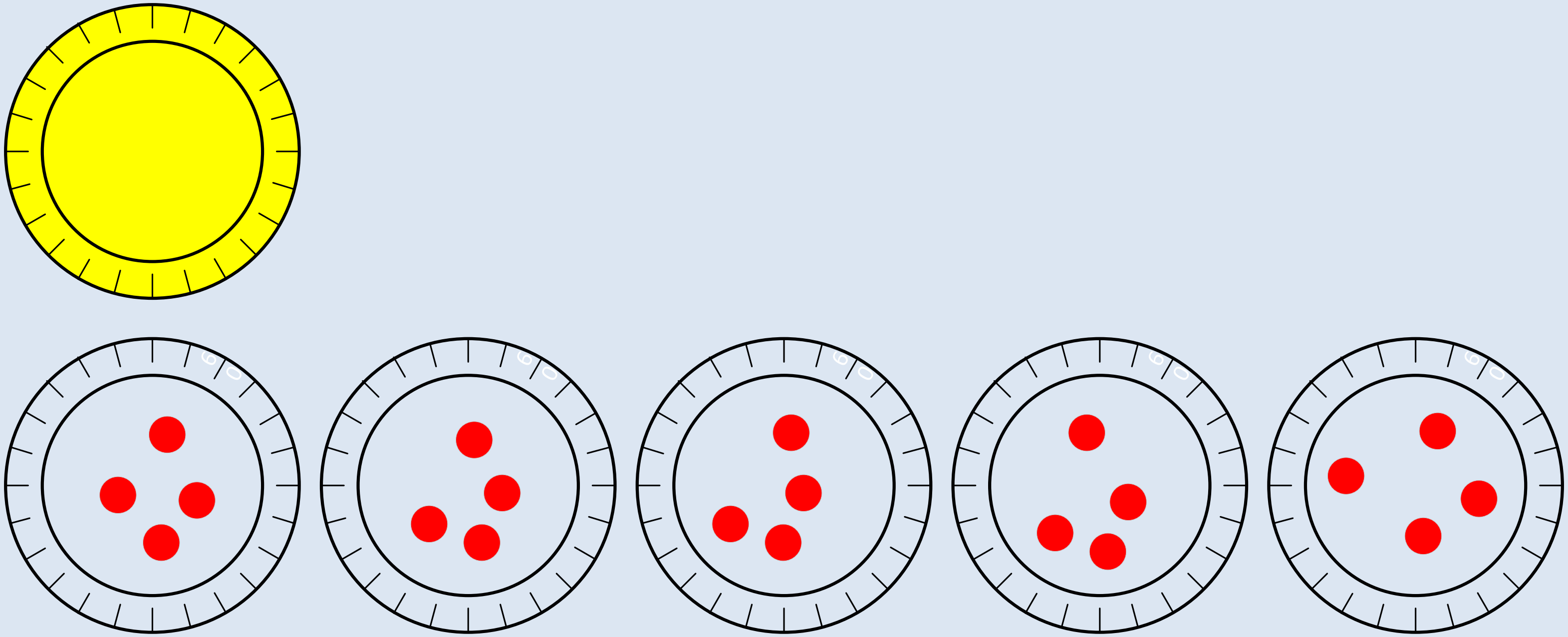
Tina picked 3 apples. Adair picked 4 times as many as Tina. How many apples did Adair pick?

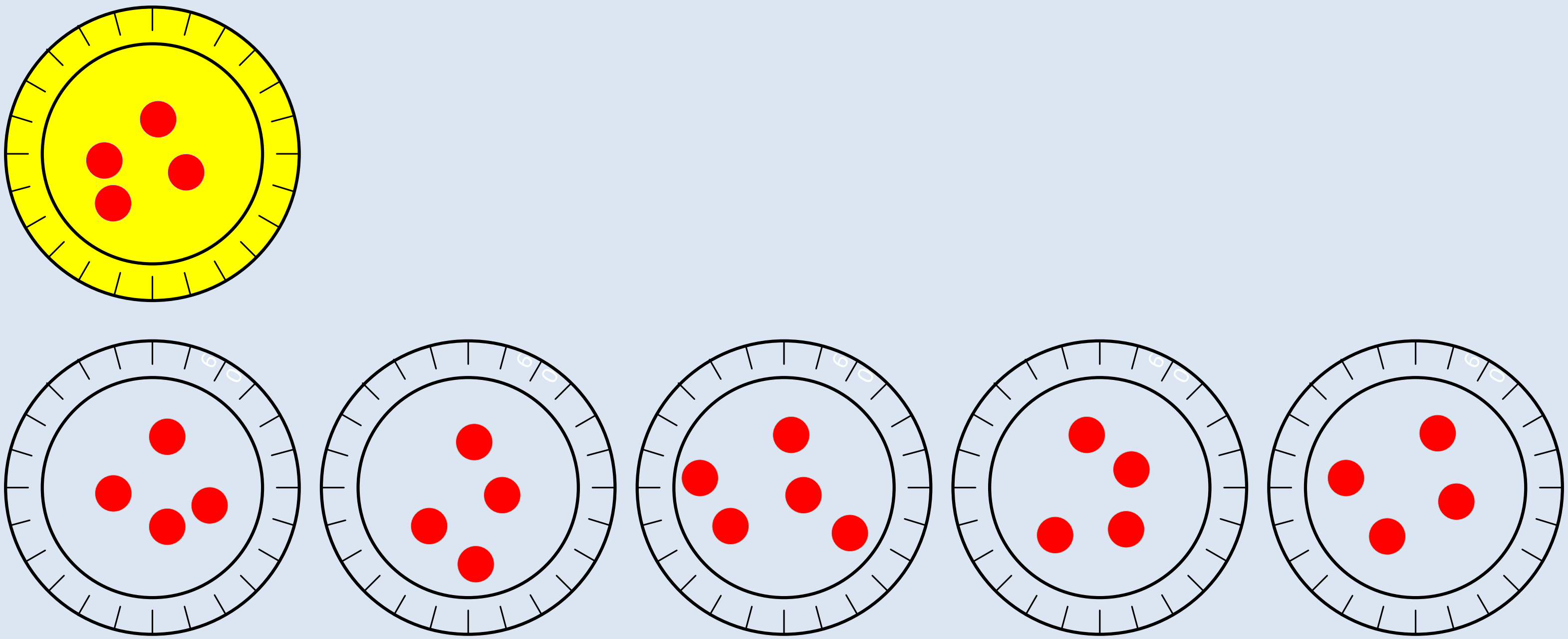
$4 \times 3 = 12$ Compare - Unknown Product

The next day Adair picked 5 times as many apples as Tina. If Adair picked 20 apples, how many apples did Tina pick?

Imagining the Situation







Always Add Abstract/Symbolic Representation

The next day Adair picked 5 times as many apples as Tina. If Adair picked 20 apples, how many apples did Tina pick?

20 ÷ 5 = ? Compare - Size of groups unknown

The third day Tina picked 16 apples and Adair only picked 4. Tina picked how many times as many apples as Adair on the third day?

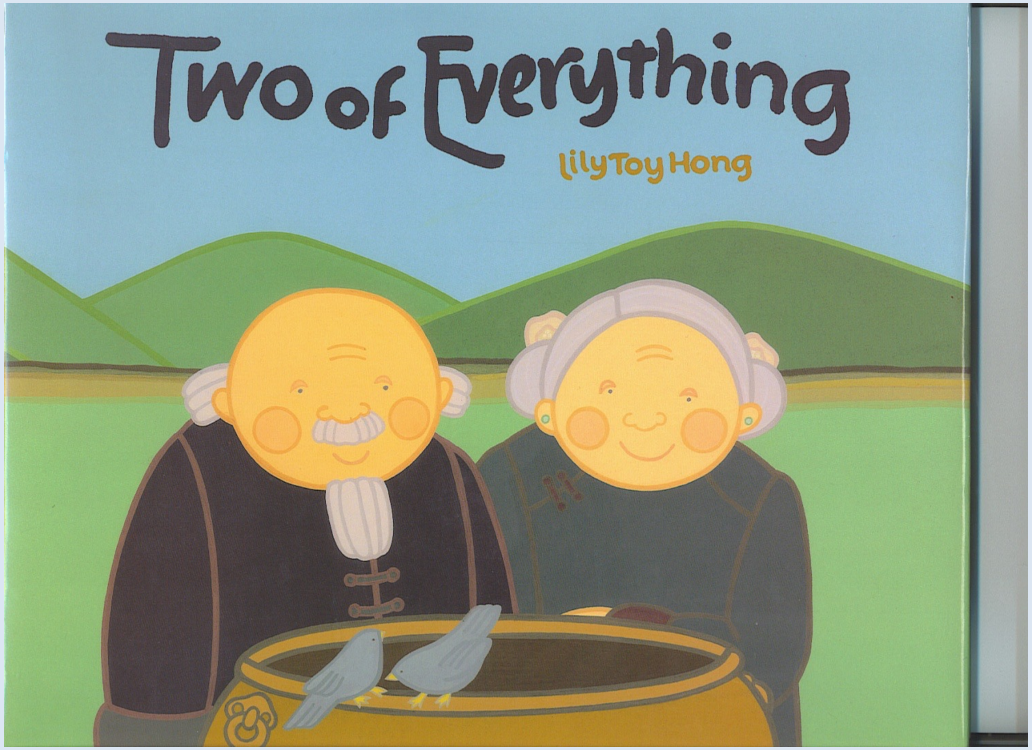
$16 \div 4 = ?$ Compare - Number of groups unknown

What is the Best Order for Teaching the Facts Conceptually?

