NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS
CELEBRATING 100 YEARS

# Supporting Students' Pathways through the Cycle of Inquiry and Justification


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*#NCTM100*


*Department of Mathematical Sciences*

*University of Delaware*

*www.pisc.udel.edu*


*June 10, 2020*

**Peter Darker**
@IAmLeoGlaze

Understand that when Black ppl talk about racism, we're talking about institutional racism. Know that racism extends far beyond police brutality.

Housing
Healthcare/healthy food
Employment
$
Financial
Education
Criminal justice
Voting
Wealth building
etc...

Fight for it ALL.

2:27 PM · Jun 5, 2020 · Twitter for iPhone

# Systemic Racism

NONWHITE SCHOOL DISTRICTS GET

$23 BILLION

LESS THAN WHITE DISTRICTS

DESPITE SERVING THE SAME NUMBER OF STUDENTS

https://edbuild.org/content/23-billion#US

# Differential Opportunities to Learn

- The typical pedagogy faced by African American students is often directive, controlling, and debilitating.

From: *Opportunities to Learn Geometry: Listening to the Voices of Three African American High School Students* (Strutchens & Westbrook, 2009)

# Distrust
## in an Urban Mathematics Classroom

(Woodson & Id-Deen, 2016)

- "I'm not just talking, I am doing the assignment you gave us…Like when she sees us talking, and immediately says, 'stop talking,' she doesn't even think we're talking about math. **We're talking about math**…"

# Distrust

## in an Urban Mathematics Classroom

(Woodson & Id-Deen, 2016)

- "She doesn't trust that we will understand the math, so she always gives us worksheets. They are easy, and **I know I can do harder work.**"

# NCTM Presidents' Statement

As educators, teachers of mathematics, and a Council, we reiterate our position:

- We support the use of mathematics as an analytic tool to challenge power, privilege, and oppression.
- We encourage all educators to challenge systems of oppression that privilege some while disadvantaging others.
- We encourage educators to create socially and emotionally safe spaces for themselves, their students, and colleagues. (Wilkerson & Berry, 2020)

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BLACK LIVES MATTER

# Goals for the Session

1. Discuss opportunities-to-learn that support students' positive mathematical identities and agency
2. Explore ways to engage students in the Cycle of Inquiry & Justification
3. Consider classroom norms that must be in place to support Inquiry & Justification

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Join at
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# POLL

One Word, Two Questions

- How do you feel about teaching mathematical proof in one word?
- How do you think your students might feel about learning mathematical proof, in one word?

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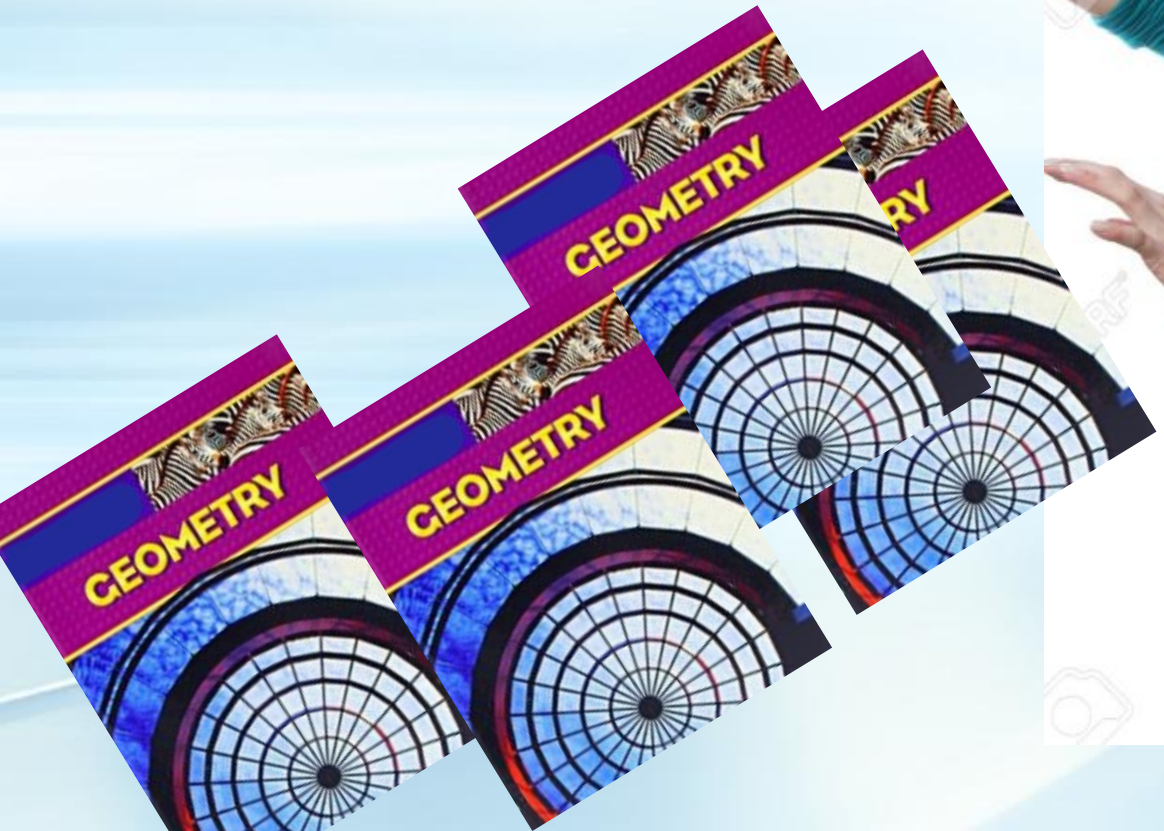



"As I was preparing to student teach, I looked forward to teaching mathematics but silently hoped not to teach geometry. Of course, that turned out to be the only subject my supervising teacher taught, but it was an amazing experience!"

- Trena Wilkerson, President's Message

SOUND OFF!

# Four Teacher-Friendly Postulates for Thriving in a Sea of Change

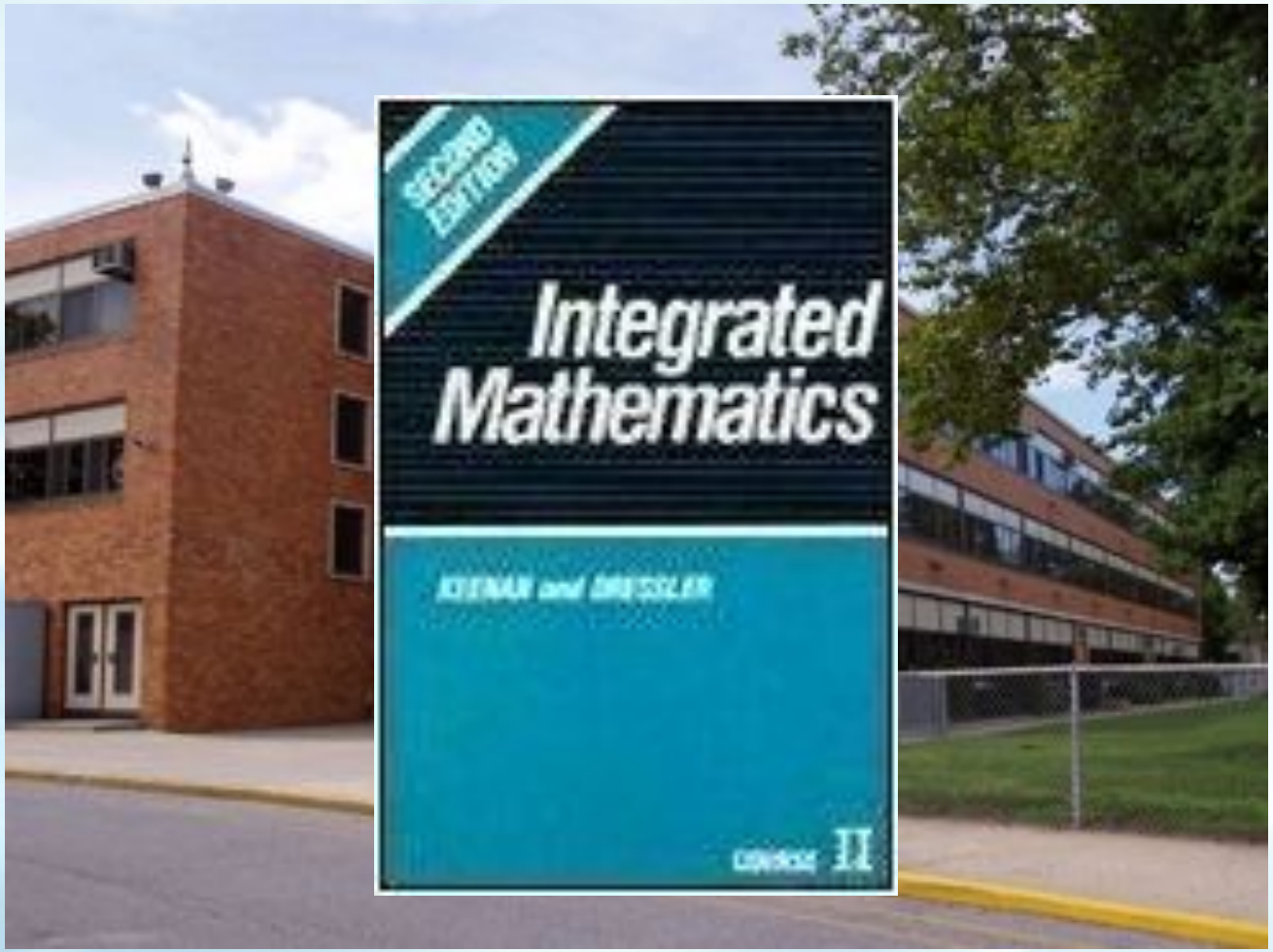
By Steven Leinwand
Connecticut Department of Education
Hartford, Connecticut

