NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS
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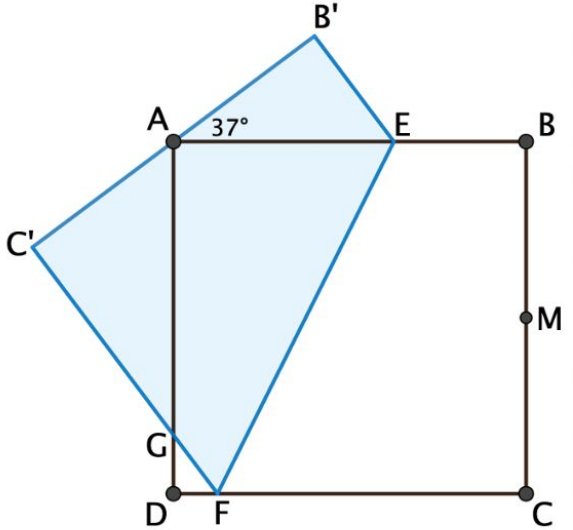
Origami & Mathematics

Please work on the problem below while we wait to begin:

Do Now

When the midpoint M of one side of a square paper is folded over to a vertex as shown on the right, a 37° angle is created. Find the measures of all the angles in the diagram.



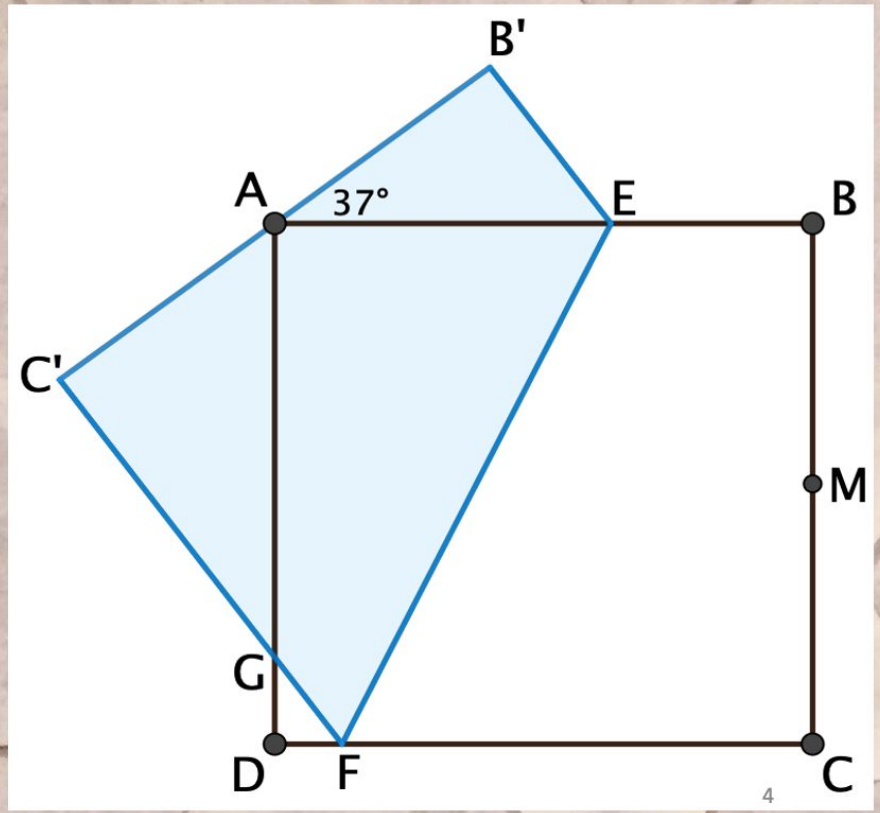
NCTM 2020
Presented by Joy Hsiao
Brooklyn Technical High School

Today's goals

- Share my experiences
- Share my students' work
- Build your experiences
- Q&A (last 5-10 minutes)

Review Problem

(Answers can be found on the handout.)



