Supporting Students' Productive Struggle

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NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

Expectations for students	Teacher actions to support students	Classroom-based indicators of success
Most tasks that promote reasoning and problem solving take time to solve, and frustration may occur, but perseverance in the face of initial difficulty is important.	Use tasks that promote reasoning and problem solving; explicitly encourage students to persevere; find ways to support students without removing all the challenges in a task.	Students are engaged in the tasks and do not give up. The teacher supports students when they are "stuck" but does so in a way that keeps the thinking and reasoning at a high level.
Correct solutions are important, but so is being able to explain and discuss how one thought about and solved particular tasks.	Ask students to explain and justify how they solved a task. Value the quality of the explanation as much as the final solution.	Students explain how they solved a task and provide mathematical justifications for their reasoning.
Everyone has a responsibility and an obligation to make sense of mathematics by asking questions of peers and the teacher when he or she does not understand.	Give students the opportunity to discuss and determine the validity and appropriateness of strategies and solutions.	Students question and critique the reasoning of their peers and reflect on their own understanding.
Diagrams, sketches, and hands-on materials are important tools to use in making sense of tasks.	Give students access to tools that will support their thinking processes.	Students are able to use tools to solve tasks that they cannot solve without them.
Communicating about one's thinking during a task makes it possible for others to help that person make progress on the task.	Ask students to explain their thinking and pose questions that are based on students' reasoning, rather than on the way that the teacher is thinking about the task.	Students explain their thinking about a task to their peers and the teacher. The teacher asks probing questions based on the students' thinking.

Fig. 20. Redefining student and teacher success. Adapted from Smith (2000, p. 382).

Analyzing Teaching and Learning 2.1
Comparing Goal Statements

Review goal statements A, B, and C (shown below), written for a lesson on proportional relationships, and consider:

- How are they the same and how are they different?
- How might the differences matter?

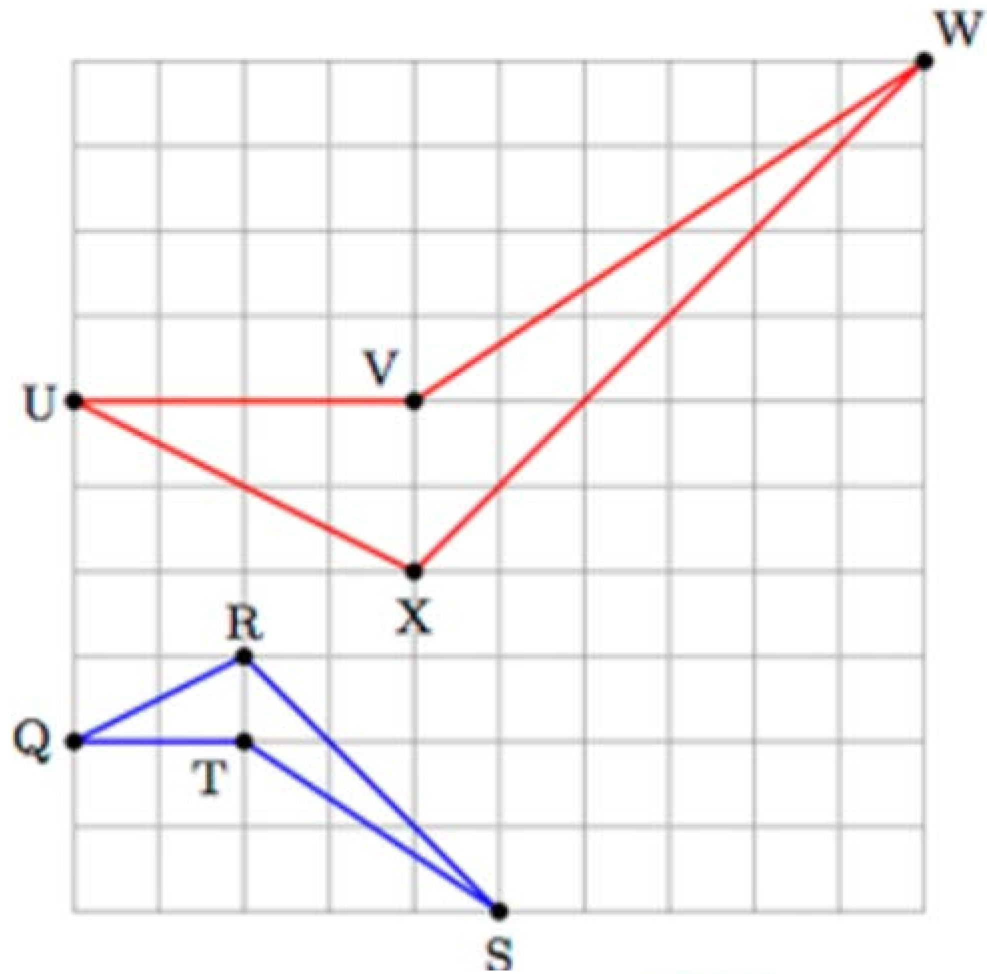
Goal A: Students will learn the procedure (cross multiplication) for finding the missing value in a proportional situation.