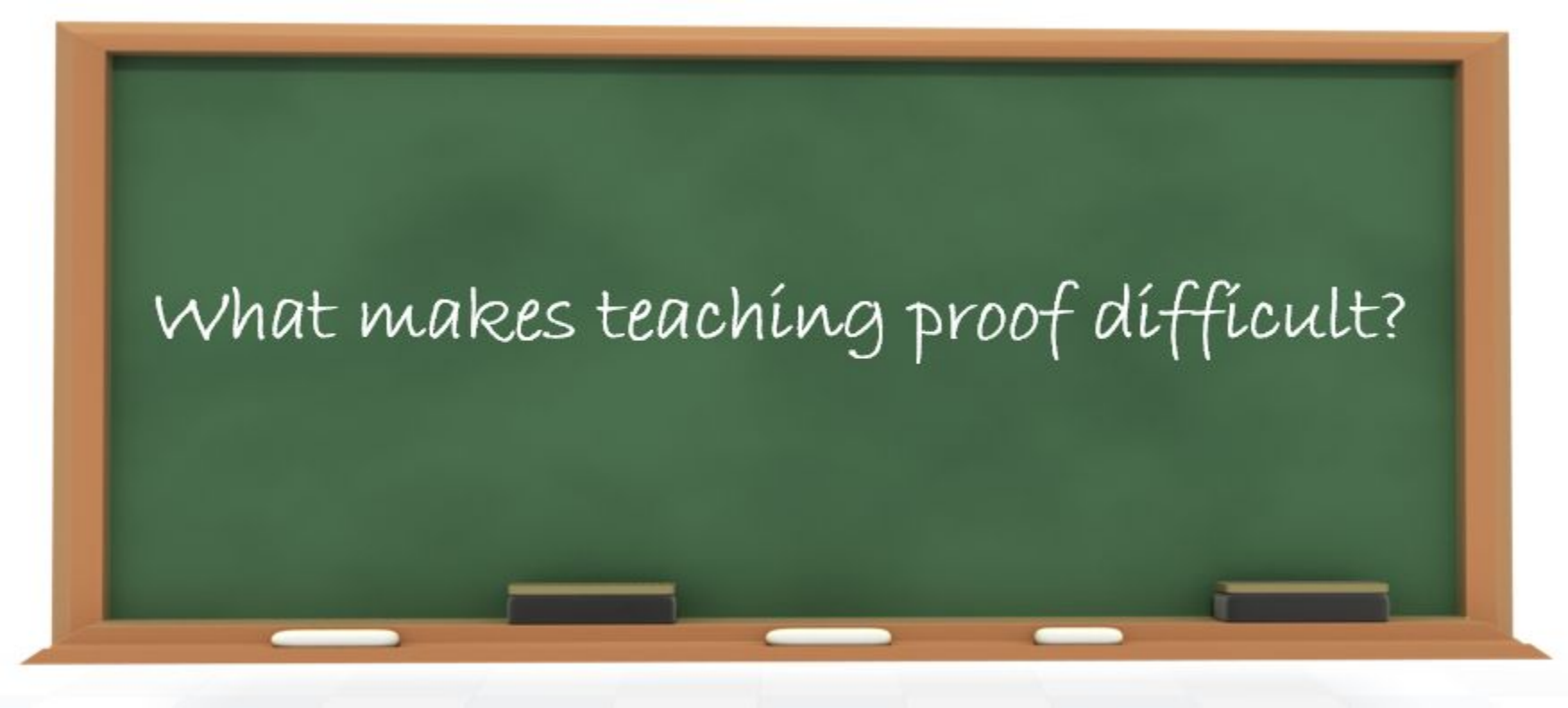
10 YEARS of MATHEMATICS TEACHER

- "It is unreasonable to ask a professional to change much more than 10% a year, but it is unprofessional to change by much less than 10% a year."

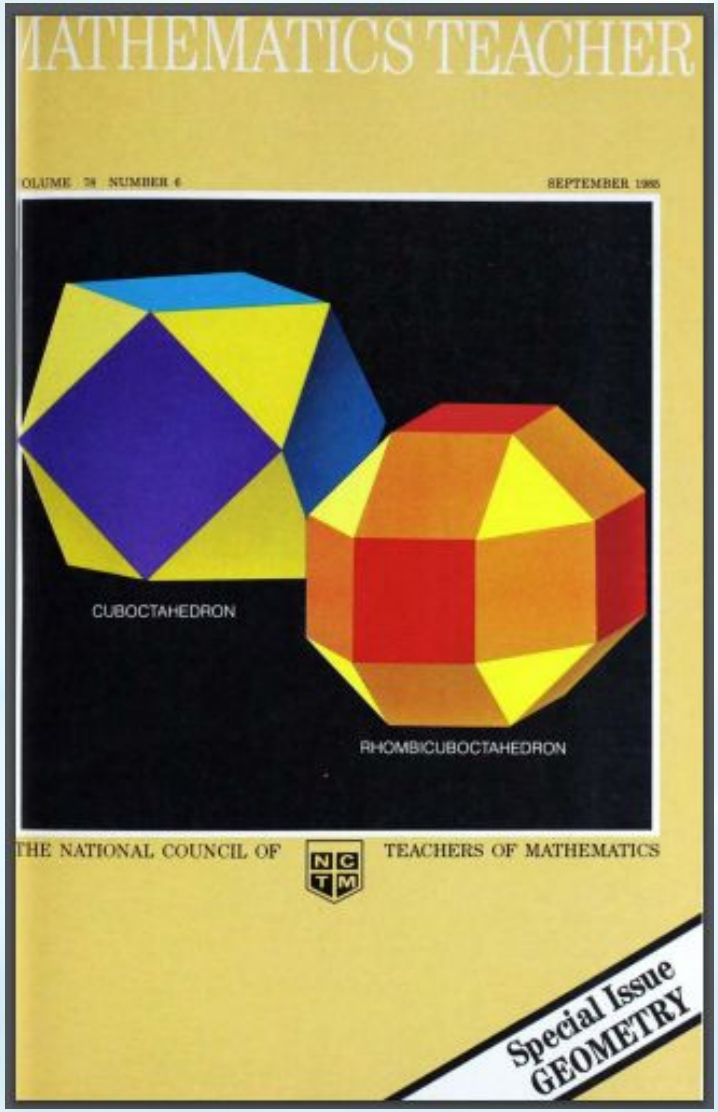
- "If you don't feel inadequate, you're probably not doing the job."