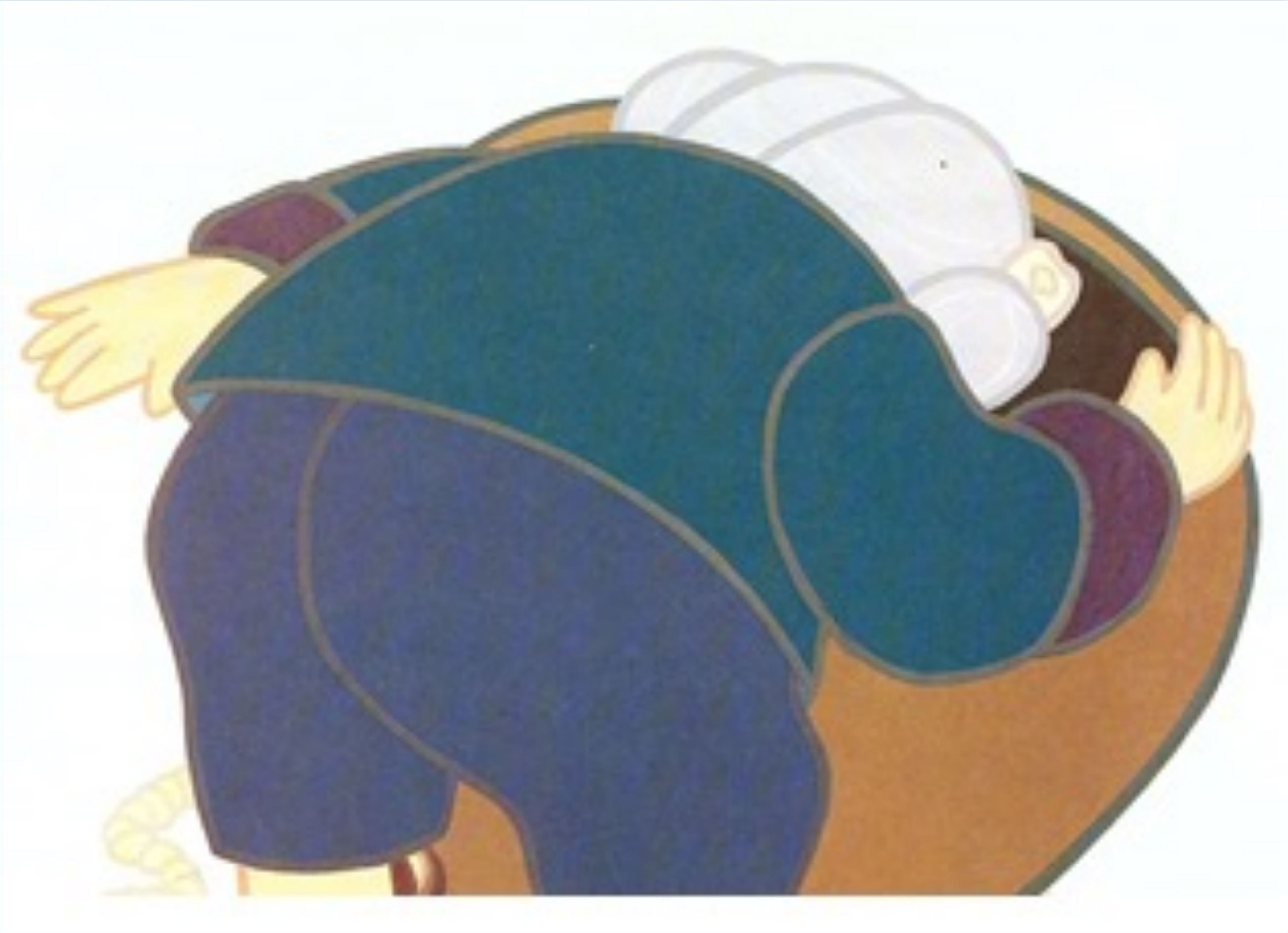
- Should we start with 0? Then 1? Then 2 and so on?
- What is the trajectory we should consider?