Goal B: Students will be able to (SWBAT) use cross multiplication to find the missing value in problems where the quantities being compared are in a proportional relationship.

Goal C: Students will recognize that a proportion consists of two ratios that are equivalent to each other (e.g., $\frac{a}{b}=\frac{ax}{bx}$) and that missing values in the proportion can be found by determining the scale factor x that relates the two ratios or by determining the unit rate – the relationship (multiplicative) between a and b and recognizing that ax and bx must have the same relationship as a and b.

Solve the task.

- Consider the pair of polygons.
- Explain whether or not one the polygons are similar.
- If you finish quickly, explain a different way.



https://www.illustrativemathematics.org/content-standards/7/G/A/1/tasks/1201

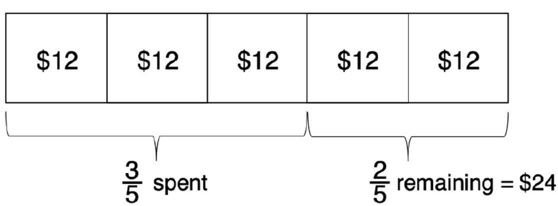
Ms. Flahive and Ms. Ramirez teach fifth grade and plan their lessons collaboratively. Their current instructional unit focuses on fractions. They have selected the Shopping Trip task shown below because they think it will be accessible to their students yet provoke some struggle and challenge, since a solution pathway is not straightforward. The mathematics goal for students is to draw on and apply their understanding of how to build non-unit fractions from unit fractions and to use visual representations to solve a multi-step word problem:

Shopping Trip Task

Joseph went to the mall with his friends to spend the money that he had received for his birthday. When he got home, he had \$24 remaining. He had spent $3/5$ of his birthday money at the mall on video games and food. How much money did he spend? How much money had he received for his birthday?

When Ms. Flahive and Ms. Ramirez present the problem in their classrooms, both teachers see students struggling to get started. Some students in both classrooms immediately raise their hands, saying, "I don't get it," or "I don't know what to do."

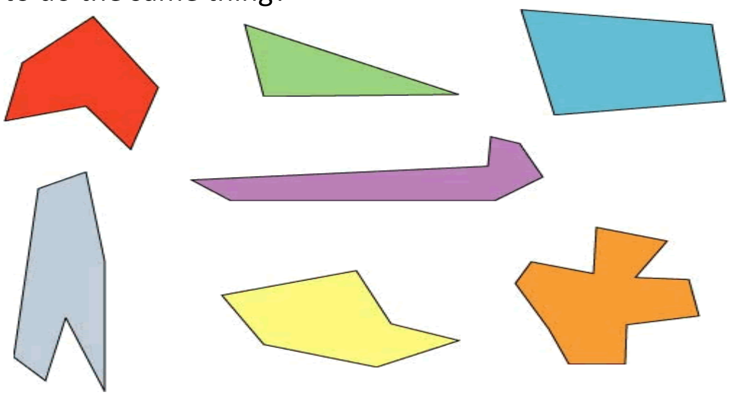
Ms. Flahive is very directive in her response to her students. She tells them to draw a rectangle and shows them how to divide it into fifths to represent what Joseph had spent and what he had left. She then guides her students step by step until they have labeled each one-fifth of the rectangle as worth \$12, as shown below. Finally, she tells the students to use the information in the diagram to figure out the answers to the questions.