Originally Published in 1994

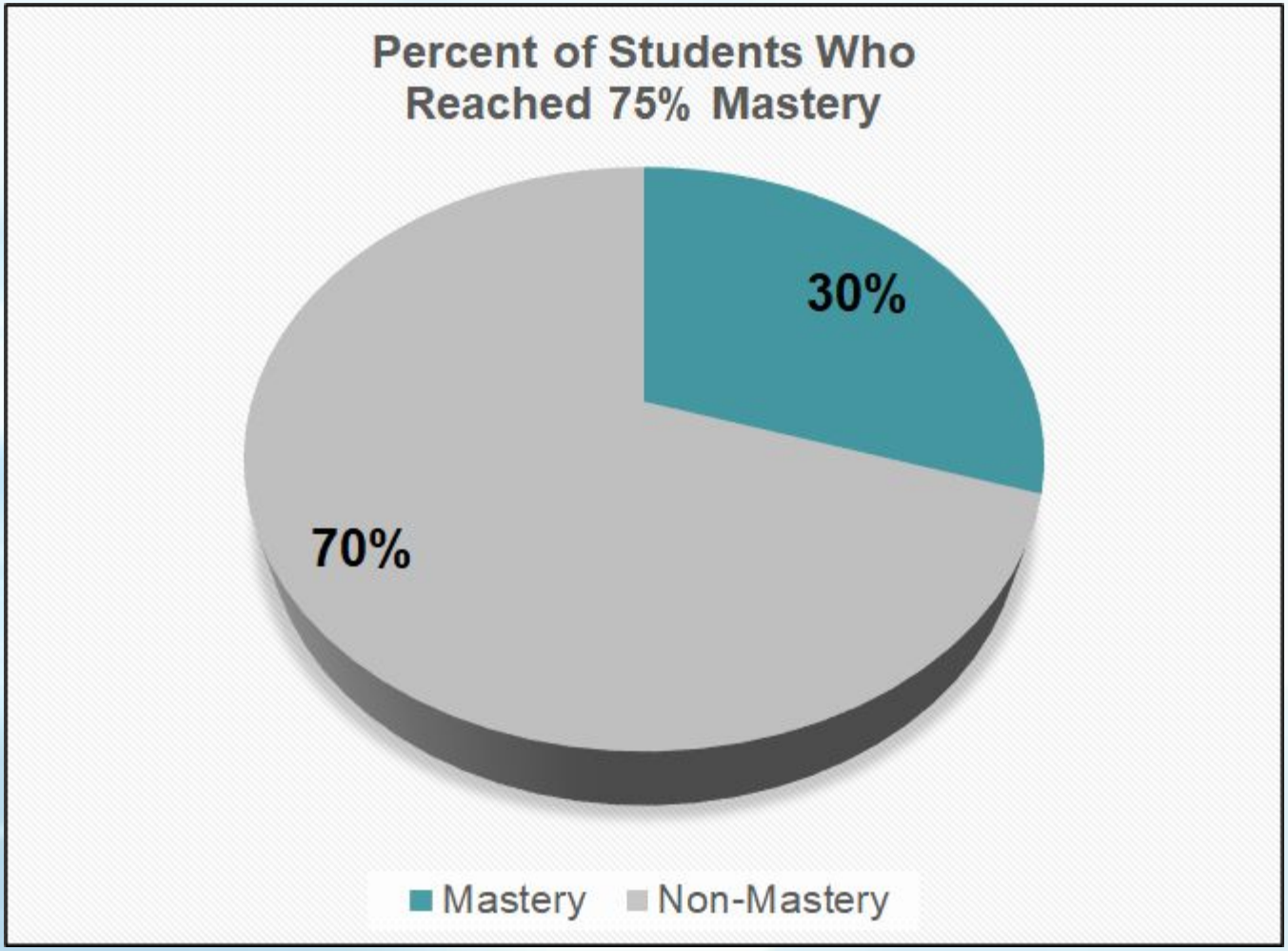

SECOND EDITION
Integrated Mathematics
KEENAN and DRESSLER
COURSE II

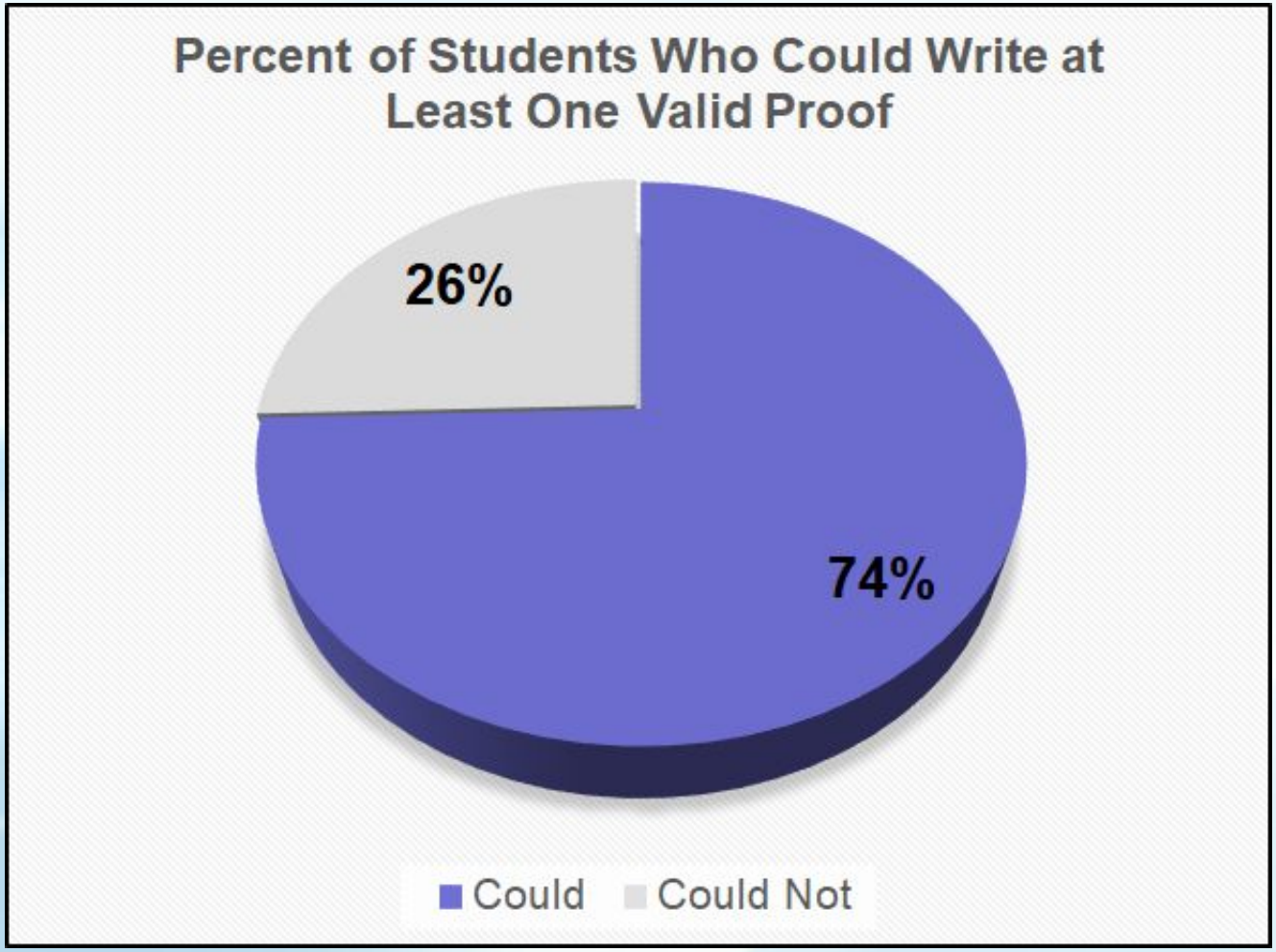

What makes teaching proof difficult?

# *How well do students write geometry proofs?*

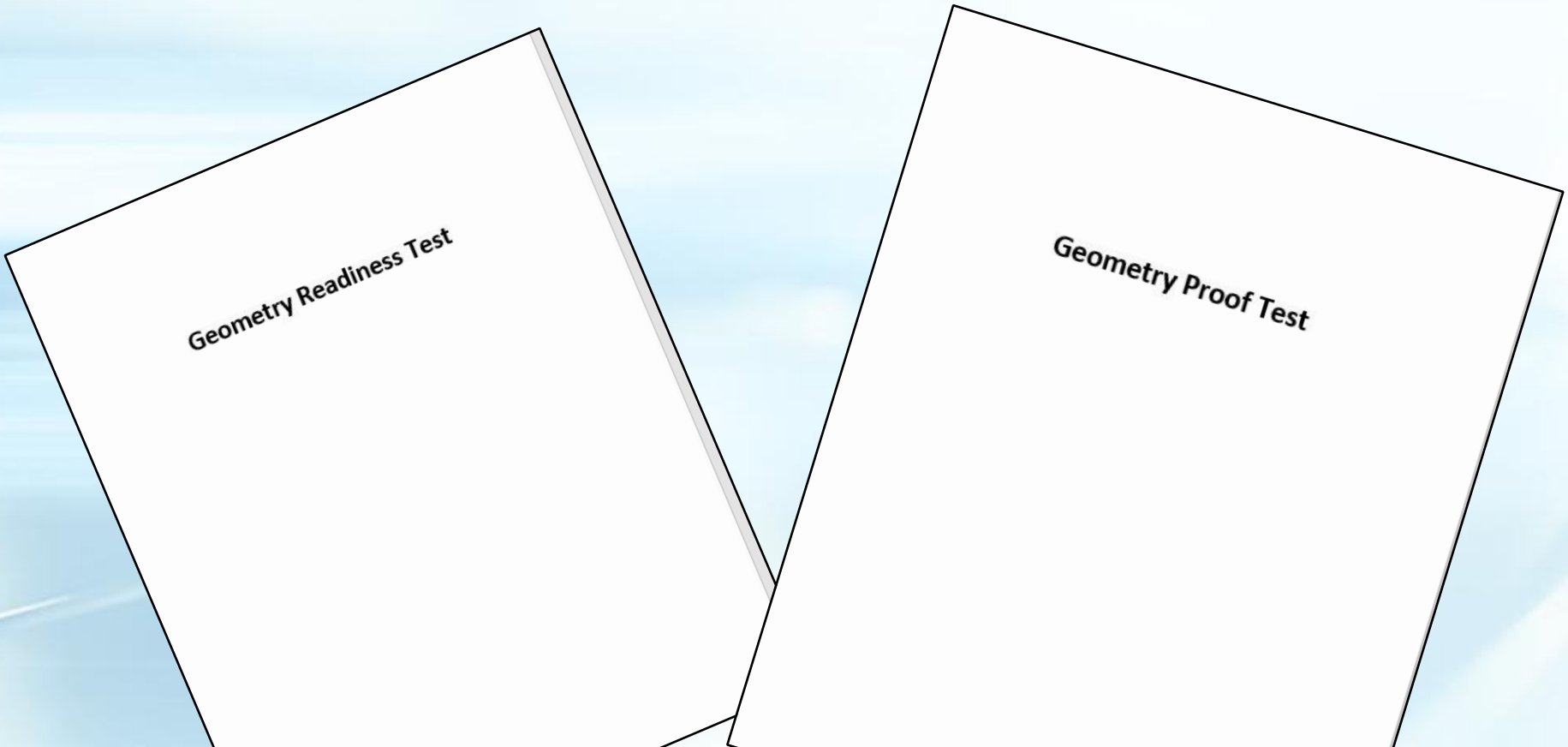


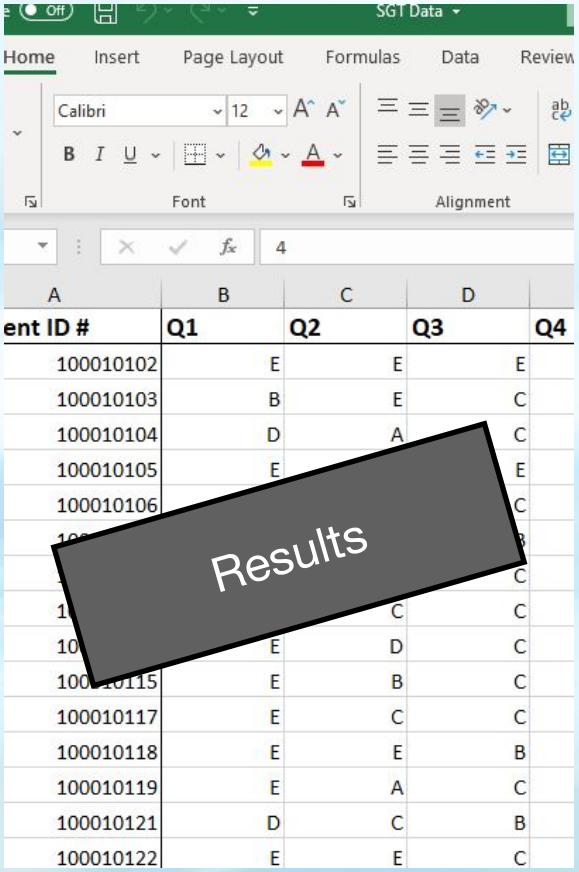

Sharon Senk (1985)