Two of Everything – Lily Toy Hong

- Tell the story of Mr. Haktak's magical doubling pot – a book for all grades



A flattering picture of Mrs. Haktak



A Hard Day's Work

In pot – and Out pot

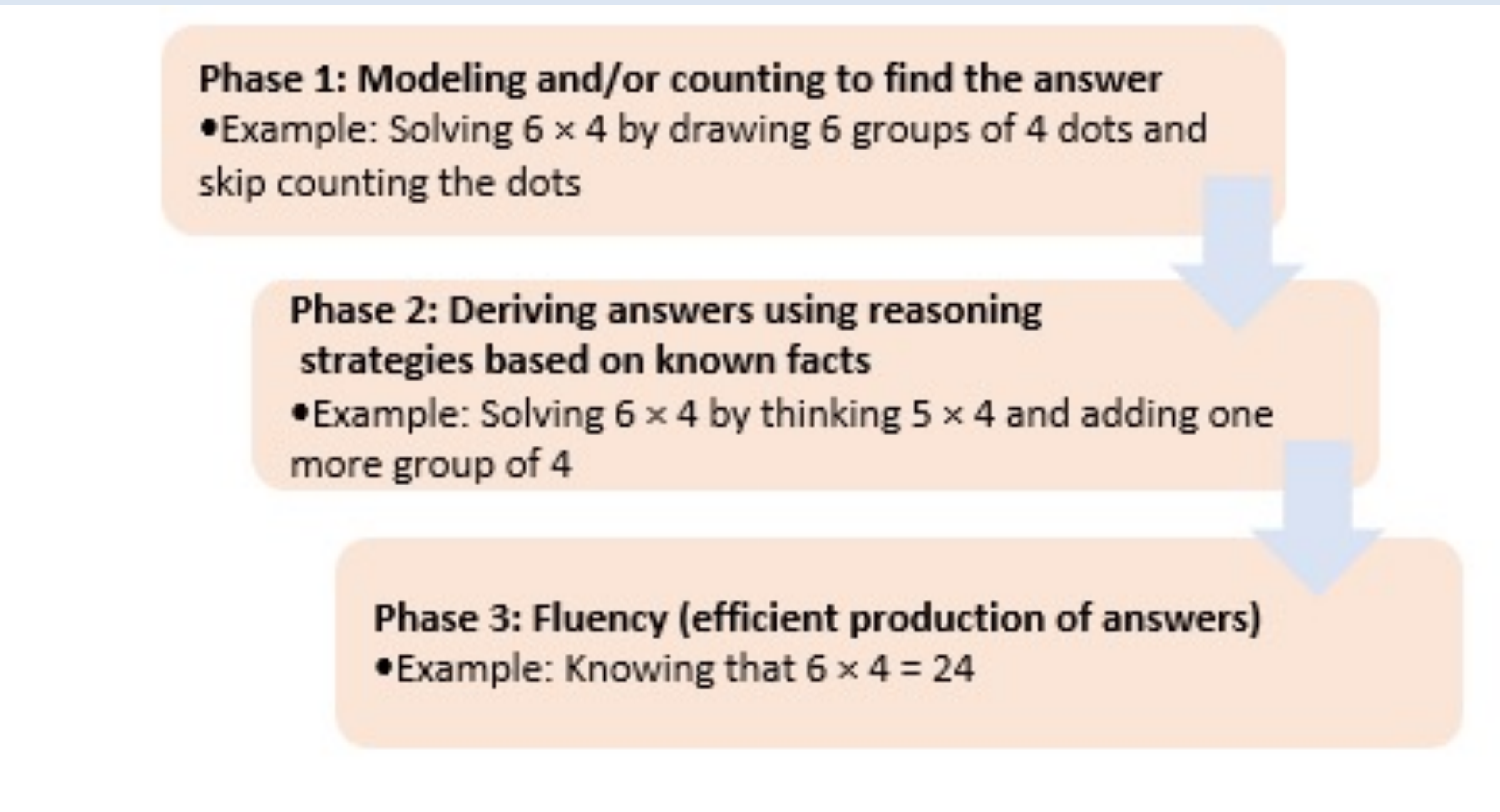
Function Table – Finding the Rule

In	Out
1	2
2	4
3	6
4	8
5	
20	
n	

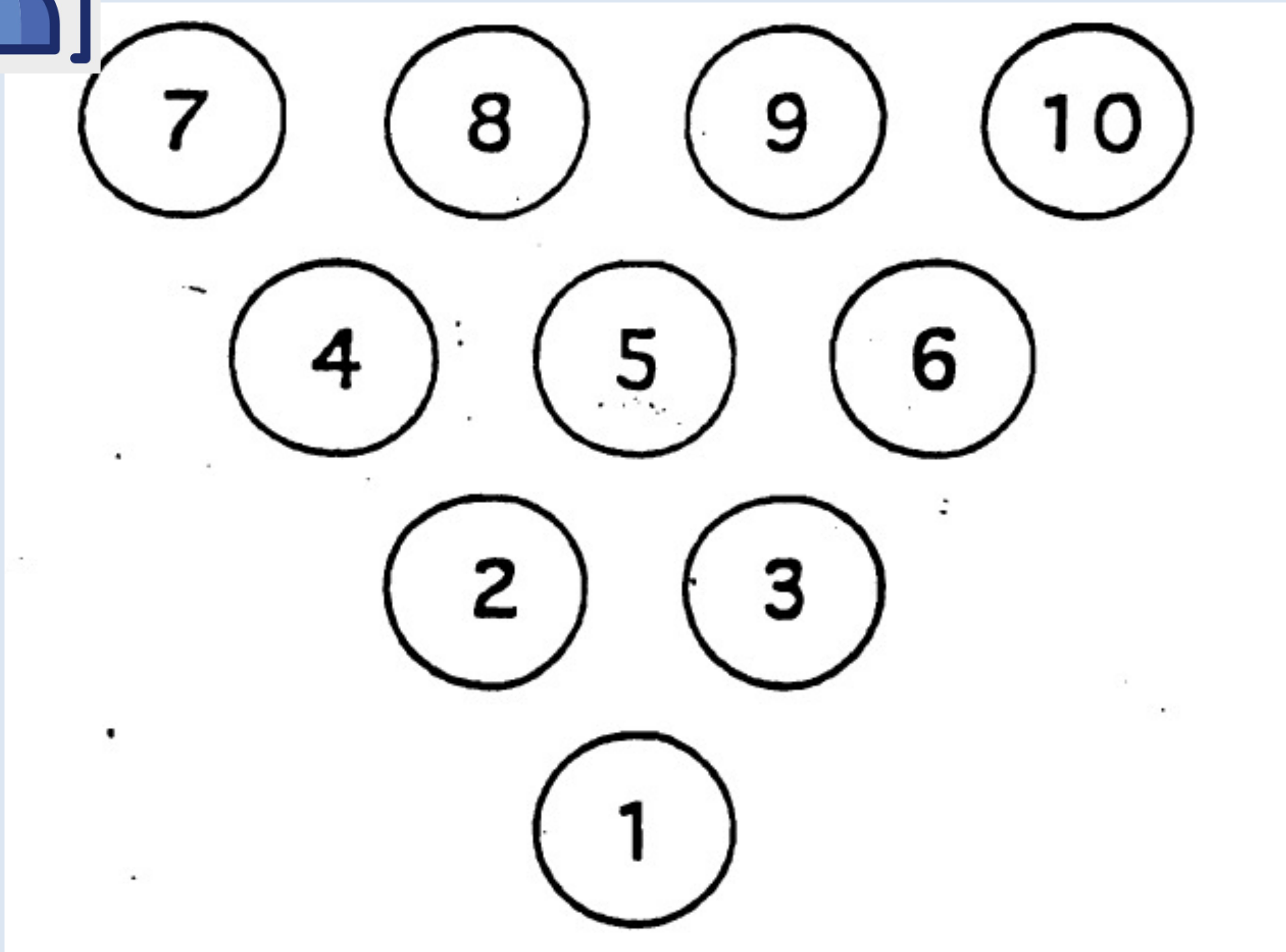
Understand that a function is a rule that assigns to each input exactly one output- enhancing algebraic thinking

Van de Walle, J., Karp, K., & Bay Williams, J. (2016). *Elementary and Middle School Mathematics: Teaching developmentally*. New York: Pearson.

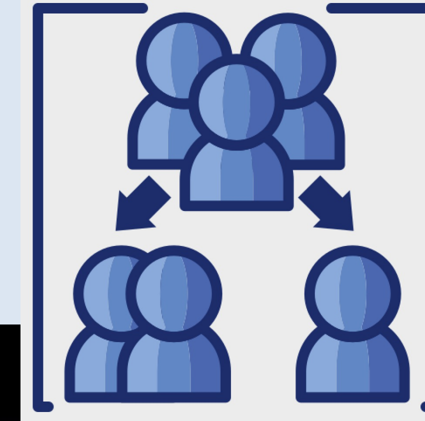
Gaining Fluency in Basic Facts



Bowl A Fact

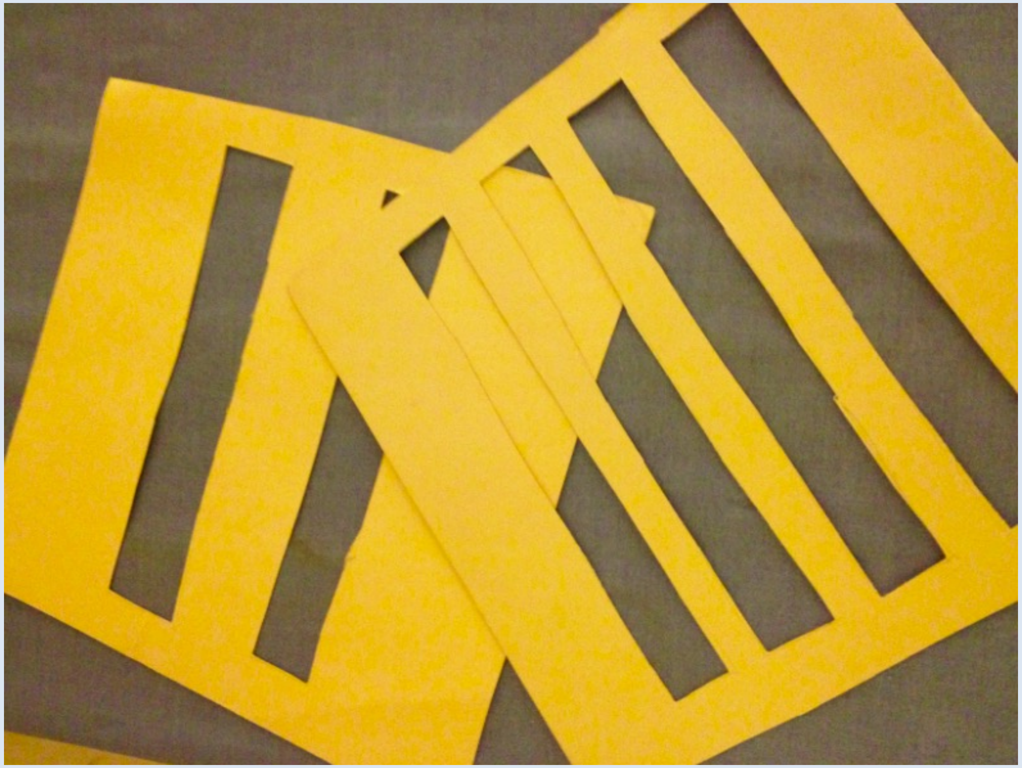


Salute



What about the Array?