Ms. Ramirez approaches her students' struggles very differently. After she sees them struggling, she has them stop working on the problem and asks all the students to write down two things that they know about the problem and one thing that they wish they knew because it would help them make progress in solving the problem. Then Ms. Ramirez initiates a short class discussion in which several ideas are offered for what to do next. Suggestions include drawing a tape diagram or number line showing fifths, or just picking a number, such as \$50 and proceeding through trial and error. Ms. Ramirez encourages the students to consider the various ideas that have been shared as they continue working on the task.

Fig. 21. Two teachers' responses to students' struggles to solve a multi-step word problem involving fractions

Amanda claims to have an amazing talent. “Draw any polygon. Don’t show it to me. Just tell me the number of sides it has and I can tell you the sum of its interior angles.”
Is Amanda’s claim legitimate? Does she really have an amazing gift, or is it possible for anyone to do the same thing?



1. Working individually, investigate the sum of the interior angles of at least two polygons with 4, 5, 6, 7 or 8 sides. Use a straight-edge to draw several polygons. Make sure that some are irregular polygons. Subdivide each polygon into triangles so you can use what you already know about angle measures to determine the sum of the interior angles of your polygon. Organize and record your results.
2. As a group, combine your results on a single recording sheet and answer these questions:
   a. How did group members subdivide their polygons into triangles? Did everyone do it in the same way? If different, how did that affect your calculations?
   b. Does whether the polygon is regular or irregular affect the sum of the angle measures? Why or why not?
   c. What patterns did you notice as you explored this problem?
   d. What is the relationship between the number of sides of the polygon and the sum of the measures of the interior angles of the polygon? Express this relationship algebraically and explain how you know that your expression will work for ANY convex polygon.

Adapted from “Amazing Amanda”, Institute for Learning, University of Pittsburgh, 2007.

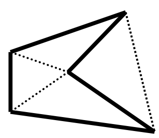
Investigating Teacher Interventions

Read the mini-dialogues shown below, then:

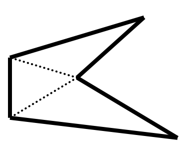
- **Discuss the nature of each student's struggle.**
- **Identify what the teacher does to help students move beyond the impasse they had reached.**

Determine whether or not the teacher supported students' productive struggle

Dialogue 1
A student made the drawing shown below.



T: What did you do here?
S: I drew a polygon with 5 sides.
T: Then what?
S: I divided it into triangles. And I got 4 triangles. But I don't think it is right because when I asked around, no one else had 4.
T: Your triangles can't go outside the polygon. If you take your picture and just get rid of one of your diagonals, you will have the right number of triangles.
S: *(Student erases one of the diagonals.)*



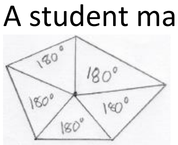
T: That's right. So how many triangles to you have now?
S: 3.
T: Okay. So now you just need to multiply 3 x 180 and you will be set. So now try another one using this method.

Dialogue 2
A student made the table shown below.

# of Sides	Degrees of Interior Angles
3	180
4	360
5	540
6	720
7	900
8	1080

T: Tell me how you constructed your table.
S: I decided to try all of the polygons from 3 to 8. I knew that the 3-sided polygon – a triangle – had angles that summed to 180 degrees because we did that last week. Then I drew polygons with more sides on scrap paper. I subdivided each polygon into non-overlapping triangles. Then I counted the number of triangles in each polygon and multiplied by 180.
T: Why did you multiply by 180?
S: Because the angles of each triangle sum up to 180 so to find the sum of all the angles in a polygon you need to multiply the number of triangles in the polygon by 180.
T: So how does this help you determine the relationship between the number of sides of the polygon and the sum of measures of the interior angles?
S: I am not sure. I know that you multiply the number of triangles in the polygon by 180 like I said, so I guess I need to figure out how many triangles there are in each polygon. Maybe I will add a column to the table to keep track of this.
T: That sounds like a good plan. I will check back in with you later.

Dialogue 3
A student made the drawing shown below.



T: So tell me about your drawing?
S: I made a 5-sided polygon and subdivided into 5 non-overlapping triangles.
T: And then what?
S: Well since each triangle has angles that sum to 180 degrees, I multiplied 180 by 5 and got 720. (*Student sounds unsure of herself.*)
T: So what is the problem?
S: I think it is too big. I took out my protractor and did a rough measure of the angles and I got closer to 500.
T: Nice way to check if you answer is reasonable. So let's take a closer look at your diagram. Can you show me where the angles of the triangles are?