Percent of Students Who Reached 75% Mastery
30%
70%
Mastery
Non-Mastery


Percent of Students Who Could Write at Least One Valid Proof
26%
74%
Could
Could Not

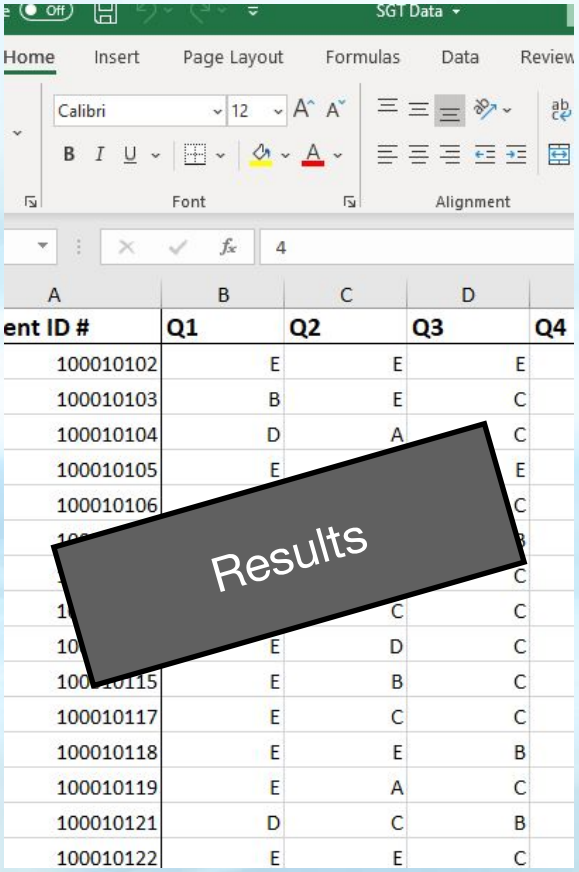
# How well can students do proof now compared to in 1985?






# RESEARCH FINDING

Today's students in our sample were more ready to engage in reasoning and proof, but the post-test outcomes were significantly worse than in 1985.




| ent ID # | Q1 | Q2 | Q3 | Q4 |
|---|---|---|---|---|
| 100010102 | E | E | E | |
| 100010103 | B | E | C | |
| 100010104 | D | A | C | |
| 100010105 | E | | E | |
| 100010106 | | | C | |
| | | | C | |
| 1 | | C | C | |
| 10 | E | D | C | |
| 100010115 | E | B | C | |
| 100010117 | E | C | C | |
| 100010118 | E | E | B | |
| 100010119 | E | A | C | |
| 100010121 | D | C | B | |
| 100010122 | E | E | C | |

# RESEARCH FINDING

Today's students in our sample were more ready to engage in reasoning and proof, but the post-test outcomes were significantly worse than in 1985. *Why might this be?*

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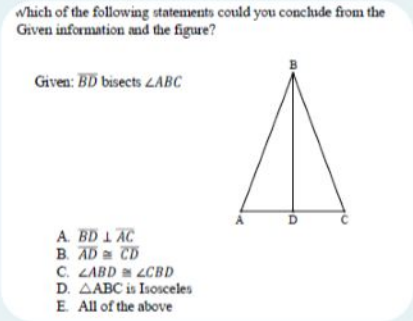
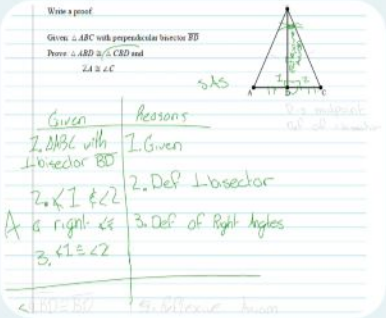
# *Proof in Secondary Classrooms* (PISC) Project (Cirillo, PI)


Which of the following statements could you conclude from the Given information and the figure?

Given: $\overline{BD}$ bisects $\angle ABC$



A. $\overline{BD} \perp \overline{AC}$
B. $\overline{AD} \cong \overline{CD}$
C. $\angle ABD \cong \angle CBD$
D. $\triangle$ABC is Isosceles
E. All of the above


Write a proof.

Given: $\triangle ABC$ with perpendicular bisector $\overline{BD}$

Prove: $\triangle ABD \cong \triangle CBD$ and

$\angle A \cong \angle C$

SAS

| Given | Reasons |
|---|---|
| 1. $\triangle ABC$ with ⊥bisector $\overline{BD}$ | 1. Given |
| 2. & 1 & $\angle 2$ | 2. Def ⊥bisector |
| A & right ∠s | 3. Def of Right Angles |
| 3. $\angle 1 \cong \angle 2$ | |

**Assessments**

Pre-Tests
(n = 1550 )

Post-Tests
(n = 1278)

**Classroom Observations**

(n = 294)

**PD Meetings**

(n = 35)

**Interviews**

Student
(n = 62)

Teacher
(n = 28)

Funded by the National Science Foundation

# Proof in School Mathematics is <u>not</u> too hard.

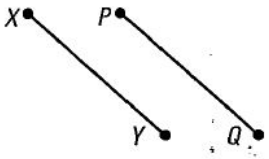
# Proof in School Mathematics is not too hard.

## It's too boring!

## It's too easy!

**EXAMPLE 1** ***Symmetric Property of Segment Congruence***

You can prove the Symmetric Property of Segment Congruence as follows.




**GIVEN** ▷ $\overline{PQ} \cong \overline{XY}$

**PROVE** ▷ $\overline{XY} \cong \overline{PQ}$

| Statements | Reasons |
|---|---|
| 1. $\overline{PQ} \cong \overline{XY}$ | 1. Given |
| 2. $PQ = XY$ | 2. Definition of congruent segments |
| 3. $XY = PQ$ | 3. Symmetric property of equality |
| 4. $\overline{XY} \cong \overline{PQ}$ | 4. Definition of congruent segments |

# Proof in School Mathematics is not too hard.

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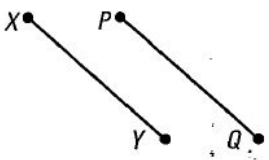


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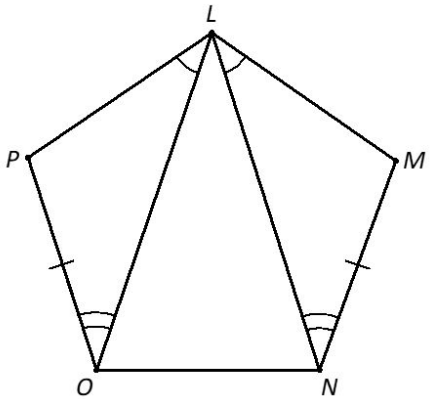
| 4. $\overline{XY} \cong \overline{PQ}$ | 4. Definition of congruent segments |

## It's too <u>easy</u>!

Given the figure shown, write a two-column proof to prove $\angle LON \cong \angle LNO$.



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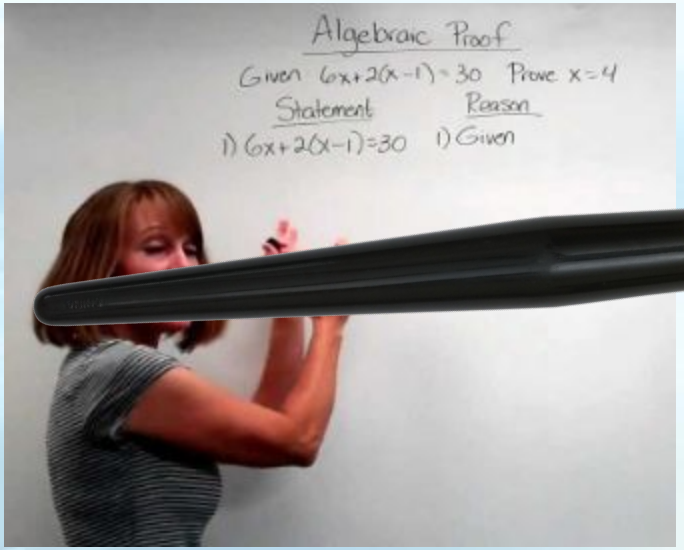
# Reasoning & Proof in School Mathematics Needs a Makeover

**BEFORE**




**AFTER**

# Learning to Think…

"Teachers and programs should focus on reasoning and sense-making. or students may not learn to think."