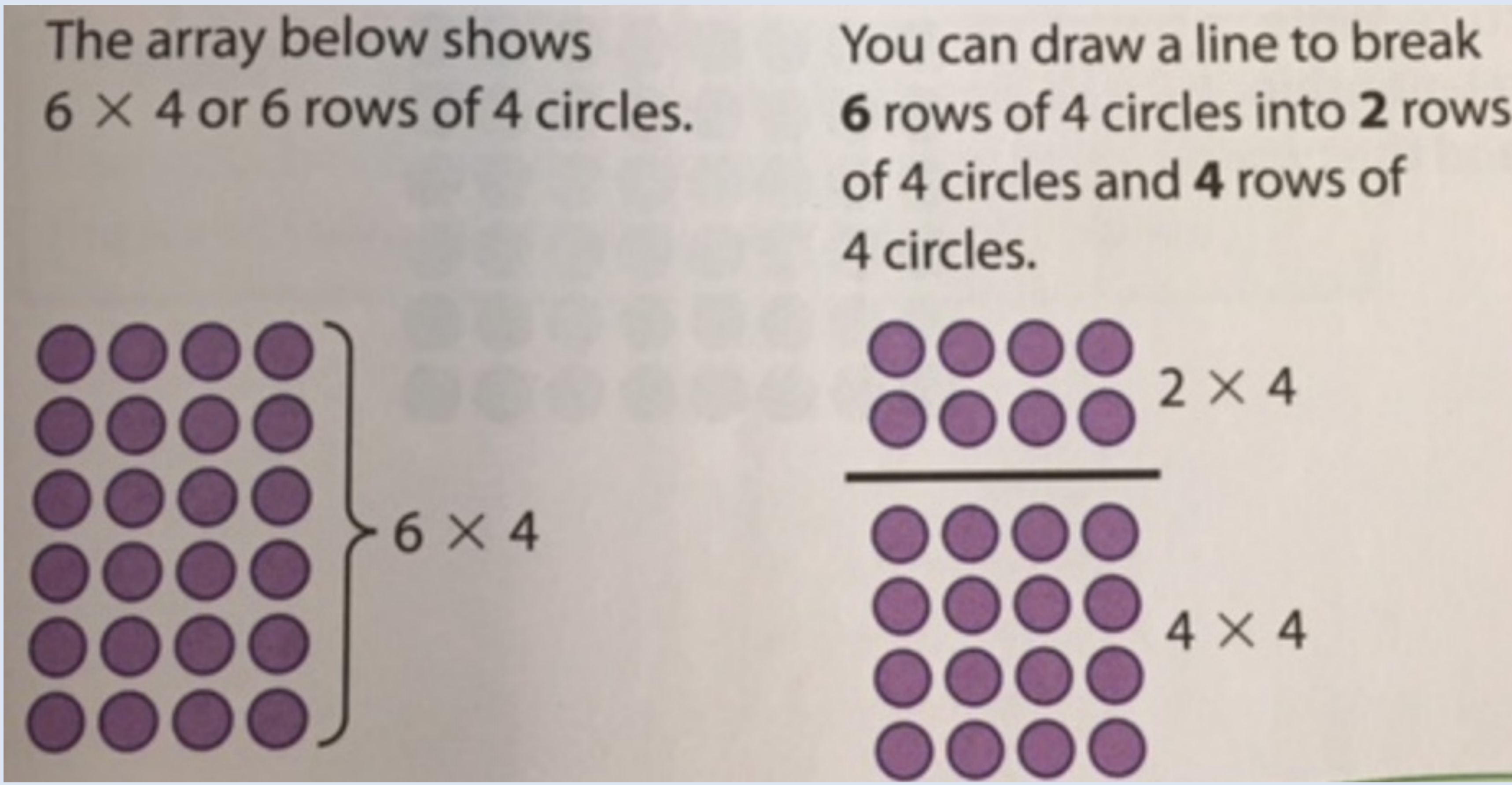
- How can folders help?



Common Multiplication and Division Situations

	Unknown Product	Group Size Unknown ("How many in each group?" Division)	Number of Groups Unknown ("How many groups?" Division)
	$3 \times 6 = ?$	$3 \times ? = 18$, and $18 \div 3 = ?$	$? \times 6 = 18$, and $18 \div 6 = ?$
Equal Groups	There are 3 bags with 6 plums in each bag. How many plums are there in all? *Measurement example*. You need 3 lengths of string, each 6 inches long. How much string will you need altogether?	If 18 plums are shared equally into 3 bags, then how many plums will be in each bag? *Measurement example*. You have 18 inches of string, which you will cut into 3 equal pieces. How long will each piece of string be?	If 18 plums are to be packed 6 to a bag, then how many bags are needed? *Measurement example*. You have 18 inches of string, which you will cut into pieces that are 6 inches long. How many pieces of string will you have?
Arrays,[4] Area[5]	There are 3 rows of apples with 6 apples in each row. How many apples are there? *Area example*. What is the area of a 3 cm by 6 cm rectangle?	If 18 apples are arranged into 3 equal rows, how many apples will be in each row? *Area example*. A rectangle has area 18 square centimeters. If one side is 3 cm long, how long is a side next to it?	If 18 apples are arranged into equal rows of 6 apples, how many rows will there be? *Area example*. A rectangle has area 18 square centimeters. If one side is 6 cm long, how long is a side next to it?
Compare	A blue hat costs \$6. A red hat costs 3 times as much as the blue hat. How much does the red hat cost? *Measurement example*. A rubber band is 6 cm long. How long will the rubber band be when it is stretched to be 3 times as long?	A red hat costs \$18 and that is 3 times as much as a blue hat costs. How much does a blue hat cost? *Measurement example*. A rubber band is stretched to be 18 cm long and that is 3 times as long as it was at first. How long was the rubber band at first?	A red hat costs \$18 and a blue hat costs \$6. How many times as much does the red hat cost as the blue hat? *Measurement example*. A rubber band was 6 cm long at first. Now it is stretched to be 18 cm long. How many times as long is the rubber band now as it was at first?
General	$a \times b = ?$	$a \times ? = p$, and $p \div a = ?$	$? \times b = p$, and $p \div b = ?$

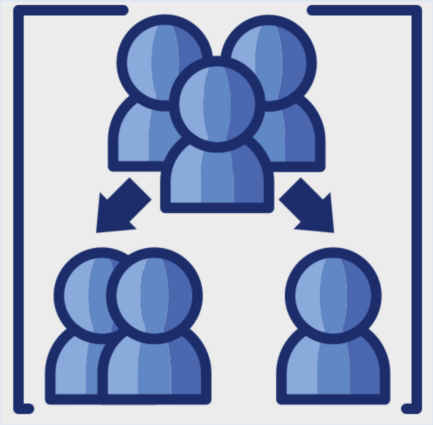
Taking Arrays to the next Level
Distributive Property in this Case!



The array below shows 6×4 or 6 rows of 4 circles.

You can draw a line to break **6** rows of 4 circles into **2** rows of 4 circles and **4** rows of 4 circles.