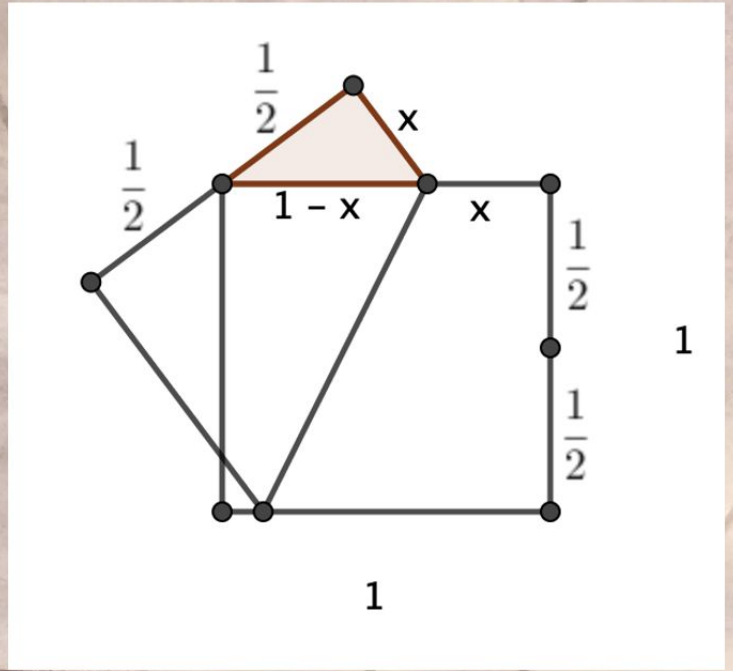
Q: What are the side lengths of the right triangles?

Solution on the next page

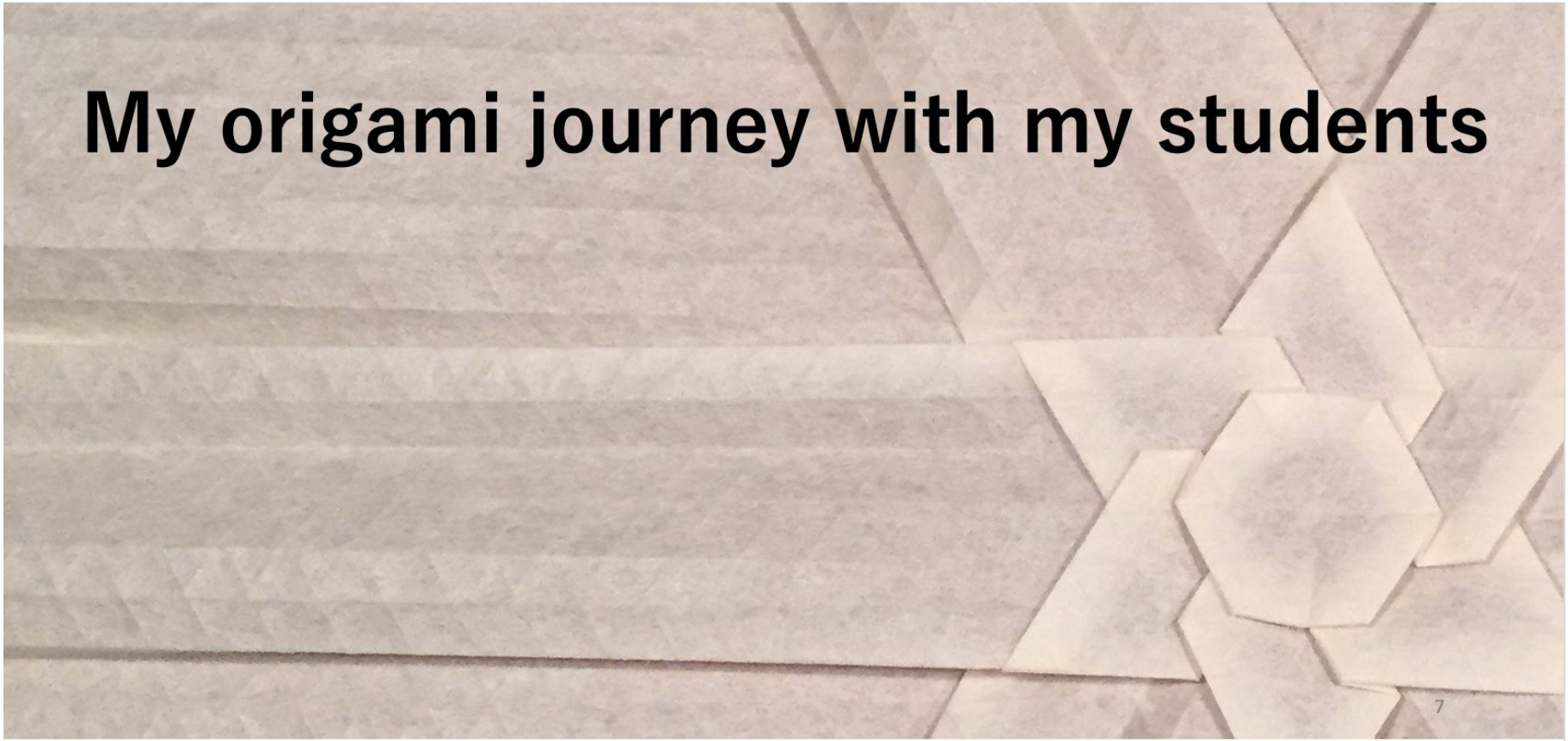
Two extension problems:

