Supporting Students’ Pathways through the Cycle of Inquiry and Justification

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#NCTM100

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June 10, 2020
Understand that when Black ppl talk about racism, we’re talking about institutional racism. Know that racism extends far beyond police brutality.

Housing
Healthcare/healthy food
Employment
$
Financial
Education
Criminal justice
Voting
Wealth building
etc...

Fight for it ALL.
Systemic Racism

Nonwhite school districts get $23 billion less than white districts despite serving the same number of students.

https://edbuild.org/content/23-billion#US
Differential Opportunities to Learn

• The typical pedagogy faced by African American students is often directive, controlling, and debilitating.

From: Opportunities to Learn Geometry: Listening to the Voices of Three African American High School Students (Strutchens & Westbrook, 2009)
Distrust in an Urban Mathematics Classroom

(Woodson & Id-Deen, 2016)

• “I’m not just talking, I am doing the assignment you gave us…Like when she sees us talking, and immediately says, ‘stop talking,’ she doesn’t even think we’re talking about math. We’re talking about math…”
Distrust in an Urban Mathematics Classroom
(Woodson & Id-Deen, 2016)

• “She doesn’t trust that we will understand the math, so she always gives us worksheets. They are easy, and I know I can do harder work.”
NCTM Presidents’ Statement

As educators, teachers of mathematics, and a Council, we reiterate our position:

• We support the use of mathematics as an analytic tool to challenge power, privilege, and oppression.

• We encourage all educators to challenge systems of oppression that privilege some while disadvantaging others.

• We encourage educators to create socially and emotionally safe spaces for themselves, their students, and colleagues.  
  (Wilkerson & Berry, 2020)
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  (Wilkerson & Berry, 2020)
Goals for the Session

1. Discuss opportunities-to-learn that support students’ positive mathematical identities and agency

2. Explore ways to engage students in the Cycle of Inquiry & Justification

3. Consider classroom norms that must be in place to support Inquiry & Justification
One Word, Two Questions

• How do you feel about teaching mathematical proof in one word?

• How do you think your students might feel about learning mathematical proof, in one word?
“As I was preparing to student teach, I looked forward to teaching mathematics but silently hoped not to teach geometry. Of course, that turned out to be the only subject my supervising teacher taught, but it was an amazing experience!”

- Trena Wilkerson, President’s Message
• “It is unreasonable to ask a professional to change much more than 10% a year, but it is unprofessional to change by much less than 10% a year.”

• “If you don’t feel inadequate, you’re probably not doing the job.”

Originally Published in 1994
What makes teaching proof difficult?
How well do students write geometry proofs?

Sharon Senk (1985)

www.pisc.udel.edu
Percent of Students Who Reached 75% Mastery

- Mastery: 30%
- Non-Mastery: 70%
Percent of Students Who Could Write at Least One Valid Proof

- Could: 74%
- Could Not: 26%
How well can students do proof now compared to in 1985?
RESEARCH FINDING

Today’s students in our sample were more ready to engage in reasoning and proof, but the post-test outcomes were significantly worse than in 1985.
Today’s students in our sample were more ready to engage in reasoning and proof, but the post-test outcomes were significantly worse than in 1985. Why might this be?
Proof in Secondary Classrooms (PISC) Project (Cirillo, PI)

Assessments
- Pre-Tests (n = 1550)
- Post-Tests (n = 1278)

Classroom Observations (n = 294)

PD Meetings (n = 35)

Interviews
- Student (n = 62)
- Teacher (n = 28)

Funded by the National Science Foundation
Proof in School Mathematics is not too hard.
Proof in School Mathematics
is not too hard.

It’s too boring!

It’s too easy!

---

**Example 1: Symmetric Property of Segment Congruence**

You can prove the Symmetric Property of Segment Congruence as follows.

**Given** \( \overline{PQ} \cong \overline{XY} \)

**Prove** \( \overline{XY} \cong \overline{PQ} \)

<table>
<thead>
<tr>
<th>Statements</th>
<th>Reasons</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. ( \overline{PQ} \cong \overline{XY} )</td>
<td>1. Given</td>
</tr>
<tr>
<td>2. ( \overline{PQ} = \overline{XY} )</td>
<td>2. Definition of congruent segments</td>
</tr>
<tr>
<td>3. ( \overline{XY} = \overline{XY} )</td>
<td>3. Symmetric property of equality</td>
</tr>
<tr>
<td>4. ( \overline{XY} \cong \overline{PQ} )</td>
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</tr>
<tr>
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<td>4. Definition of congruent segments</td>
</tr>
</tbody>
</table>

Given the figure shown, write a two-column proof to prove \( \angle LON \cong \angle LNO \).
Reasoning & Proof in School Mathematics Needs a Makeover

BEFORE

AFTER
“Teachers and programs should focus on reasoning and sense-making, or students may not learn to think.”