Multiplication

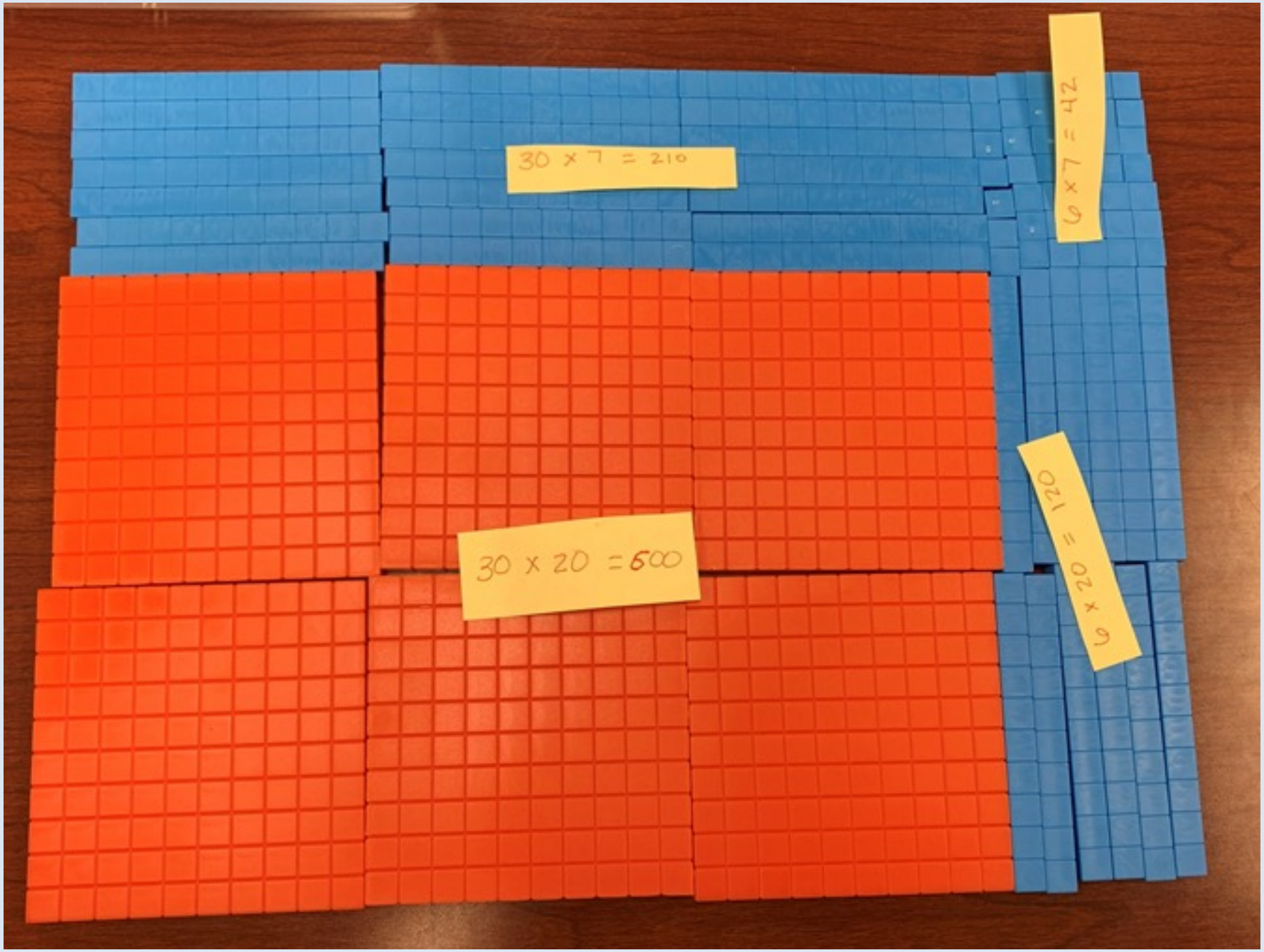


Multiply: 27 x 36

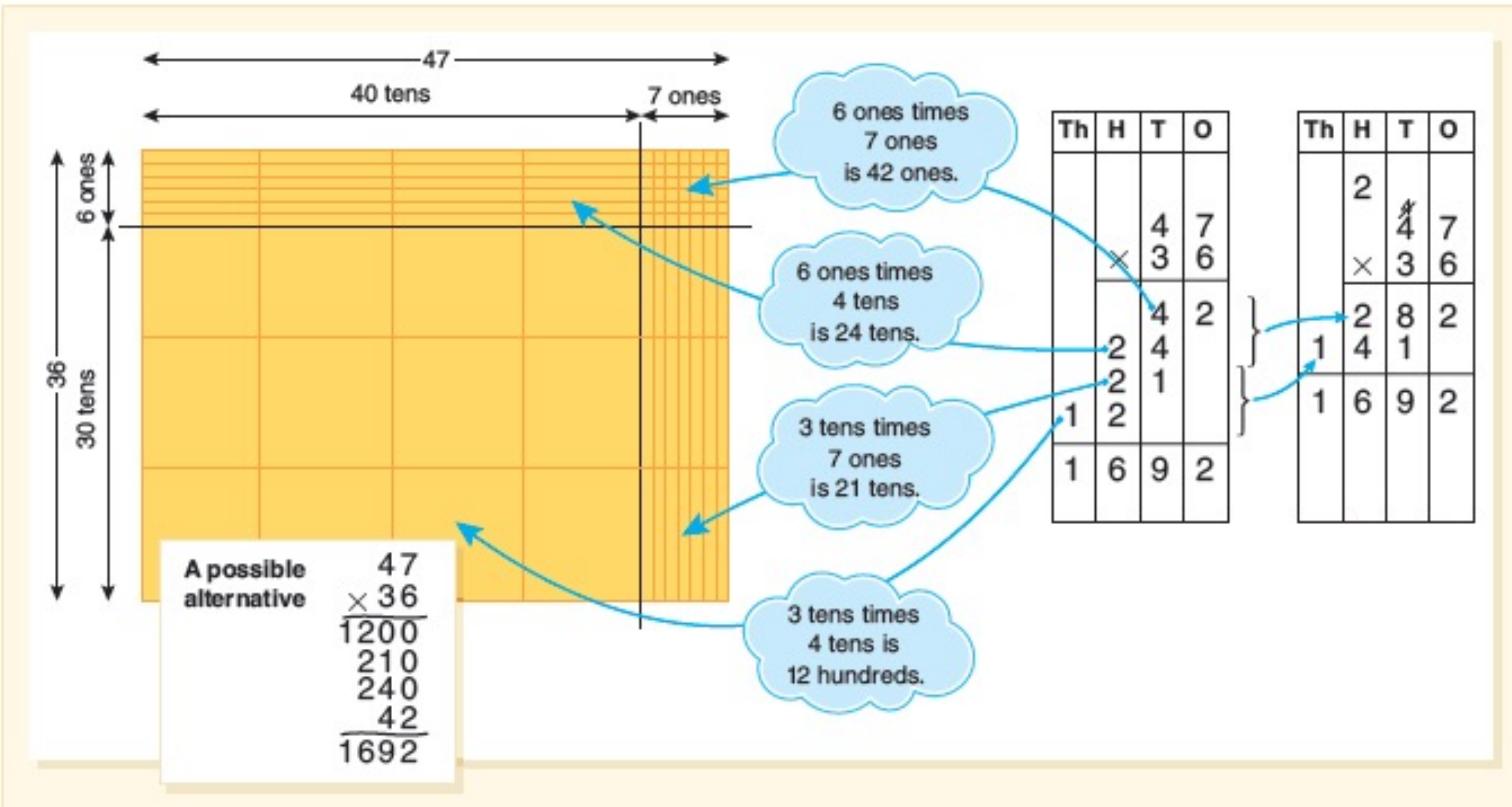
Using the base ten blocks to model the process you used. Model and sketch your process.

What is another process you could have used? Model and sketch it.

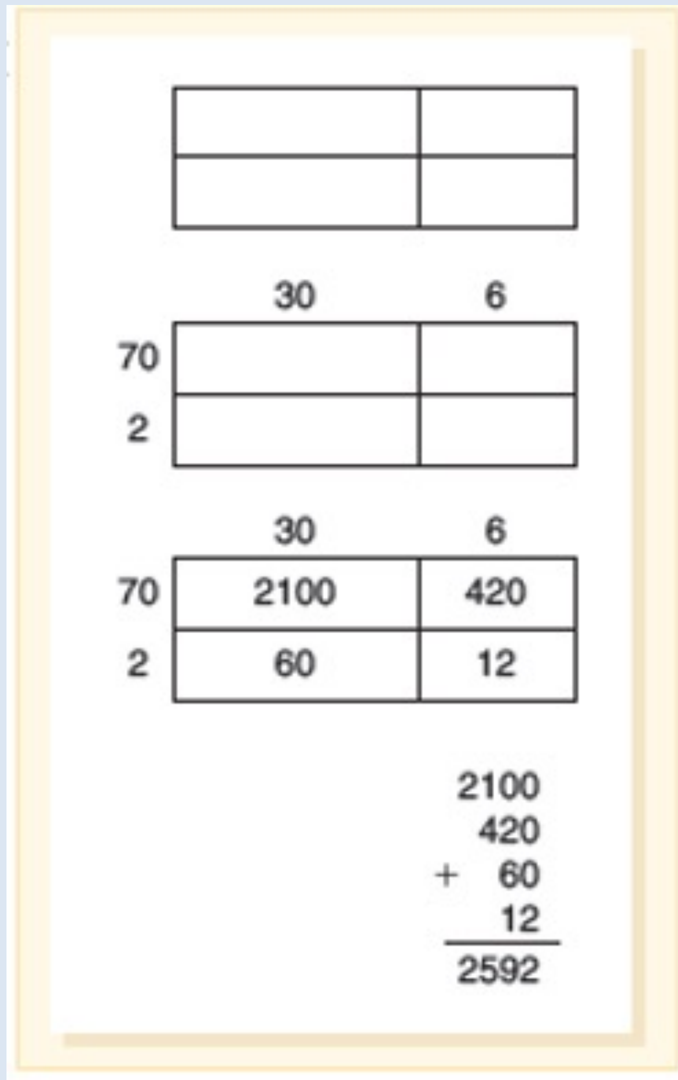
Two Digit Factors
Build the Area



Double-Digit Multiplication Problems
Build the Area! (CSA)



Move to Open Arrays – with Partial Products



Equal Sign

Are students acquiring an appropriate understanding of the equal sign when you ask them to explain their thinking?

Are they comfortable using operations on both sides of the equal sign and can use the meaning of equal as “is the same as?”