Prove

1. All three triangles are similar.
2. The triangle side lengths are multiples of 3-4-5.



My origami journey with my students



Menger Sponge (a 14" fractal cube)

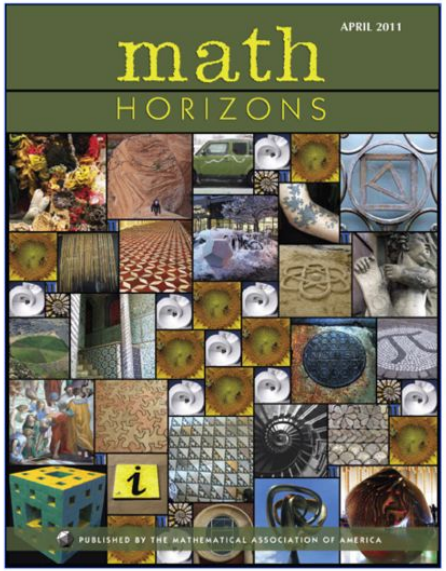
Bard HS Early College (2009)

Photo and origami by Joy Hsiao, *Menger Sponge*, 2009

Menger Sponge (a fractal)

Bard HS Early College (2009)

Photo and origami by Joy Hsiao,
Menger Sponge, 2009



Math Horizons, April 2011

Bridges Conference Towson University (2012)

Photo by Christopher Bartlett, *Bridges Conference Art Exhibition*, 2012
http://bridgesmathart.org/bridges-2012/month-long-art-exhibition

Brooklyn Tech HS (2012)

Photo by Joy Hsiao, *Brooklyn Tech*, 2012

Brooklyn Tech Gala, 2012, https://www.bthsalumni.org/news/event-photos

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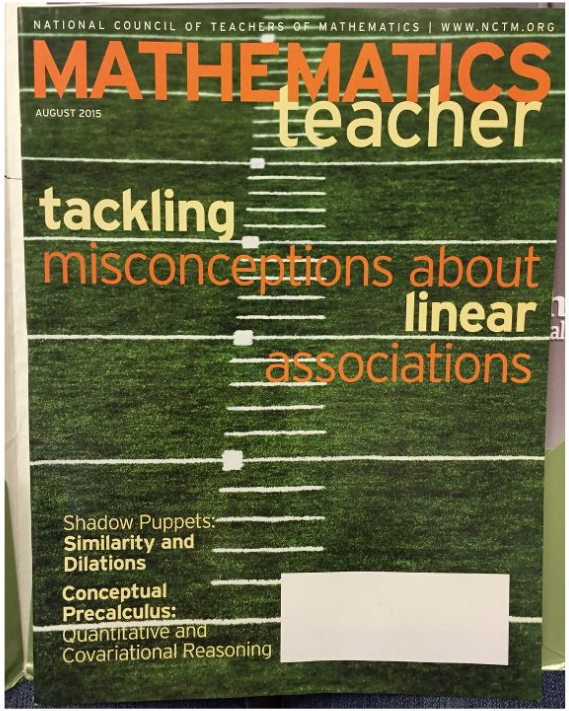
Math investigations (2015)




DELVING deeper

Joy Hsiao

Finding Fifths in Origami

Paper folding, or *origami* in Japanese, is a traditional craft that has been enjoyed by both children and adults for hundreds of years. An origami is typically made with a square piece of paper and no additional tools. A series of folds can transform a flat piece of square paper into an everyday object such as a bird, a boat, or a flower. Aligning vertices or edges makes a crease, and as new creases are made, they and their intersections become additional matching targets used to create new folds.