Cathy Seeley
Student Task – Triangle Congruence Proof

Given: $\overline{FJ} \parallel \overline{HI}$
$\overline{FI}$ bisects $\overline{JH}$ at $G$

Prove: $\triangle JFG \cong \triangle HIG$

Diagram:

1. Write a plan for the proof.

2. Write a proof.
Video 1: First Triangle Congruence Proofs

Go to: https://tinyurl.com/PISC100

Password: PISC

Watch Video 1 Only

In the chat, comment on what you notice or wonder
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Watch Video 1

In the chat, comment on what you notice or wonder
Mathematical Reasoning in Middle School

• Mathematical reasoning includes the processes of:
  – conjecturing
  – generalizing
  – justifying

• Mathematical reasoning in middle school focuses largely on informal reasoning, setting the stage for proof in high school and beyond.
Challenges of the High School Curriculum

• Lack of Focus

• Challenging to meet all goals:
  – Teach desired level of rigor
  – Develop mathematical processes and conceptual understanding
  – Cultivate positive mathematical identities

• Need to “cover” a vast number of standards
Balancing Content & Processes

Content

Access, Agency, Positive Identity

Processes
What residue remains from your lessons?

Objectives of High School Geometry:

- to develop the ability to reason logically
- to develop knowledge of geometric facts and relations
What residue remains from your lessons?

Objectives of High School Geometry:

- to develop the ability to reason logically
- to develop knowledge of geometric facts and relations
Reasoning and Proof and the Cycle of Inquiry & Justification

Inquiry  Justification
Reasoning and Proof and the Cycle of Inquiry & Justification
The Cycle of Inquiry & Justification

Inquiry
- Exploration
- Discovery
- Conjecture

Justification
- Proof
- Verification

Bass (2015)
The Cycle of Inquiry & Justification

**Inquiry**
- Exploration
- Discovery
- Conjecture

**Justification**
- Proof
- Verification

Inductive Reasoning

Deductive Reasoning

Bass (2015)
The Cycle of Inquiry & Justification

Inquiry
- Exploration
- Discovery
- Conjecture

Justification
- Proof
- Verification

Inductive Reasoning

Deductive Reasoning

Bass (2015)
Reasoning & Proof in School
Mathematics Needs a Makeover
Authenticity of Mathematics Tasks

Student Engagement
The Cycle of Inquiry & Justification

**Inquiry**
- Exploration
- Discovery
- Conjecture

**Justification**
- Proof
- Verification

Bass (2015)
Student Task – Investigating Intersecting Lines

Investigate Intersecting Lines

**Student Task**
- Fold the paper in such a way that the creases form two intersecting lines.
- Measure all the angles with your protractor.
- Complete the first row of the table on your student sheet with a sketch of your intersecting lines and what you notice about the angles formed by the intersecting lines.

<table>
<thead>
<tr>
<th>Sketch of Construction</th>
<th>What do you notice? What conjecture(s) do you have?</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
</tr>
</tbody>
</table>
# Student Task – Exploring Triangle Congruence

<table>
<thead>
<tr>
<th>Triangle</th>
<th>Triangle Congruence Strategy and Information Given</th>
<th>Congruent to Original (Yes or No?)</th>
<th>Other Notes</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

![Image of triangle task setup with tools and paper](www.pisc.udel.edu)
Videos 2 & 3 : Inquiry

Go to: https://tinyurl.com/PISC100
Password: PISC

Watch Videos 2 & 3 Only

In the chat, comment on what you notice or wonder
Videos 2 & 3: Inquiry

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Password: PISC

Watch Videos 2 & 3 Only

In the chat, comment on what you notice or wonder
The Cycle of Inquiry & Justification

Inquiry:
- Exploration
- Discovery
- Conjecture

Justification:
- Proof
- Verification

Inductive Reasoning

Deductive Reasoning

Bass (2015)
Prior to Video 4

Defining & Conjecturing

• Write a definition of parallelogram.

• Write a conjecture about the opposite angles of a parallelogram.

Translating

• Line $PQ$ is parallel to line $RS$.

Drawing Conclusions

• Given: $\overline{AD}$ bisects $\angle ABC$.