Van de Walle, J., Karp, K., & Bay Williams, J. (2013). *Elementary and Middle School Mathematics: Teaching developmentally*. New York: Pearson.

Equal Sign–Two Levels of Understanding

Operational: Students see the equal sign as signaling something they must "do" with the numbers such as "give me the answer."

Relational: Students see the equal sign as indicating two quantities are equivalent, they represent the same amount. More advanced relational thinking will lead to students generalizing rather than actually computing the individual amounts. They see the equal sign as relating to "greater than," "less than," and "not equal to."

Knuth, E. et. al (2008). The importance of equal sign understanding in the middle grades. *Mathematics Teaching in the Middle School*, *13*, 514–519.

Why is understanding the equal sign important?

Table 1 Percent of students at each grade level who provided each type of equal sign definition as their best definition ($n = 375$)

Best Definition	Grade 6	Grade 7	Grade 8
Relational	29	36	46
Operational	58	52	45
Other	7	9	8
No response/ don't know	6	3	1

Knuth, E. et. al (2008). The importance of equal sign understanding in the middle grades. *Mathematics Teaching in the Middle School*, *13*, 514–519.

Which number sentences would students say are True? False?

$27 = 27$

$22 + 5 = 4 + 23$

$25 + 1 = 27$

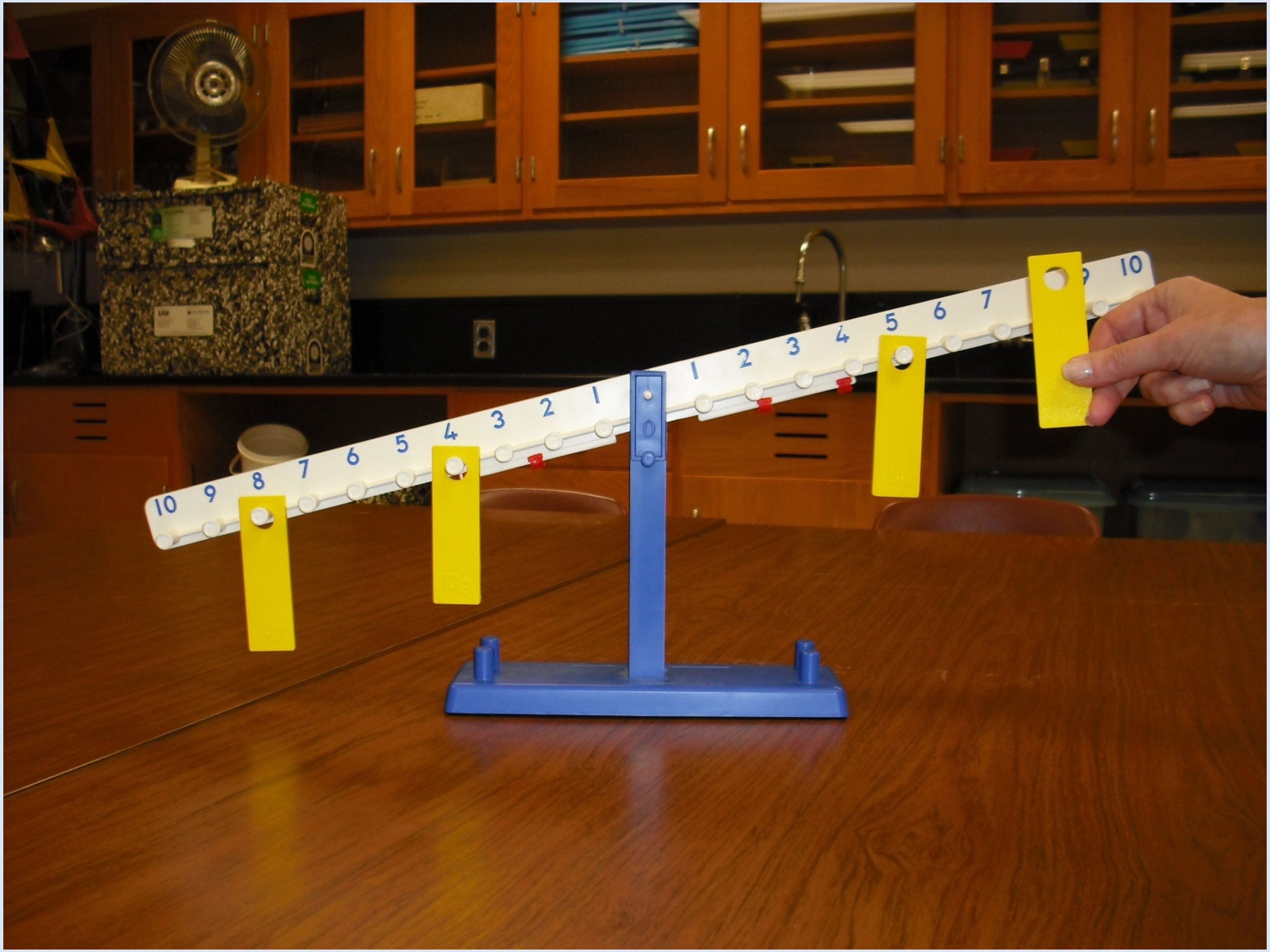
$27 = 22 + 5$

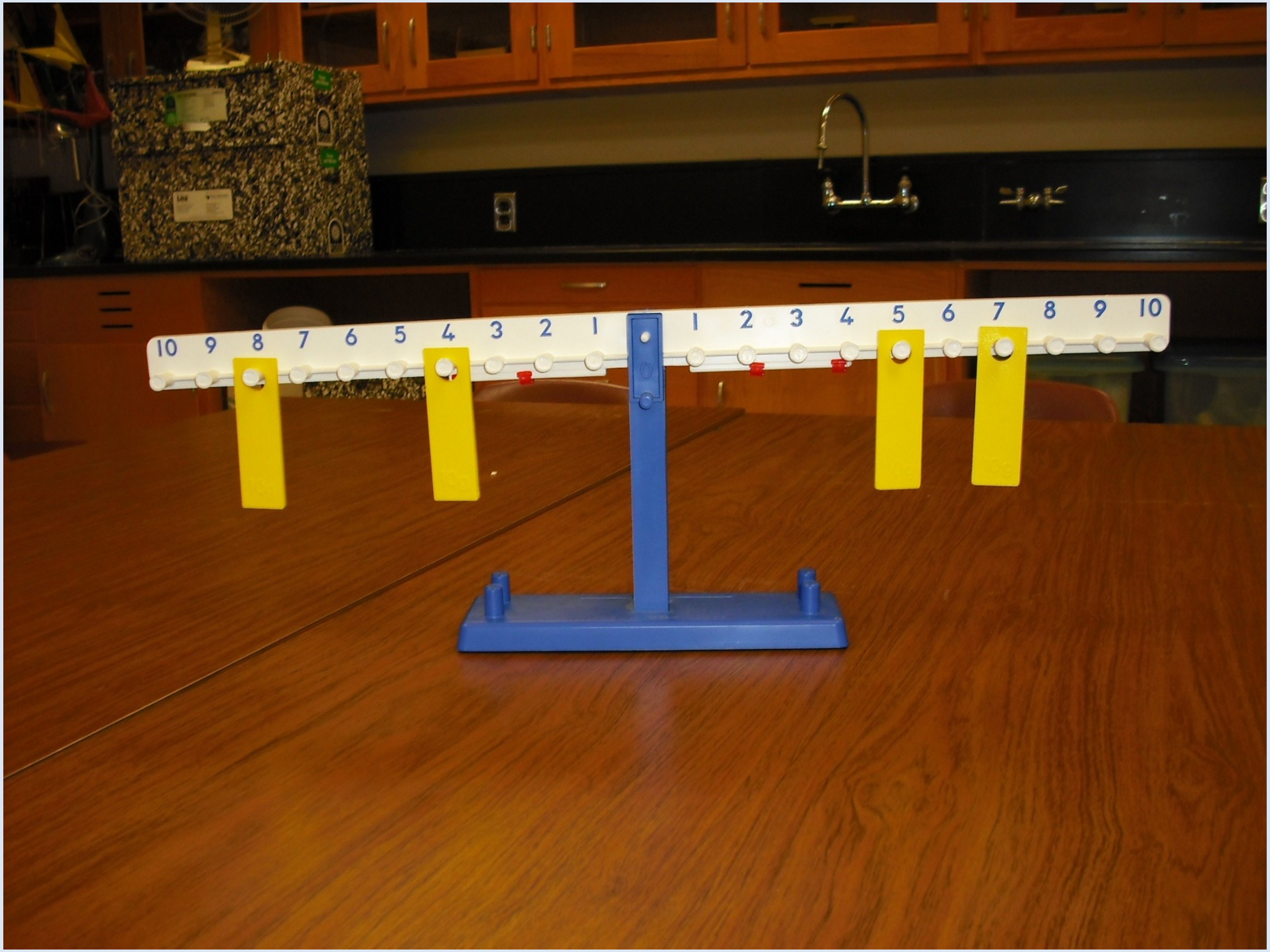
- Why?
- What would confuse them?