S: (*Student points to the angles in each triangle.*)
T: So are all the angles you just pointed to included in the interior angles of the polygon?
S: No. All these (*point to the angles formed around the center point*) are not included in the interior angles. Oh, so somehow I need to figure out how not to count these.
T: I will leave you to figure out what you know about the angles around a point and how this can help you solve your problem. I will check in with you later.

Dialogue 4
A student can't get started.

T: What have you figured out so far?
S: Nothing. I am not sure what to do.
T: The first thing I want you to do is to draw a polygon with 4 sides.
S: (*Draws a square.*)
T: Now you need to divide it into triangles, starting at one of the vertices.
S: (*Divides the square into two triangles by drawing the diagonal.*)
T: Okay. So you have two triangles. What is the sum of the angles of a triangle equal to?
S: 180?
T: So if you have two triangles, what would the sum of the angles be?
S: 360?
T: Yes! So the angles of a 4-sided polygon sum up to 360 degrees. Now try a five-sided polygon and use the same method of breaking it up into triangles that we just did.

Dialogue 5
A student can't get started.

T: What have you figured out so far?
S: Nothing. I am not sure what to do.
T: Go back through your notes and review the work you did when we proved that the sum of the angles of a triangle sum to 180.

TASK 1

Name____Date____________________

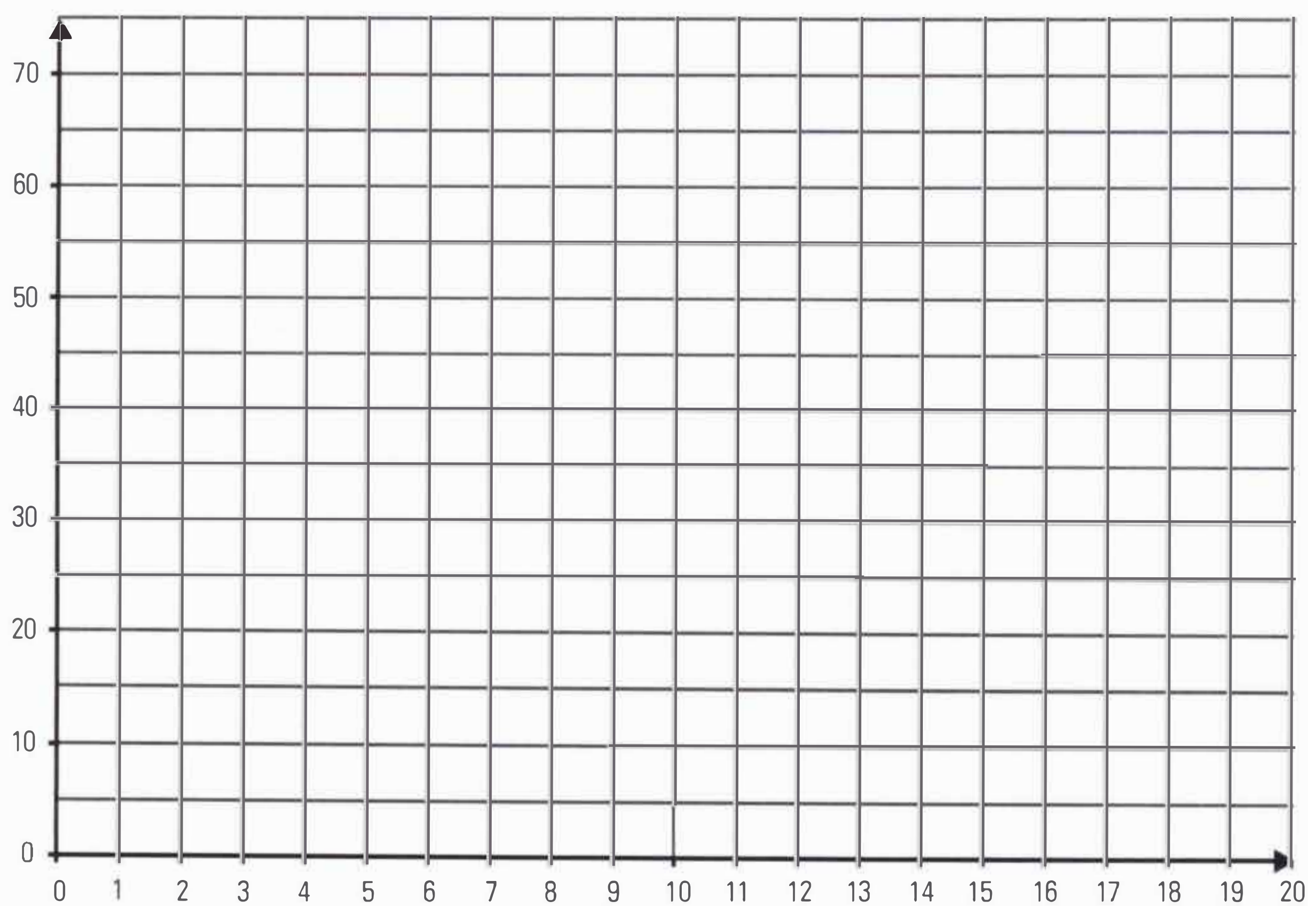
Joe's on the Beach Ice Cream

At Joe's on the Beach, single-scoop ice cream cones sell for $2.99 and ice cream cakes sell for $24.99. Rosa buys an ice cream cake for her party. She also decides to buy a single-scoop cone for each of her friends.

1. Write an equation that can be used to determine the cost *(y)* of a cake and any number of cones *(x)* that Rosa buys. Explain the meaning of the terms in your equation.