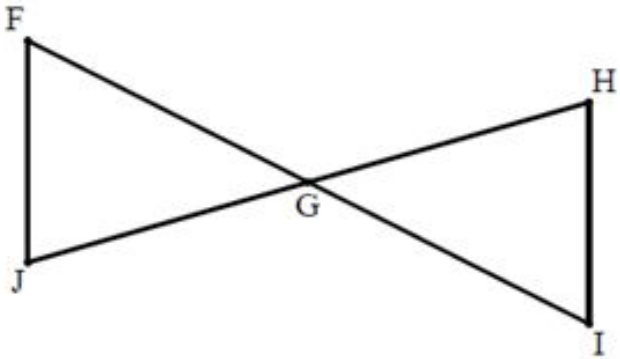
Cathy Seeley

# Student Task – Triangle Congruence Proof

**Given**: $\overline{FJ} \parallel \overline{HI}$
$\overline{FI}$ bisects $\overline{JH}$ at G

**Prove**: $\triangle JFG \cong \triangle HIG$

**Diagram**:




1. Write a plan for the proof.

2. Write a proof.

# Video 1: First Triangle Congruence Proofs

Go to:

https://tinyurl.com/PISC100

Password: PISC

Watch **Video 1** Only

In the chat, comment on what you notice or wonder

# Video 1: First Triangle Congruence Proofs

Go to:

https://tinyurl.com/PISC100

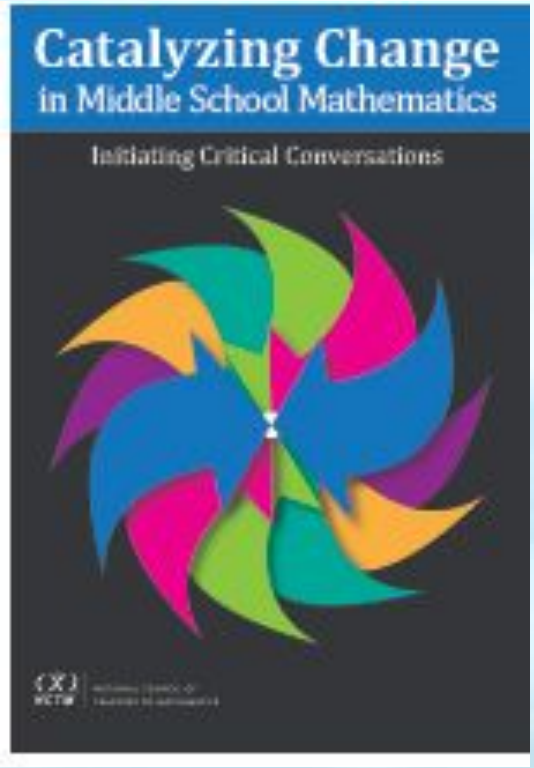
Password: PISC

Watch Video 1

In the chat, comment on what you notice or wonder

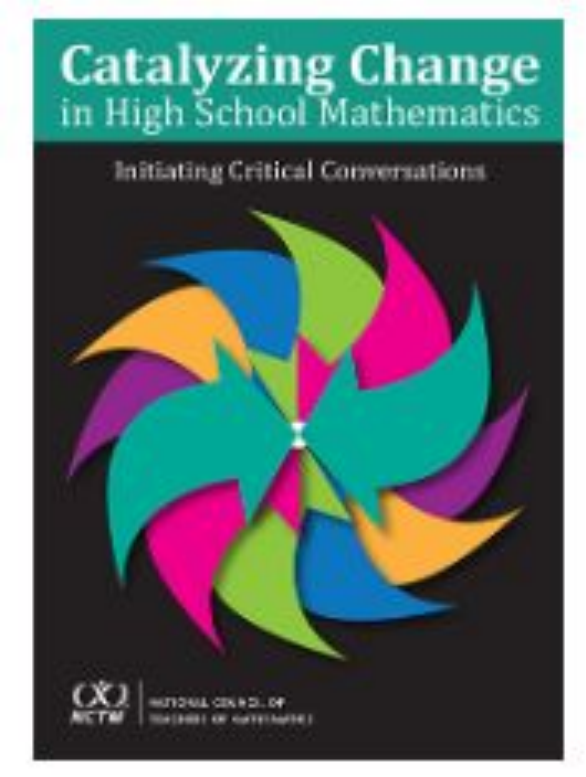
3 min

# Mathematical Reasoning in Middle School




- Mathematical reasoning includes the processes of:
  - conjecturing
  - generalizing
  - justifying
- Mathematical reasoning in middle school focuses largely on informal reasoning, setting the stage for proof in high school and beyond.

# Challenges of the High School Curriculum




- Lack of Focus
- Challenging to meet all goals:
  - Teach desired level of rigor
  - Develop mathematical processes and conceptual understanding
  - Cultivate positive mathematical identities
- Need to “cover” a vast number of standards

# Balancing Content & Processes

# What residue remains from your lessons?

## Objectives of High School Geometry:

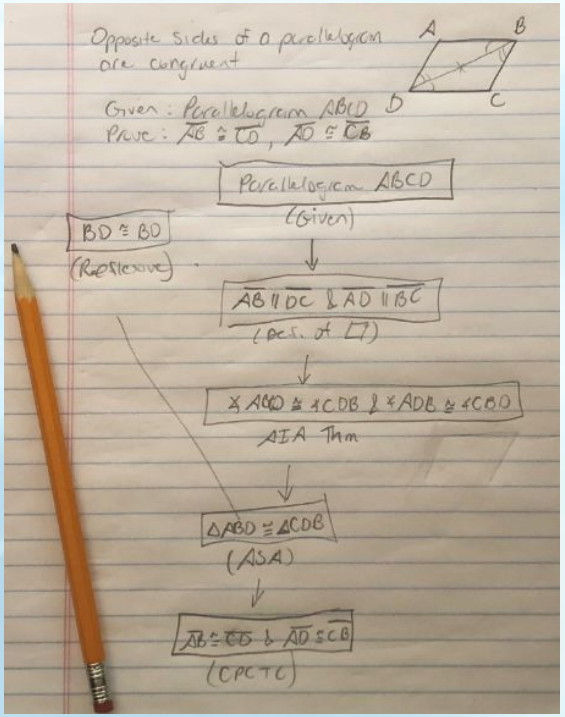
- to develop the ability to reason logically

- to develop knowledge of geometric facts and relations

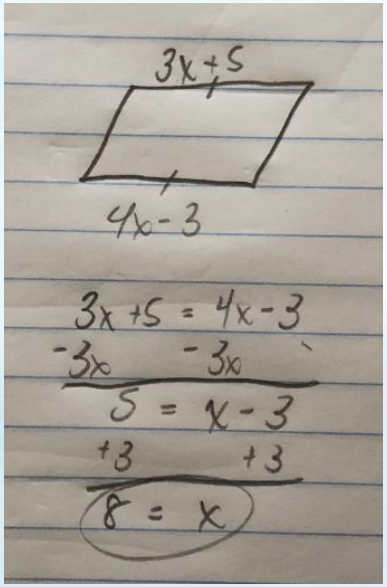
# What residue remains from your lessons?

## Objectives of High School Geometry:

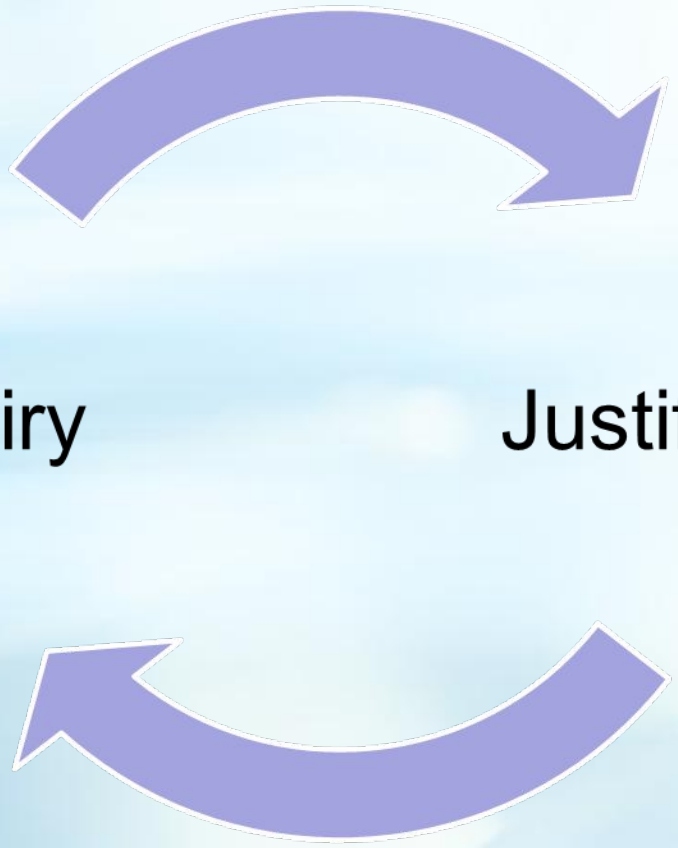
- to develop the ability to reason logically




- to develop knowledge of geometric facts and relations

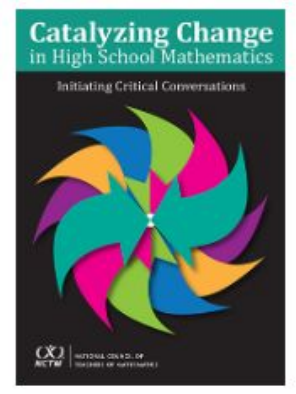


# Reasoning and Proof and the Cycle of Inquiry & Justification



# Reasoning and Proof and the Cycle of Inquiry & Justification

Inquiry




Justification

# The Cycle of Inquiry & Justification

**Inquiry**

- Exploration
- Discovery
- Conjecture

**Justification**

- Proof
- Verification

Bass (2015)

# The Cycle of Inquiry & Justification

**Inquiry**

- Exploration
- Discovery
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**Justification**

- Proof
- Verification

Inductive Reasoning

Deductive Reasoning

Bass (2015)