Mathematicians have long studied the mathematics of paper folding. They use square papers to construct mathematical shapes (for example, folding an equilateral triangle from a square paper or trisecting an angle), demonstrate geometric properties, and pose and prove problems that arise from paper folding. Computer scientists have written software that can generate crease patterns so that origamists can fold almost anything they can draw on paper. An example is Robert Lang's TreeMaker software. Interested readers may also wish to see the TED Talk video by Robert Lang (2008), "The Math and Magic of Origami," in which he introduced real-life applications of origami folding in air bags; solar panels that need to be folded to fit into rocket ships for launch but unfolded once in space; and a heart stent folded to travel through the blood vessel but unfolded once in its final destination.

The mathematics behind origami is very rich and readily accessible to high school students. They can often pose original problems and find creative solutions. The mathematical thinking needed to draw an accurate crease pattern is also challenging yet accessible with the knowledge of high school geometry. This article will examine the mathematics that arises from three paper folding models.

The geometric connections between origami folding and mathematics, such as shapes and symmetries, can easily be observed. An origami model folded from a square piece of paper may have one, two, or four lines of symmetry that mimic the symmetries found in the original square paper. A finished origami that has five or six symmetry lines is even more intriguing. For convenience, some models require the use of a regular pentagonal or hexagonal paper, for example, to produce a five-point star or a six-point snowflake, respectively.

In this article, I will share the problems I have encountered from origami folding, my closer examination of these models through a mathematical lens, and surprises uncovered as a result of the investigations.

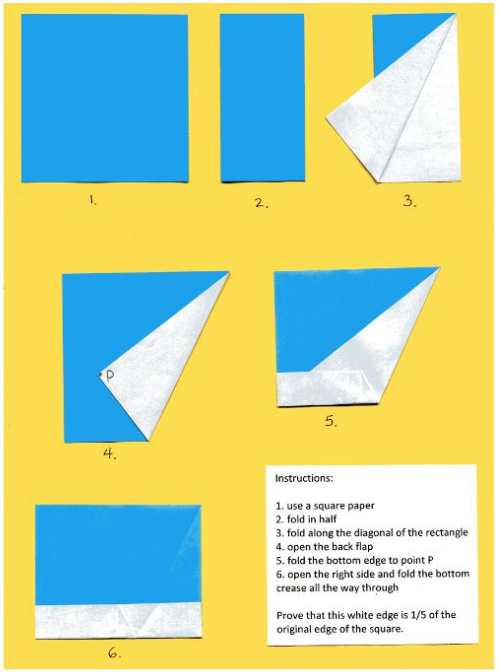
Edited by Maria Fung

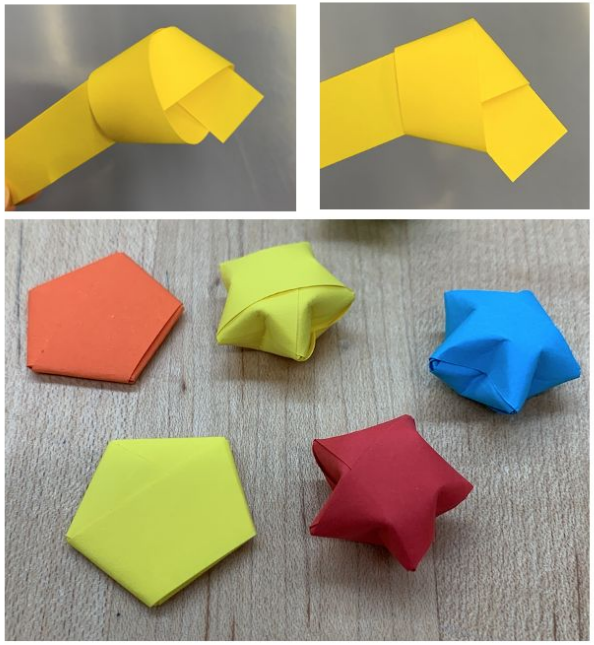
Delving Deeper offers a forum for classroom teachers to share the mathematics from their own work with the journal's readership. It appears in every issue of *Mathematics Teacher*. Manuscripts for the department should be submitted via http://mt.msubmit.net. For more information on submitting manuscripts, visit http://www.nctm.org/mtcalls.