2. Complete Parts A-B:

   A. Sketch a graph that models the problem situation.



(Question #2 continued)

TASK 1

B. Explain how you know your graph models the problem situation.

3. Complete Parts A-C:

   A. How does the total cost increase with the number of cones bought?

   B. How does this appear in the equation?

   C. How does this appear in the graph?

TASK 1
APPLICATION

Name________________________________Date______________

Joe's on the Beach Ice Cream

Recall from *Task 1: Joe's on the Beach Ice Cream*:

> *At Joe's on the Beach, single-scoop ice cream cones sell for $2.99 and ice cream cakes sell for $24.99. Rosa buys an ice cream cake for her party. She also decides to buy a single-scoop cone for each of her friends.*

If Joe's on the Beach reduces the cost of the ice cream cake to $17.99:

1. How does this affect the equation we created?

2. How does this affect the graph we created?

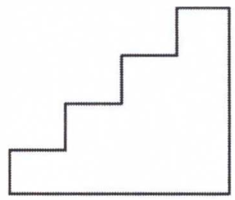
3. How does this affect the table we created?

Fictional Stairs

NAME________________________

Measure the following sets of stairs to determine rise/run. Use the chart in the Steps and Slopes Activity Sheet to document your measurements. Be sure to use the scale of $\frac{1}{8}$"=1" actual to determine the actual measurements of the stairs.

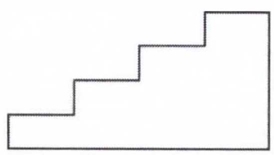
1.



Actual Rise: ________________

Actual Run: ________________

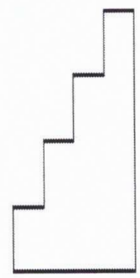
2.



Actual Rise: ________________

Actual Run: ________________

3.



Actual Rise: ________________

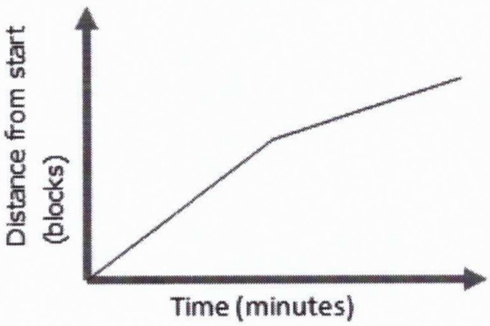
Actual Run: ________________

Matching Graphs

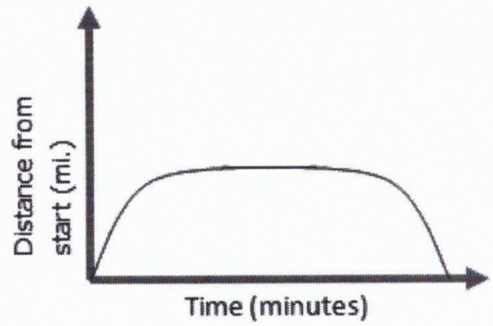
NAME______________________________

Getting Started

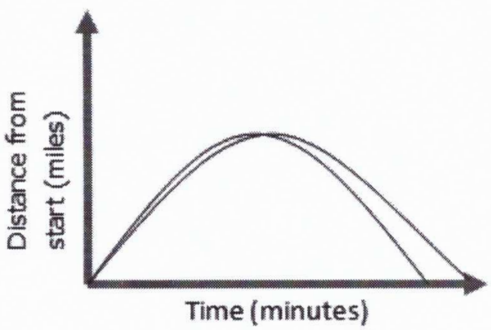
A.