What can you conclude?
Addressing MISCONCEPTIONS in Secondary Geometry Proof

Research suggests that teachers struggle to find effective ways to introduce proof. In 1960, in an article in this journal, Smith argued that being aware of student misconceptions in geometry is the first step in preparing to address the fundamental challenges of learning to prove. Through careful study, he identified and analyzed “three serious learning difficulties” that students have in connection with (1) a lack of familiarity with geometric figures, (2) not seeing the meaning of the if-then relationship, and (3) an inadequate understanding of the meaning of proof (p. 100). Smith found that when these difficulties were attended to capriciously, student results improved.

Years later, in 1983, Saxe detailed findings from her study of 1,520 students, in which she found that only 50 percent of students in a full-year geometry course that covered proof reached a 75 percent mastery of proof. Overall, 28 percent of the sample could not write a single valid proof. Consequently, Saxe recommended (p. 411) that we must immediately look for more effective ways to teach proof in geometry, making the following suggestions:

- Pay special attention to teaching students to start a chain of reasoning.
- Place greater emphasis on the meaning of proof than we do currently.
- Teach students how, why, and when they can transform a diagram in a proof.
Inquiry – Exploration & Discovery

Conjecturing About Parallelograms

Author: Jenifer Hummer, Kelly Curtis
Topic: Parallelogram

Move the blue points on the figure to change the size of the parallelogram. Use the check boxes on the right to display specific measurements.

Given: Parallelogram ABCD

Diagonals:
AC=12.1  BD=5.4
AE=6  BE=2.7
EC=6  ED=2.7
Student Task – Conjecturing About the Diagonals of a Parallelogram

1. Use inductive reasoning to explore the situation.
   Use the diagram from [http://tinyurl.com/ConjecturingDay1](http://tinyurl.com/ConjecturingDay1) to fill in the table based on the various parallelogram configurations. Drag point on the parallelogram to create different configurations.

<table>
<thead>
<tr>
<th>Configuration 1</th>
<th>Configuration 2</th>
<th>Configuration 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>AC =</td>
<td>AC =</td>
<td>AC =</td>
</tr>
<tr>
<td>BD =</td>
<td>BD =</td>
<td>BD =</td>
</tr>
<tr>
<td>AE =</td>
<td>AE =</td>
<td>AE =</td>
</tr>
<tr>
<td>EC =</td>
<td>EC =</td>
<td>EC =</td>
</tr>
<tr>
<td>BE =</td>
<td>BE =</td>
<td>BE =</td>
</tr>
<tr>
<td>ED =</td>
<td>ED =</td>
<td>ED =</td>
</tr>
</tbody>
</table>

2. Write a conjecture.
   Write a conjecture about the diagonals of a parallelogram.

www.pisc.udel.edu
Video 4: Conjecturing about Parallelograms

Go to: https://tinyurl.com/PISC100

Password: PISC

Watch Video 4

In the chat, comment on what you notice or wonder
Video 4: Conjecturing about Parallelograms

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Password: PISC
Watch Video 4

In the chat, comment on what you notice or wonder
3. Develop a particular instance of your conjecture.
   a. Rewrite your conjecture as a conditional statement (i.e., If..., then...).
   b. Sketch and label a diagram for the proof.
   c. Use the conditional statement to write the “Given” and “Prove” statements for the conjecture about the diagonals of a parallelogram.

   **Conditional Statement:**

   **Given:**

   **Diagram:**

   **Prove:**

4. Make a plan for the proof. Sketch and mark a diagram, and brainstorm a plan for proving the conjecture.
Student Task – Proving the Conjecture

3. Develop a particular instance of your conjecture.
   a. Rewrite your conjecture as a conditional statement (i.e., If..., then...).
   b. Sketch and label a diagram for the proof.
   c. Use the conditional statement to write the “Given” and “Prove” statements for the conjecture about the diagonals of a parallelogram.

   Conditional Statement:

   Given:
   Prove:

4. Make a plan for the proof. Sketch and mark a diagram, and brainstorm a plan for proving your conjecture.
Conjecture → Conditional

The diagonals of a parallelogram bisect each other.

If ________________________________,
then ________________________________.
The diagonals of a parallelogram bisect each other.

If a quadrilateral is a parallelogram, then the diagonals bisect each other.
Student Post-Test – Results

Rhionna conjectured that: The diagonals of a parallelogram bisect each other. Rewrite Rhionna’s conjecture as an “If _____ , then _____” statement.