Department editors
Brian M. Dean, bdean@pasco.k12.fl.us, District School Board of Pasco County, FL, and **Daniel Ness**, dness@gc.cuny.edu, Queens College, City University of New York

Hsiao J., Finding Fifths in Origami, *Mathematics Teacher*, *Vol. 109*, August 2015

Math investigations (2015)





Photos and origami by Joy Hsiao,
Stars and origami diagram, 2020

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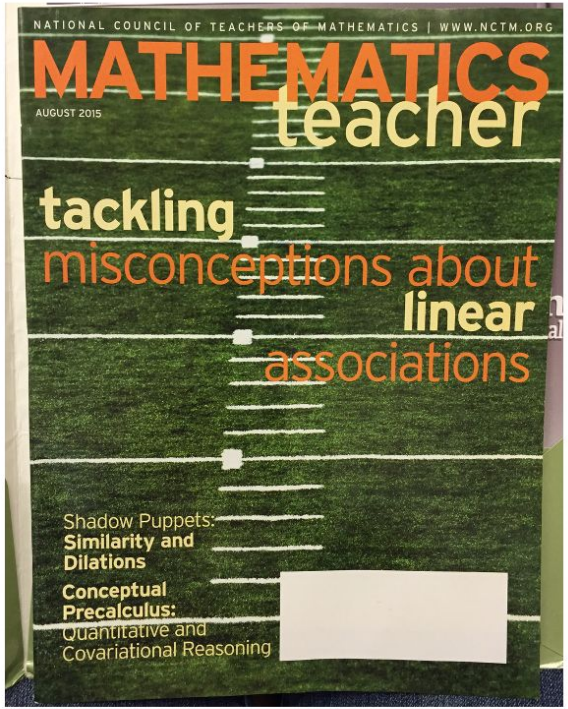
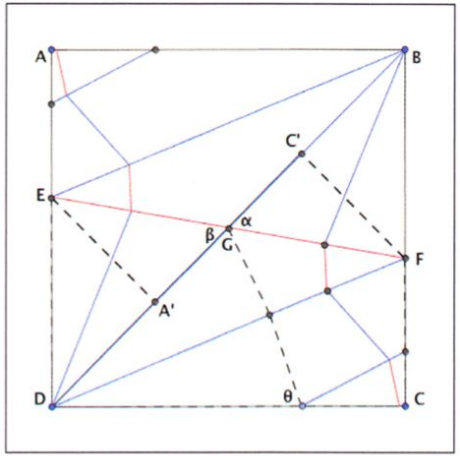
Math investigations (2015)

- Algebra
- Geometry
- Trigonometry





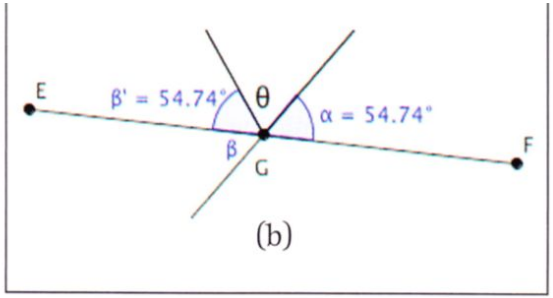


Fig. 5 The angle measure of interest is approximately 70.52°, not the ideal 72°.

Hsiao J., Finding Fifths in Origami, *Mathematics Teacher*, *Vol. 109*, August 2015