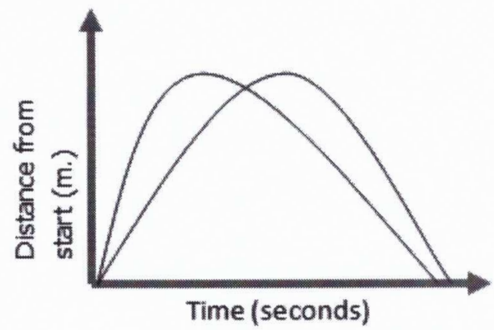
B.



C.



D.



1. Match each of the following scenarios with the appropriate graph shown above.

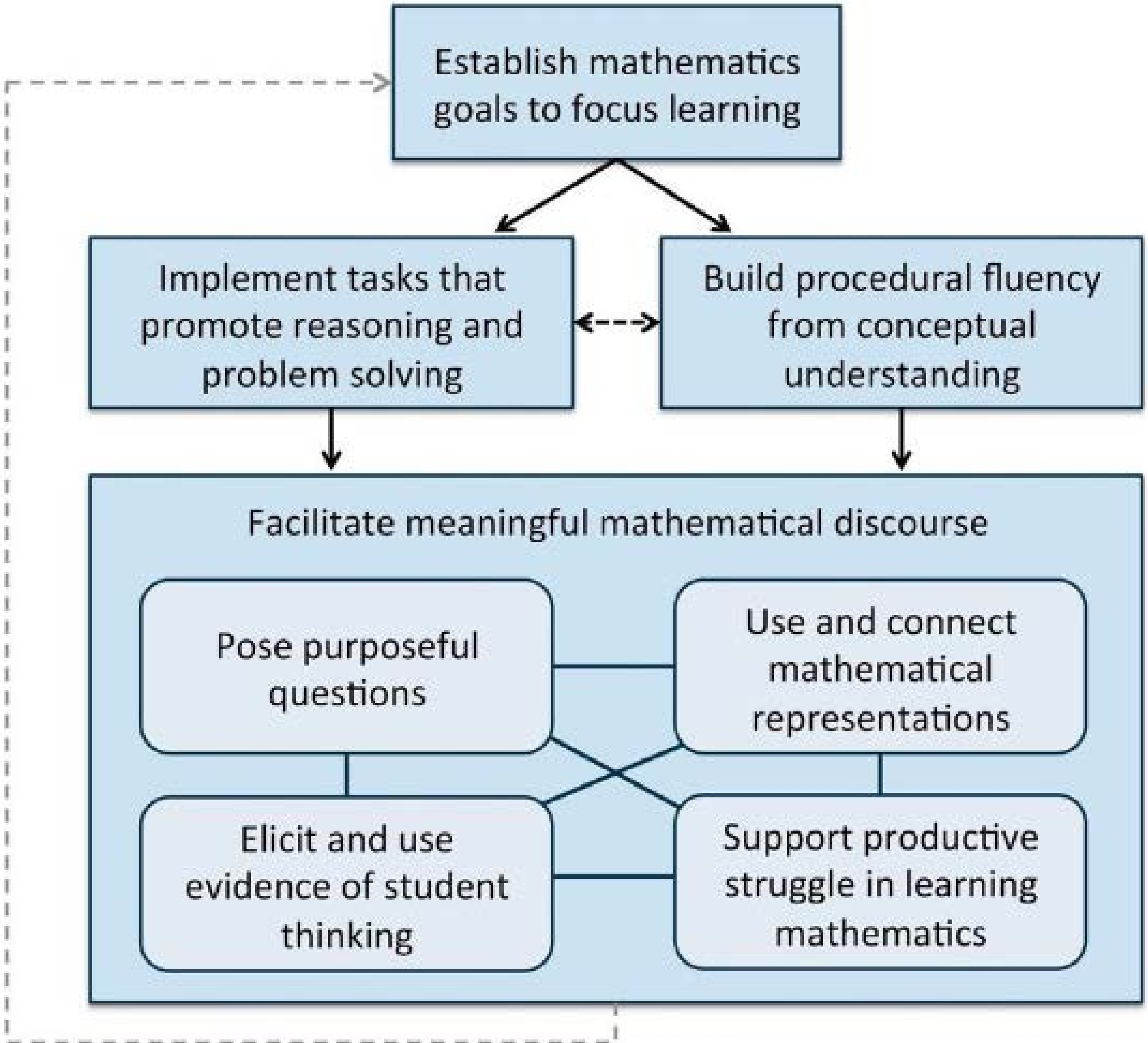
- Kelly and Cindy sprinted from one end of the gym to the other end. They then jogged back to where they started. Cindy sprinted faster than Kelly.
- John walked to school. The second half of his walk was uphill, so he walked at a slower pace.
- Rachel drove to the dentist's office. Her appointment lasted for 45 minutes, and then she headed home.
- Mike and Ron rode their bikes from Middletown to Centerville and back. Soon after they began, Mike was always ahead of Ron.