How is the equal sign interpreted?

Given the task:

$$8 + 4 = \square + 5$$


10 9 8 7 6 5 4 3 2 1
0
1 2 3 4 5 6 7
10


10 9 8 7 6 5 4 3 2 1 0 1 2 3 4 5 6 7 8 9 10

How did other students perform on this same problem?

Table 1 Students in various grades had different responses to make the sentence $8 + 4 = \square + 5$ true.

Responses	Grades		
	1 and 2	3 and 4	5 and 6
7	5%	9%	2%
12	58%	49%	10%
17	13%	25%	21%
12 and 17	8%	10%	2%

Source: Adapted from Falkner, Levi, and Carpenter 1999, p. 132

State Mathematics Standards

Grade 1:
Understand the meaning of the equal sign, and determine if equations involving addition and subtraction are true or false. *For example, which of the following equations are true and which are false?*

6 = 6 *7 = 8 – 1*
5 + 2 = 2 + 5 *4 + 1 = 5 + 2.*

What is the long term danger?

- If middle grades students think the equal sign means “put the answer next,” what happens when they move to algebraic equations such as $3x = 2x + 3$?

- Remember all variables are written in italics!!! Be precise.

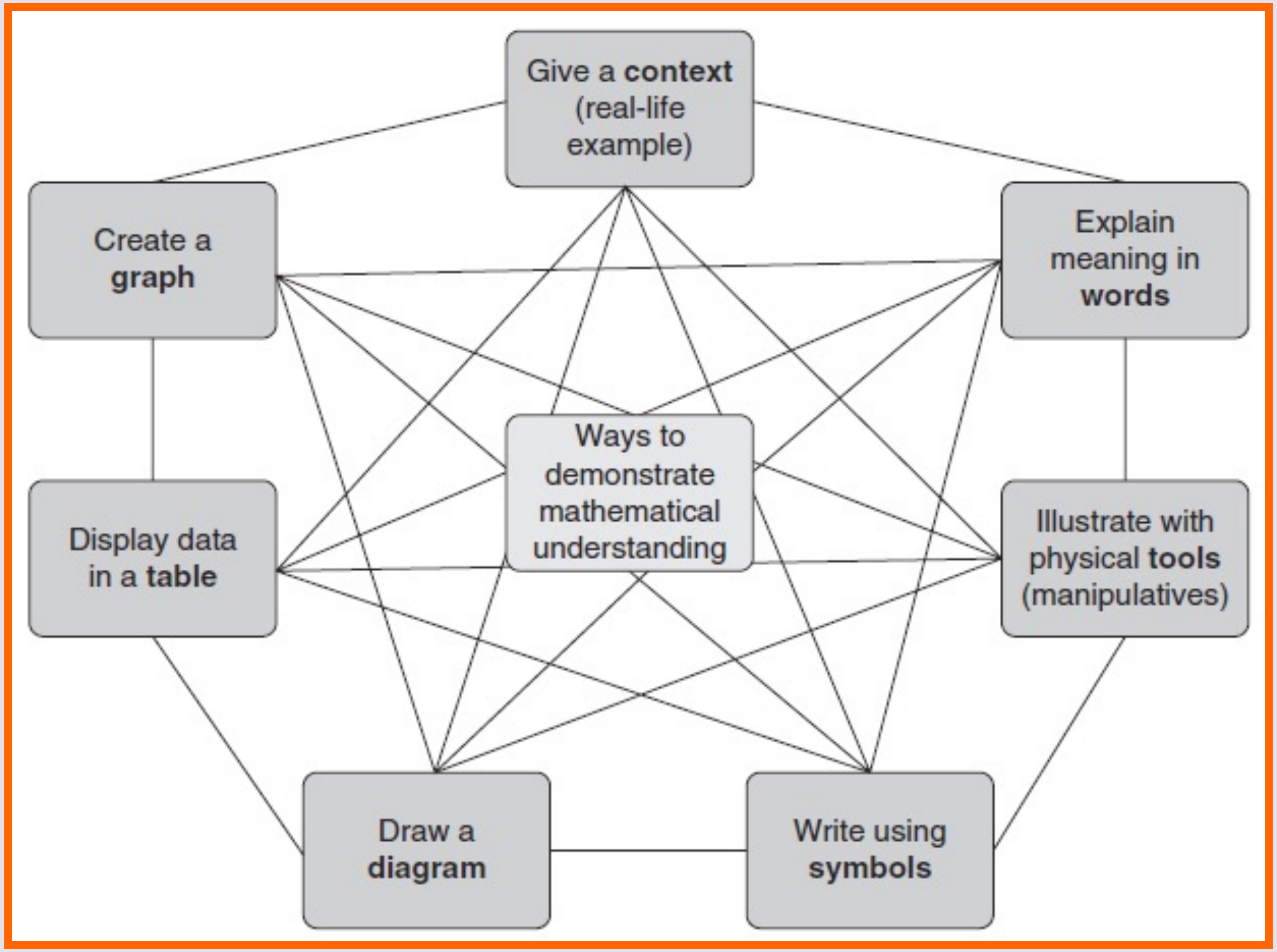
Teaching and Learning

- Telling isn't teaching
- Told isn't taught
- Listening isn't learning

Shifts in teaching and learning

Moving away from . . .	**To. . .**
Telling/showing how to do something	Building from concept to skill
Teacher-centric instruction	Student-centered instruction
Problem solving intermittently	Problem solving every day
A focus on only the answer	A focus on justifying and explaining
Showing the steps	Explaining the reasoning
Problems that require only fast calculations	Problems that require thinking

Multiple Representations



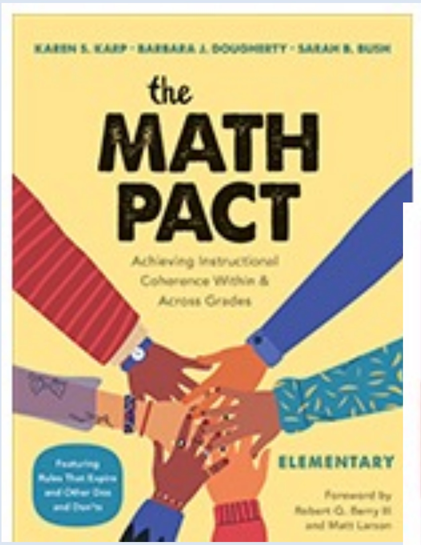
From *Teaching Student Centered Mathematics* (2014) (Volumes I - III)

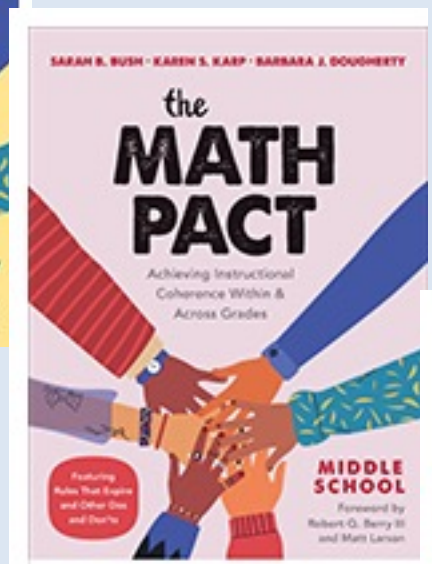
A Concluding Thought

We expect that the very best doctors will treat the most grievously ill patients. It should be no different in education. Great teachers have the skills to help the students who struggle the most. (Larson, 2011)

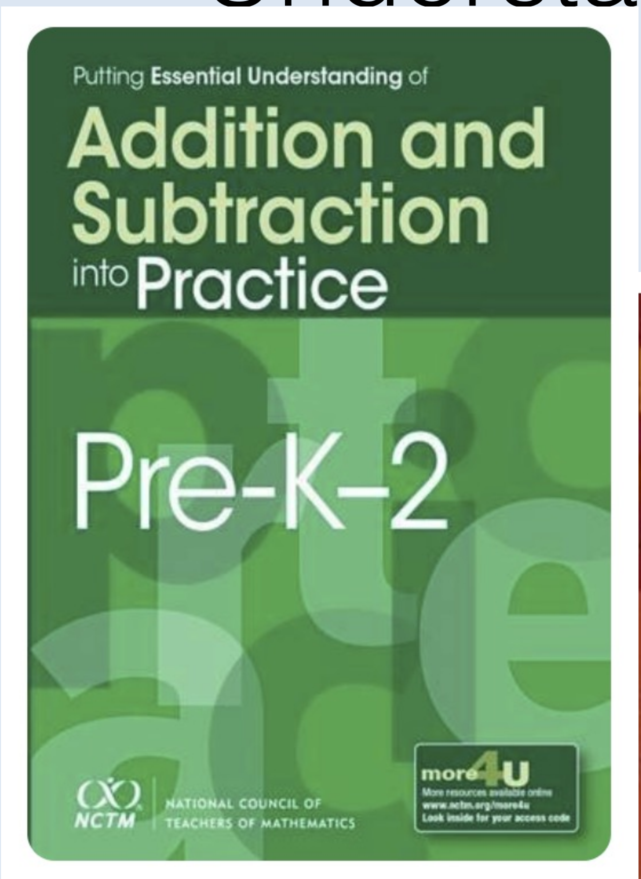
Larson, M. (2011).. Supporting students' achievement of the Common Core. NCTM Conference presentation. Indianapolis, IN.