# The Cycle of Inquiry & Justification

**Inquiry**

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- Discovery
- Conjecture

**Justification**

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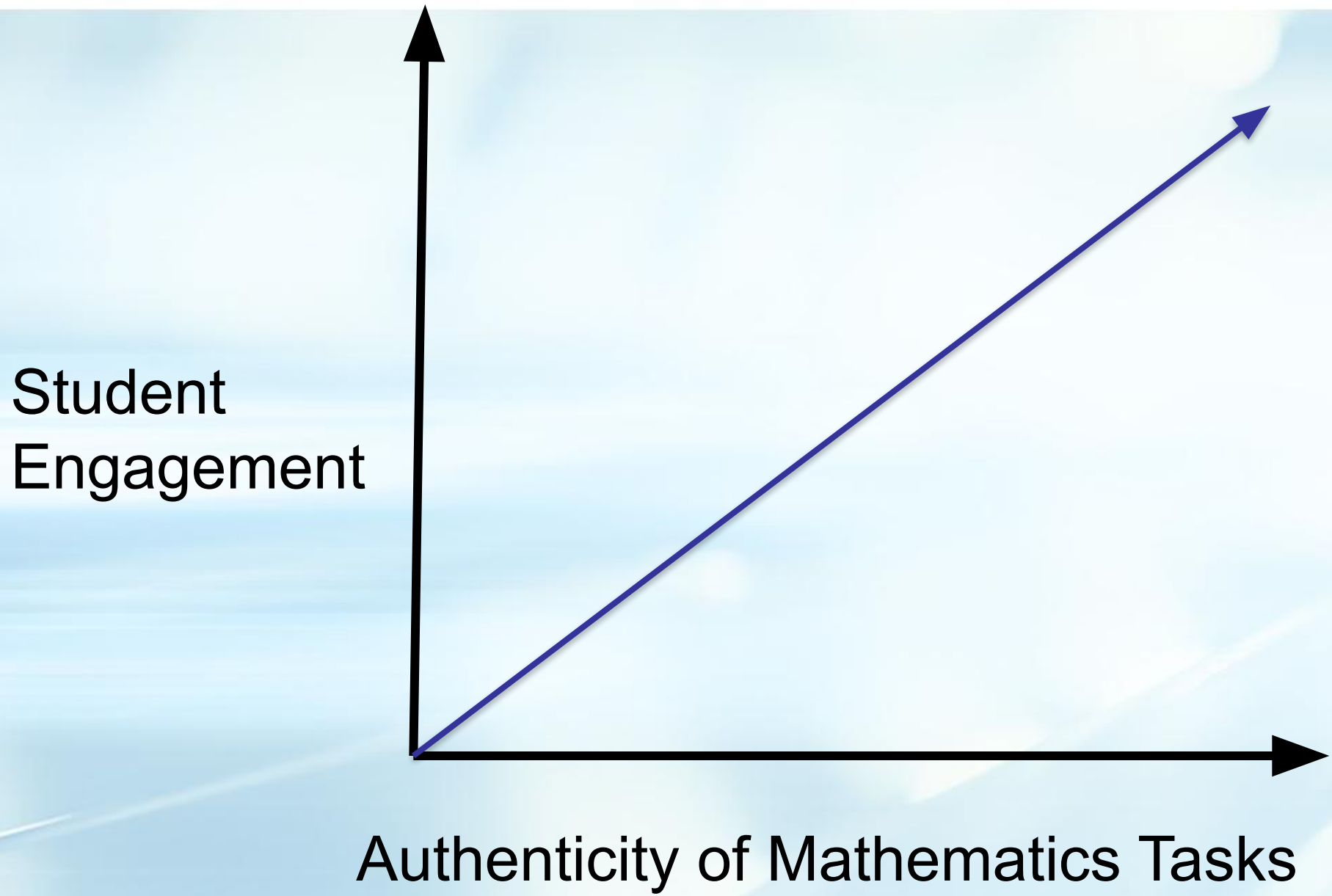
- Verification

Inductive Reasoning

Deductive Reasoning

Bass (2015)

# Reasoning & Proof in School Mathematics Needs a Makeover


Student
Engagement
Authenticity of Mathematics Tasks

# The Cycle of Inquiry & Justification

**Inquiry**

- Exploration
- Discovery
- Conjecture

**Justification**

- Proof
- Verification

Inductive Reasoning

Deductive Reasoning

Bass (2015)

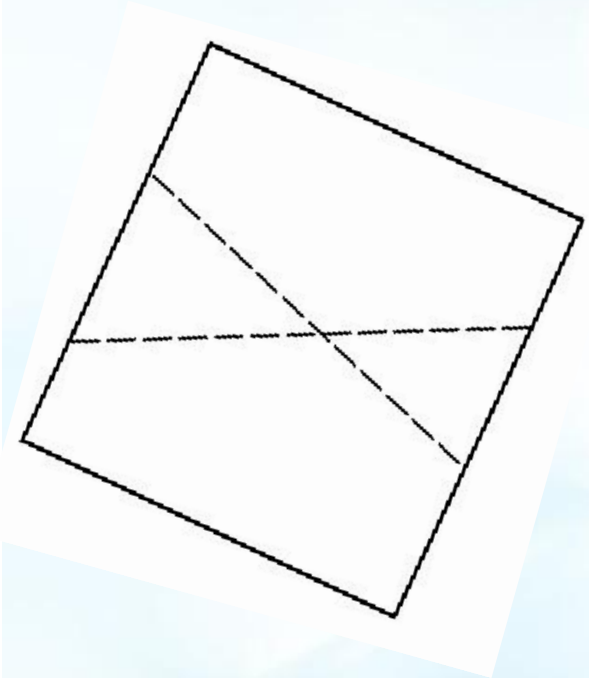
# Student Task – Investigating Intersecting Lines

## Investigate Intersecting Lines

### Student Task

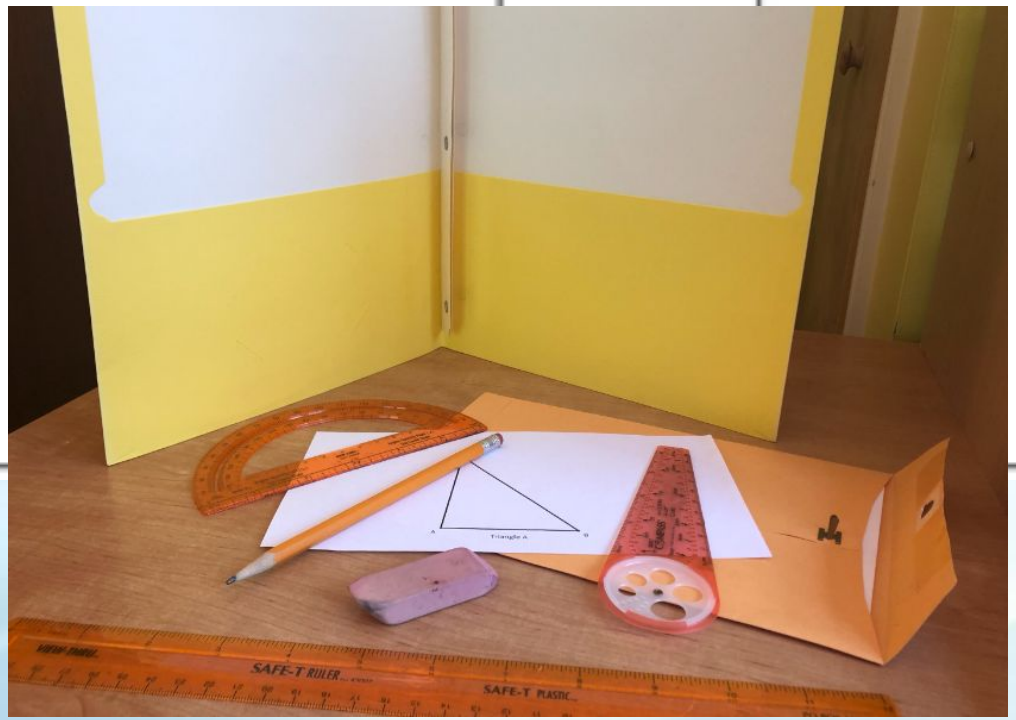
- Fold the paper in such a way that the creases form two intersecting lines.
- Measure all the angles with your protractor.
- Complete the first row of the table on your student sheet with a sketch of your intersecting lines and what you notice about the angles formed by the intersecting lines.

| Sketch of Construction | What do you notice?<br>What conjecture(s) do you have? |
|---|---|
| | |



# Student Task – Exploring Triangle Congruence

| Triangle | Triangle Congruence Strategy and Information Given | Congruent to Original (Yes or No?) | Other Notes |
|---|---|---|---|
| | | | |



# Videos 2 & 3 : Inquiry

Go to: https://tinyurl.com/PISC100

Password: PISC

Watch **Videos 2 & 3** Only

In the chat, comment on
what you notice or wonder

# Videos 2 & 3 : Inquiry

Go to: https://tinyurl.com/PISC100

Password: PISC

Watch **Videos 2 & 3** Only

In the chat, comment on what you notice or wonder

4 min

# The Cycle of Inquiry & Justification

**Inquiry**

- Exploration
- Discovery
- Conjecture

**Justification**

- Proof
- Verification

Inductive Reasoning

Deductive Reasoning

Bass (2015)

# Prior to Video 4

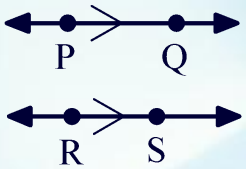
## Defining & Conjecturing

- Write a definition of parallelogram.
- Write a conjecture about the opposite angles of a parallelogram.

## Translating

- Line PQ is parallel to line RS

$\overleftrightarrow{PQ} \parallel \overleftrightarrow{RS}$




## Drawing Conclusions

- Given: $\overrightarrow{AD}$ bisects $\angle ABC$

What can you conclude?

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# *Mathematics Teacher* (Cirillo & Hummer, April, 2019)

# Addressing MISCONCEPTIONS in Secondary Geometry Proof

Michelle Cirillo and Jenifer Hummer

MIKE CHERNEY/GETTY IMAGES

**Use these ideas to diagnose and address common conceptual obstacles that inhibit students' success.**

Research suggests that teachers struggle to find effective ways to introduce proof. In 1940, in an article in this journal, Smith argued that being aware of student misconceptions in geometry is the first step in preparing to address the fundamental challenges of learning to prove. Through careful study, he identified and analyzed "three serious learning difficulties" that students have in connection with (1) a lack of familiarity with geometric figures, (2) not sensing the meaning of the if-then relationship, and (3) an inadequate understanding of the meaning of proof (p. 100). Smith found that when these difficulties were attended to explicitly, student results improved.