2. How did you match each story with a graph? Write a 1 - 2 sentence explanation for each story.

The Petoskey Population

The population of Petoskey, Michigan, was 6,076 in 1990 and was growing at the rate of 3.7% per year. The city planners want to know what the population will be in the year 2025. Write and evaluate an expression to estimate this population. (Source: Holt Algebra 2 [Schultz et al. 2004, p. 415])

Boston, M., Dillon, F., Smith, M. S., & Miller, S. (2017). *Taking Action: Implementing Effective Mathematics Teaching Practices Grades 9-12.* Reston, VA: NCTM.



Mathematics Teaching Framework

Appendix C
Mathematics Task Framework
Levels of Cognitive Demand

Lower-Level Demands	**Higher-Level Demands**
<u>Memorization</u> • involve either reproducing previously learned facts, rules, formulae or definitions OR committing facts, rules, formulae or definitions to memory. • cannot be solved using procedures because a procedure does not exist or because the time frame in which the task is being completed is too short to use a procedure. • are not ambiguous. Such tasks involve exact reproduction of previously-seen material and what is to be reproduced is clearly and directly stated. • have no connection to the concepts or meaning that underlie the facts, rules, formulae or definitions being learned or reproduced.	<u>Procedures With Connections</u> • focus students' attention on the use of procedures for the purpose of developing deeper levels of understanding of mathematical concepts and ideas. • suggest pathways to follow (explicitly or implicitly) that are broad general procedures that have close connections to underlying conceptual ideas as opposed to narrow algorithms that are opaque with respect to underlying concepts. • usually are represented in multiple ways (e.g., visual diagrams, manipulatives, symbols, problem situations). Making connections among multiple representations helps to develop meaning. • require some degree of cognitive effort. Although general procedures may be followed, they cannot be followed mindlessly. Students need to engage with the conceptual ideas that underlie the procedures in order to successfully complete the task and develop understanding.
<u>Procedures Without Connections</u> • are algorithmic. Use of the procedure is either specifically called for or its use is evident based on prior instruction, experience, or` placement of the task. • require limited cognitive demand for successful completion. There is little ambiguity about what needs to be done and how to do it. • have no connection to the concepts or meaning that underlie the procedure being used. • are focused on producing correct answers rather than developing mathematical understanding. • require no explanations or explanations that focuses solely on describing the procedure that was used.	<u>Doing Mathematics</u> • require complex and non-algorithmic thinking (i.e., there is not a predictable, well-rehearsed approach or pathway explicitly suggested by the task, task instructions, or a worked-out example). • require students to explore and understand the nature of mathematical concepts, processes, or relationships. • demand self-monitoring or self-regulation of one's own cognitive processes. • require students to access relevant knowledge and experiences and make appropriate use of them in working through the task. • require students to analyze the task and actively examine task constraints that may limit possible solution strategies and solutions. • require considerable cognitive effort and may involve some level of anxiety for the student due to the unpredictable nature of the solution process required.

*These characteristics are derived from the work of Doyle on academic tasks (1988), Resnick on high-level thinking skills (1987), and from the examination and categorization of hundreds of tasks used in QUASAR classrooms (Stein, Grover, & Henningsen, 1996; Stein, Lane, and Silver, 1996).

Stein, Smith, Henningsen, & Silver, 2000, p.16