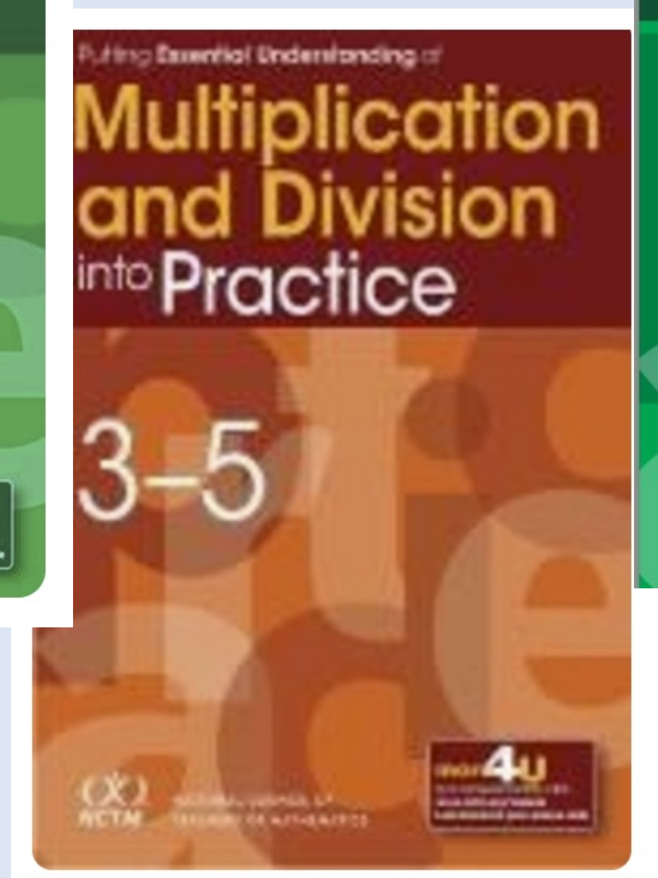
NCTM Resources: Math Pact Series

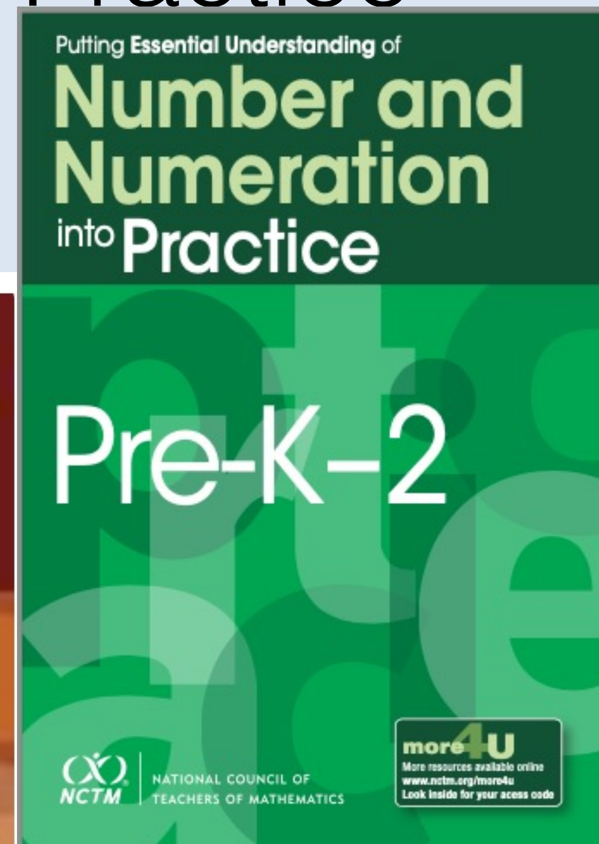




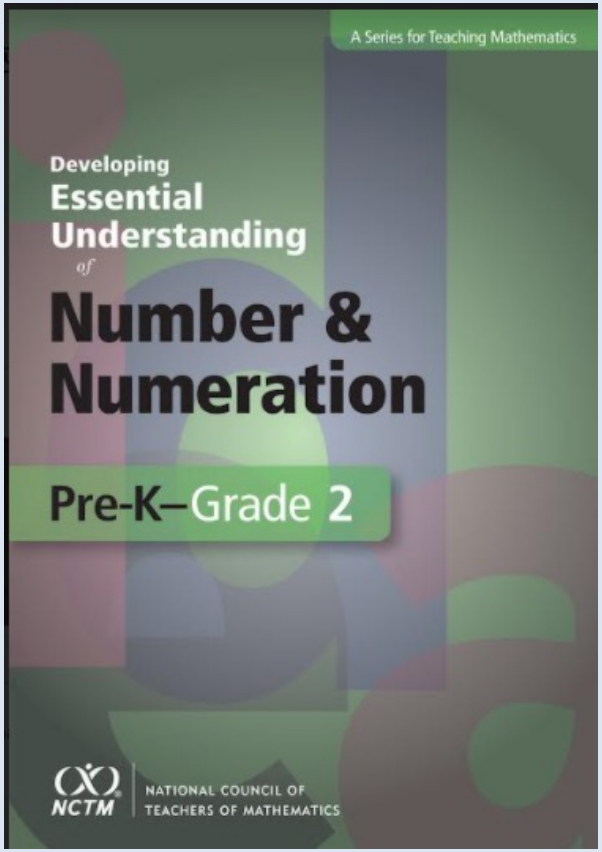
NCTM Resources: Putting Essential Understandings Into Practice

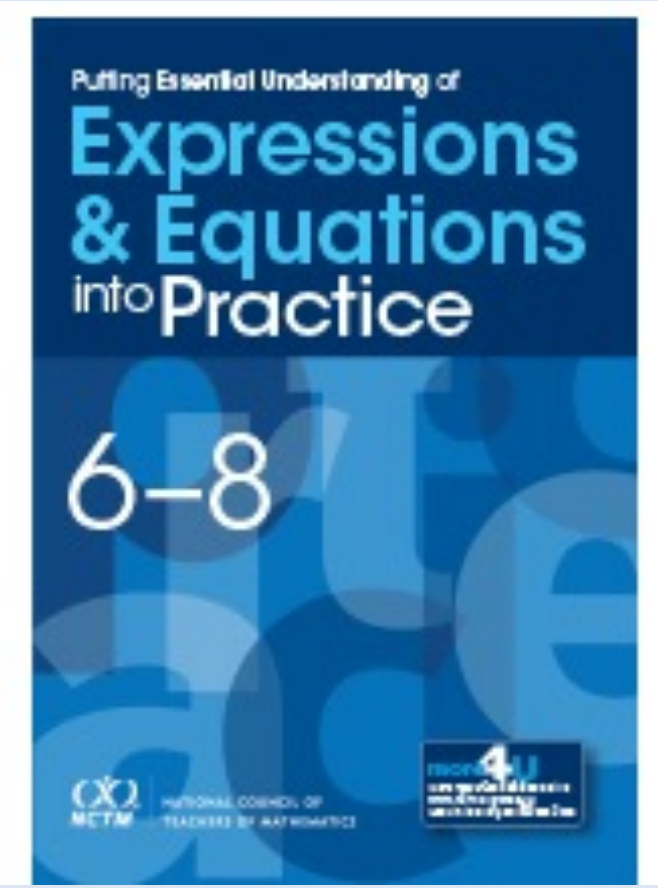






NCTM Resources: Putting Essential Understandings Into Practice





Questions?

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