Years later, in 1985, Senk detailed findings from her study of 1520 students, in which she found that only 30 percent of students in a full-year geometry course that covered proof reached a 75 percent mastery of proof. Overall, 29 percent of the sample could not write a single valid proof. Consequently, Senk recommended (p. 455) that we must immediately look for more effective ways to teach proof in geometry, making the following suggestions:

- Pay special attention to teaching students to start a chain of reasoning.
- Place greater emphasis on the meaning of proof than we do currently.
- Teach students how, why, and when they can transform a diagram in a proof.





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# Inquiry – Exploration & Discovery

## Conjecturing About Parallelograms

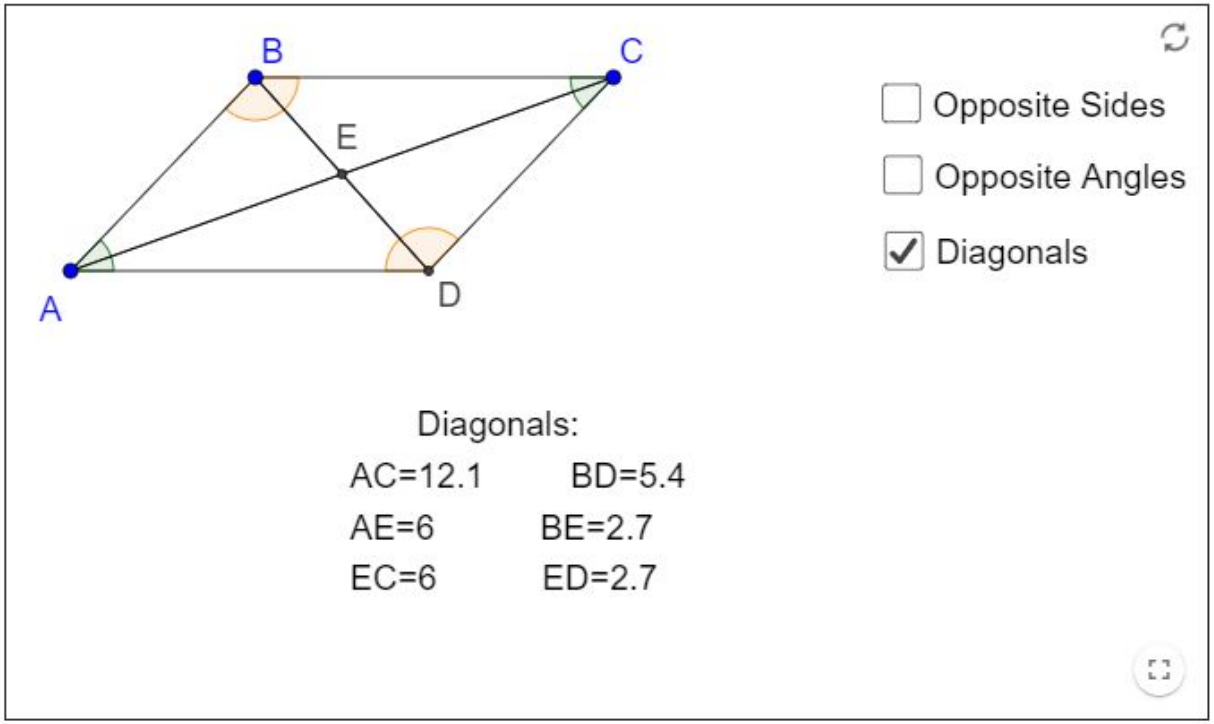
**Author:** Jenifer Hummer, Kelly Curtis

**Topic:** Parallelogram

GeoGebra

Move the blue points on the figure to change the size of the parallelogram. Use the check boxes on the right to display specific measurements.

Given: Parallelogram ABCD



# Student Task – Conjecturing About the Diagonals of a Parallelogram

1. **Use inductive reasoning to *explore* the situation.**
   Use the diagram from http://tinyurl.com/ConjecturingDay1 to fill in the table based on the various parallelogram configurations. Drag point on the parallelogram to create different configurations.

| Configuration 1 | Configuration 2 | Configuration 3 |
|---|---|---|
| AC = | AC = | AC = |
| BD = | BD = | BD = |
| AE = | AE = | AE = |
| EC = | EC = | EC = |
| BE = | BE = | BE = |
| ED = | ED = | ED = |

2. **Write a *conjecture.***
   Write a conjecture about the diagonals of a parallelogram.

# Video 4: Conjecturing about Parallelograms

Go to:

https://tinyurl.com/PISC100

Password: PISC

Watch Video 4

In the chat, comment on what you notice or wonder

# Video 4: Conjecturing about Parallelograms

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https://tinyurl.com/PISC100

Password: PISC

Watch Video 4

In the chat, comment on what you notice or wonder

3 min

# Student Task – Proving the Conjecture

3. **Develop a particular instance of your conjecture.**
   a. Rewrite your conjecture as a conditional statement (i.e., If…, then…).
   b. Sketch and label a diagram for the proof.
   c. Use the conditional statement to write the "Given" and "Prove" statements for the conjecture about the diagonals of a parallelogram.

**Conditional Statement:**

**Given**: **Diagram**:

**Prove**:

4. **Make a *plan* for the proof.** Sketch and mark a diagram, and brainstorm a plan for proving the conjecture.

# Student Task – Proving the Conjecture

3. **Develop a particular instance of your conjecture.**
   a. Rewrite your conjecture as a conditional statement (i.e., If…, then…).
   b. Sketch and label a diagram for the proof.
   c. Use the conditional statement to write the "Given" and "Prove" statements for the conjecture about the diagonals of a parallelogram.

Conjecture

**Conditional Statement:**

Conditional Statement (If…, then…)

**Given**:

**Diagram**:

Given, Figure, Prove

**Prove**:

4. **Make a *plan* for the proof.** Sketch and mark a diagram, and brainstorm a plan for p
conjecture.

Proof

# Conjecture → Conditional

The diagonals of a parallelogram bisect each other.

If _____,

then _____.

# Conjecture → Conditional

The diagonals of a parallelogram bisect each other.

If _____ If a quadrilateral is a parallelogram _____,
then _____ the diagonals bisect each other _____.

# Student Post-Test – Results

Rhionna conjectured that: *The diagonals of a parallelogram bisect each other.*
Rewrite Rhionna's conjecture as an "If _____, then _____" statement.

If _____,

then _____.

***Scores for the Post-Test Task***

| Score | 0 | 1 | 2 | 3 | 4 | Mean |
|---|---|---|---|---|---|---|
| Percentage (n = 647) | 25% | 12% | 17% | 19% | 18% | 1.76 |

# The diagonals of a parallelogram bisect each other.

| Sample Responses |
| --- |
| *If the diagonals of a parallelogram bisect each other, then the angles are congruent.* |
| *If the diagonals of a shape bisect, then it is a parallelogram.* |
| *If there are diagonals in a parallelogram, then they bisect each other.* |

# Student Interview Task

Determine what has been "Given" and what one would need to "Prove" for the theorem below. Draw, label, and mark a diagram that represents the theorem.

Conjecture: *If a quadrilateral is a parallelogram, then the diagonals bisect each other.*

**Given:** _____ **Diagram:**

**Prove:** _____

# Student Interview Task - Results

Determine what has been "Given" and what one would need to "Prove" for the theorem below. Draw, label, and mark a diagram that represents the theorem.

Conjecture: *If a quadrilateral is a parallelogram, then the diagonals bisect each other.*

**Given:** _____ **Diagram:**

**Prove:** _____

***Scores from Conventional Approach***

| Score | 0 | 1 | 2 | 3 | 4 | Mean |
|---|---|---|---|---|---|---|
| Percentage (n = 14) | 29 | 14 | 21 | 36 | 0 | 1.64 |

***Scores from Inquiry & Justification Approach***

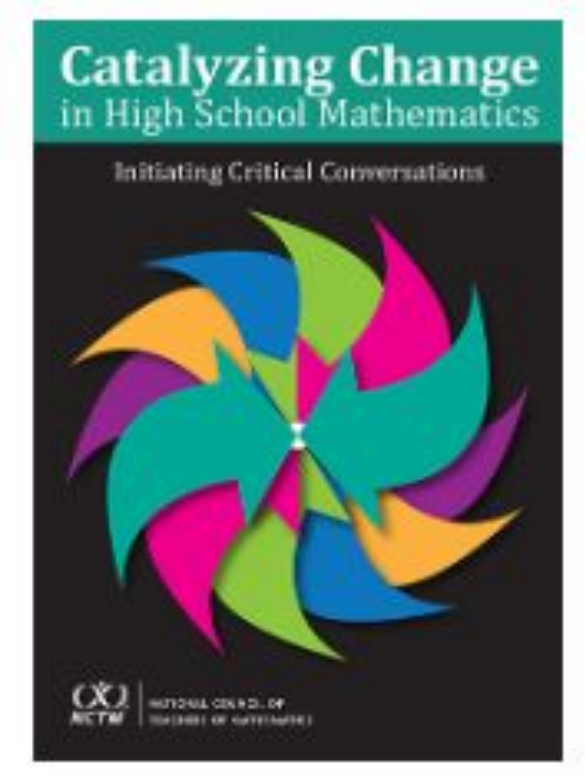
| Score | 0 | 1 | 2 | 3 | 4 | Mean |
|---|---|---|---|---|---|---|
| Percentage (n = 14) | 7 | 14 | 7 | 14 | 57 | 3 |