If _______________________________________________________________________
then _______________________________________________________________________

Scores for the Post-Test Task

<table>
<thead>
<tr>
<th>Score</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>Mean</th>
</tr>
</thead>
<tbody>
<tr>
<td>Percentage (n = 647)</td>
<td>25%</td>
<td>12%</td>
<td>17%</td>
<td>19%</td>
<td>18%</td>
<td>1.76</td>
</tr>
</tbody>
</table>
The diagonals of a parallelogram bisect each other.

Sample Responses

*If the diagonals of a parallelogram bisect each other, then the angles are congruent.*

*If the diagonals of a shape bisect, then it is a parallelogram.*

*If there are diagonals in a parallelogram, then they bisect each other.*
Student Interview Task

Determine what has been “Given” and what one would need to “Prove” for the theorem below. Draw, label, and mark a diagram that represents the theorem.

Conjecture: If a quadrilateral is a parallelogram, then the diagonals bisect each other.

Given: ____________________________________________________________________________  Diagram:

Prove: ____________________________________________________________________________
**Student Interview Task - Results**

Determine what has been “Given” and what one would need to “Prove” for the theorem below. Draw, label, and mark a diagram that represents the theorem.

Conjecture: *If a quadrilateral is a parallelogram, then the diagonals bisect each other.*

<table>
<thead>
<tr>
<th>Given:</th>
<th>Diagram:</th>
</tr>
</thead>
</table>

**Scores from Conventional Approach**

<table>
<thead>
<tr>
<th>Score</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>Mean</th>
</tr>
</thead>
<tbody>
<tr>
<td>Percentage (n = 14)</td>
<td>29</td>
<td>14</td>
<td>21</td>
<td>36</td>
<td>0</td>
<td>1.64</td>
</tr>
</tbody>
</table>

**Scores from Inquiry & Justification Approach**

<table>
<thead>
<tr>
<th>Score</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>Mean</th>
</tr>
</thead>
<tbody>
<tr>
<td>Percentage (n = 14)</td>
<td>7</td>
<td>14</td>
<td>7</td>
<td>14</td>
<td>57</td>
<td>3</td>
</tr>
</tbody>
</table>
Student Interview Task – Common Errors

If a quadrilateral is a parallelogram, then the diagonals bisect each other.

<table>
<thead>
<tr>
<th>Sample Responses</th>
</tr>
</thead>
<tbody>
<tr>
<td>Given: Quadrilateral ABCD</td>
</tr>
<tr>
<td>Given: $AB \cong DC$ and $AD \cong BC$</td>
</tr>
</tbody>
</table>
| Given: If a quadrilateral is a parallelogram  
Prove: The diagonals bisect each other. |
Inquiry & Justification are Central to Mathematical Thinking

"Students should recognize the process of establishing valid mathematical statements as the central act of doing mathematics, and they should see this process as the entire cycle of inquiry and justification." (p. 40)
<table>
<thead>
<tr>
<th>What I had</th>
<th>What I wanted</th>
</tr>
</thead>
<tbody>
<tr>
<td>● Direct instruction</td>
<td>● Active participation</td>
</tr>
<tr>
<td>● “Show and Tell”</td>
<td>● Collaboration</td>
</tr>
<tr>
<td>● Cut-and-paste activities</td>
<td>● Inquiry-based learning</td>
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- Direct instruction
- “Show and Tell”
- Cut-and-paste activities
- Active participation
- Collaboration
- Inquiry-based learning
What Changed?

• Students do cognitive lifting
• Wait time
• “Am I right?”
• Modeling giving constructive feedback
• Discovery activities
Resources - See Webpage

A Full 30-Hour Professional Development Program

Use code Books40 on Math Solutions webpage to receive a 40% discount.
MTLT Papers that Connect to These Ideas

Justification as an Equity Practice

See how to give students equitable access to mathematics, develop their agency, and support their engagement.

Eileen S. Mula and Megan Staples

Constructing and Unpacking Diagrams in Geometry

Develop students’ understanding of mathematical claims, vocabulary, and notation methods as well as their ability to interpret and organize informal diagrams.

Kimberly A. Cogan

In this article, we examine student difficulties in working with mathematical diagrams and unpack the role that these difficulties play in students’ understanding of geometry.
“Students’ access to mathematics and rigor is influenced by their sense of agency, which shapes how they engage, take up challenges, and see themselves as capable of learning new ideas.”

Bieda & Staples (p. 104)  
MTLT (February, 2020)
Thank you!

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Thanks to the teachers and students in the videos. Thanks to Amanda Seiwell for help with this presentation.

For more information about the project visit: [www.pisc.udel.edu](http://www.pisc.udel.edu) and follow @UDMichy on Twitter.

Email mcirillo@udel.edu for questions about or updates on the project.