# Student Interview Task – Common Errors

*If a quadrilateral is a parallelogram, then the diagonals bisect each other.*

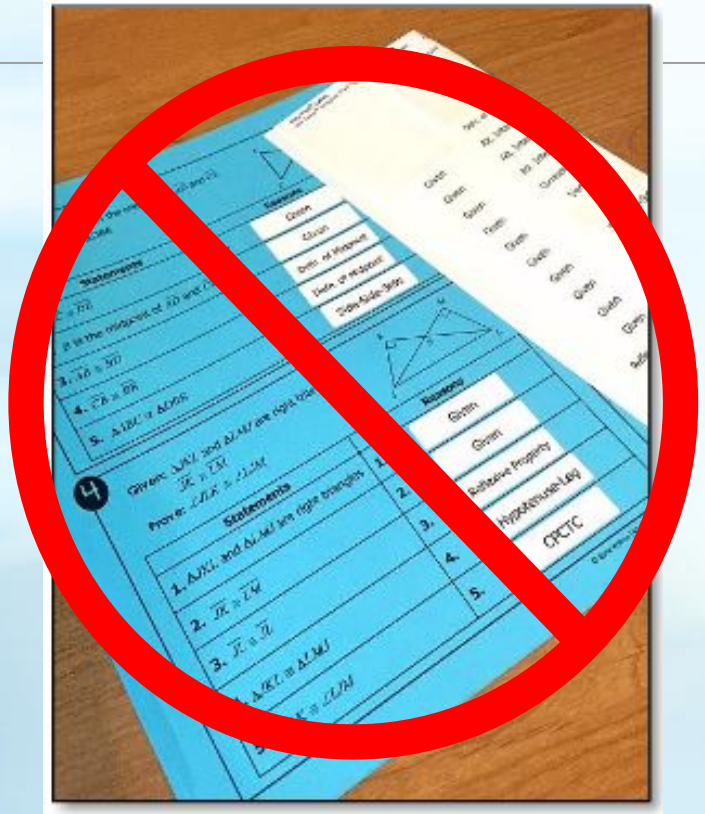
| Sample Responses |
| --- |
| *Given: Quadrilateral ABCD* |
| *Given:* $\overline{AB} \cong \overline{DC}$ *and* $\overline{AD} \cong \overline{BC}$ |
| *Given: If a quadrilateral is a parallelogram*<br>*Prove: The diagonals bisect each other.* |

# Inquiry & Justification are Central to Mathematical Thinking




"Students should recognize the process of establishing valid mathematical statements as the central act of doing mathematics, and they should see this process as the entire cycle of inquiry and justification." (p. 40)

| What I had | What I wanted |
| --- | --- |
| • Direct instruction<br>• “Show and Tell”<br>• Cut-and-paste activities | • Active participation<br>• Collaboration<br>• Inquiry-based learning |



# What Changed?

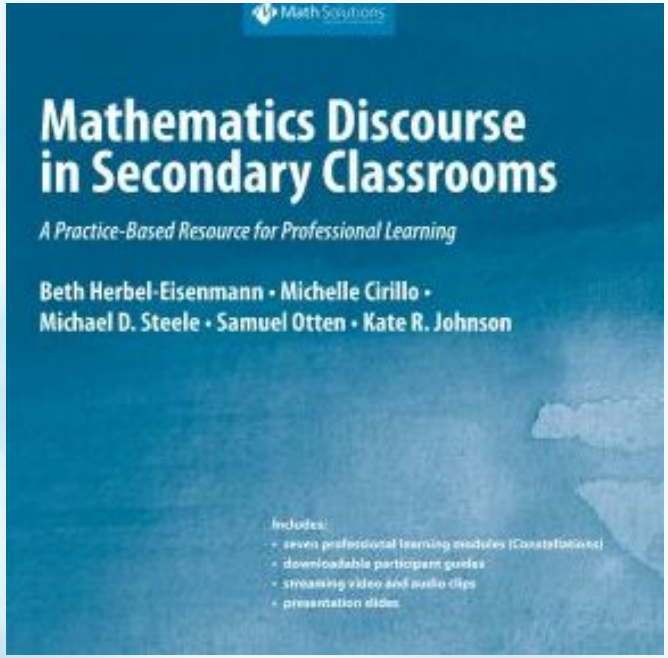
- Students do cognitive lifting
- Wait time
- “Am I right?”
- Modeling giving constructive feedback
- Discovery activities

# Before

# After

# Resources - See Webpage

## A Full 30-Hour Professional Development Program

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# MTLT Papers that Connect to These Ideas

Access digital content at nctm.org/mtlt/pub/k

## Justification as an Equity Practice

See how to give students equitable access to mathematics, develop their agency, and support their engagement.

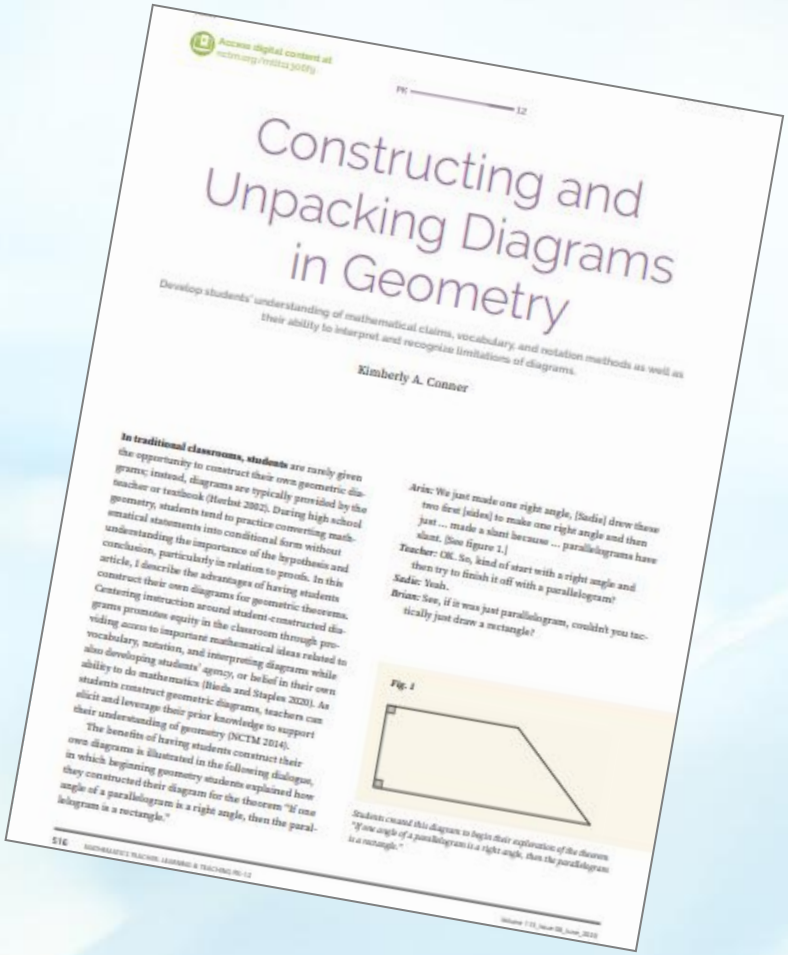
Kristen N. Bieda and Megan Staples

**Despite strong efforts, and** some progress, our profession still has great strides to make in organizing classrooms to produce more equitable outcomes in mathematics. Teaching that supports all students to learn mathematics for understanding is not a stance or an ideology. Equity is a fundamental guiding principle for school mathematics teaching:

> The question is not whether all students can succeed in mathematics but whether the adults organizing mathematics learning opportunities can alter traditional beliefs and practices to promote success for all. (NCTM 2014, p. 60)

Justification, an essential mathematical practice, is well-known for its role in promoting rigor and developing mathematical understanding. Equally powerful is its role as a discursive practice that provides students with access to mathematical thinking and reasoning (the mathematics behind the answers) and promotes





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PK ——— 12

## Constructing and Unpacking Diagrams in Geometry

Develop students' understanding of mathematical claims, vocabulary, and notation methods as well as their ability to interpret and recognize limitations of diagrams.

Kimberly A. Conner

**In traditional classrooms, students** are rarely given the opportunity to construct their own geometric diagrams; instead, diagrams are typically provided by the teacher or textbook (Herbst 2002). During high school geometry, students tend to practice converting mathematical statements into conditional form without understanding the importance of the hypothesis and conclusion, particularly in relation to proofs. In this article, I describe the advantages of having students construct their own diagrams for geometric theorems. Centering instruction around student-constructed diagrams promotes equity in the classroom through providing access to important mathematical ideas related to vocabulary, notation, and interpreting diagrams while also developing students' agency, or belief in their own ability to do mathematics (Bieda and Staples 2020). As students construct geometric diagrams, teachers can elicit and leverage their prior knowledge to support their understanding of geometry (NCTM 2014).

The benefits of having students construct their own diagrams is illustrated in the following dialogue, in which beginning geometry students explained how they constructed their diagram for the theorem "If one angle of a parallelogram is a right angle, then the parallelogram is a rectangle."

> **Aria:** We just made one right angle, [Sadie] drew these two first [sides] to make one right angle and then just ... made a slant because ... parallelograms have slant. [See figure 1.]
> **Teacher:** OK. So, kind of start with a right angle and then try to finish it off with a parallelogram?
> **Sadie:** Yeah.
> **Brian:** See, if it was just parallelogram, couldn't you tactically just draw a rectangle?

**Fig. 1**



*Students created this diagram to begin their exploration of the theorem "If one angle of a parallelogram is a right angle, then the parallelogram is a rectangle."*

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Justification, an essential mathematical practice, is well-known for its role in promoting rigor and developing mathematical understanding. Equally powerful is its role as a discursive practice that provides students with access to mathematical thinking and reasoning (the mathematics behind the answers) and promotes

"Students' access to mathematics and rigor is influenced by their sense of agency, which shapes how they engage, take up challenges, and see themselves as capable of learning new ideas."

Bieda & Staples (p. 104)
MTLT (February, 2020)

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# Thank you!

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For more information about the project visit: www.pisc.udel.edu and follow @UDMichy on Twitter.


Email mcirillo@udel.edu for questions about or updates on the project.

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