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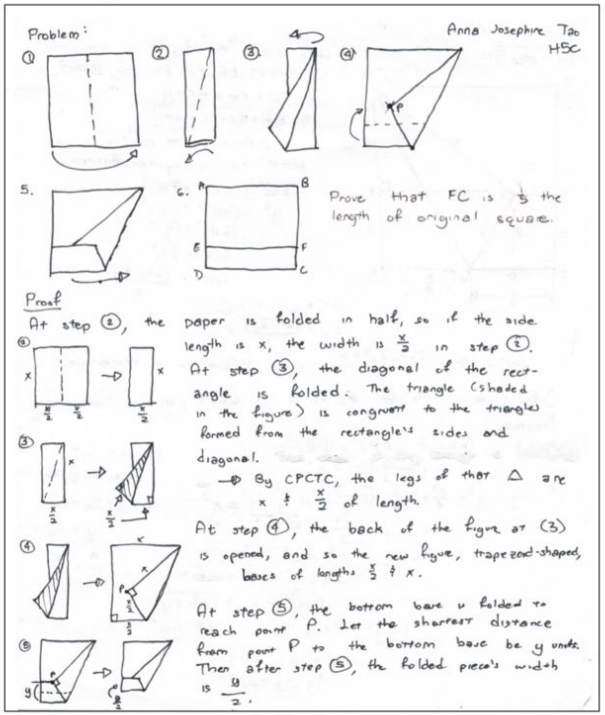
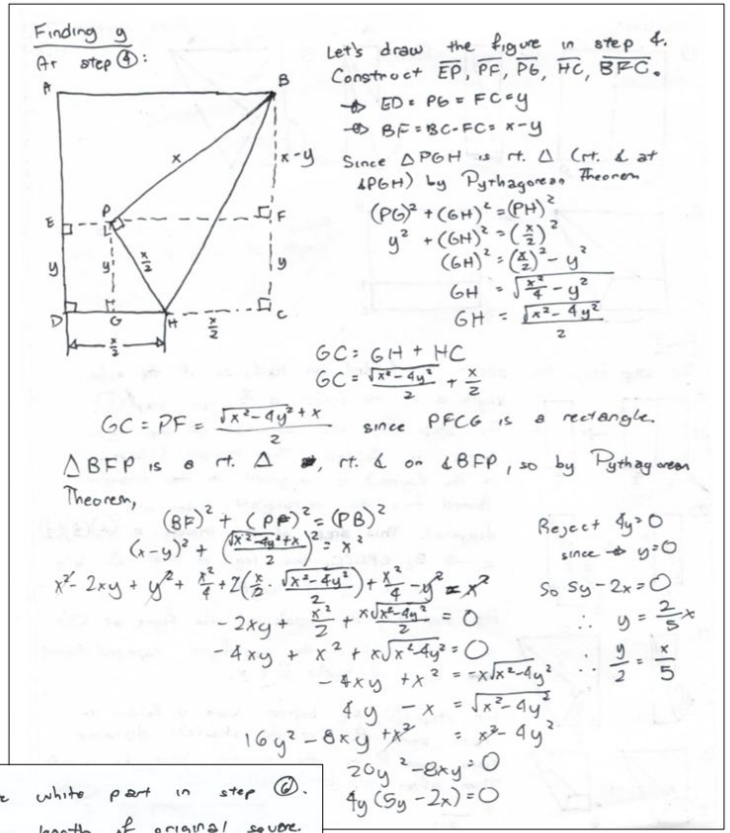
Sample student work 1



By Anna Tao,
Brooklyn Tech 2012

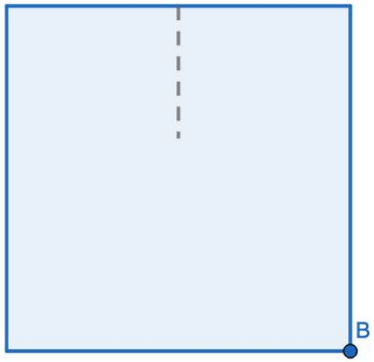


y/2 is the width of the white part in step (6).
y/2 = x/5, where x = side length of original square.
∴ the width of white part in step (6) is 1/5 the side length of the original square.
q.e.d.

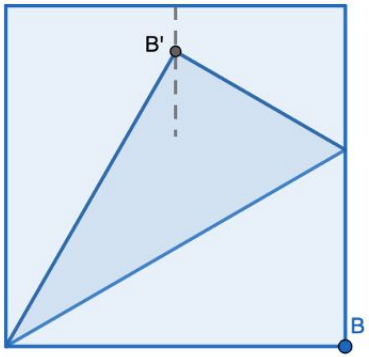
Origami Exercise

- Two-fold angle – conjecture, proof, construction of crease pattern

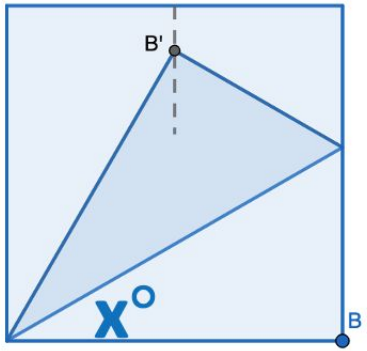
1. Pinch a midline



2. Bring the lower right corner to the midline and create a crease that ends at the lower left corner.



3. Find x.



Photos by Joy Hsiao, *Stuyvesant High School*, 2017

2017 Origami USA Annual Convention

Photos by Joy Hsiao, *Origami USA Convention*, 2017

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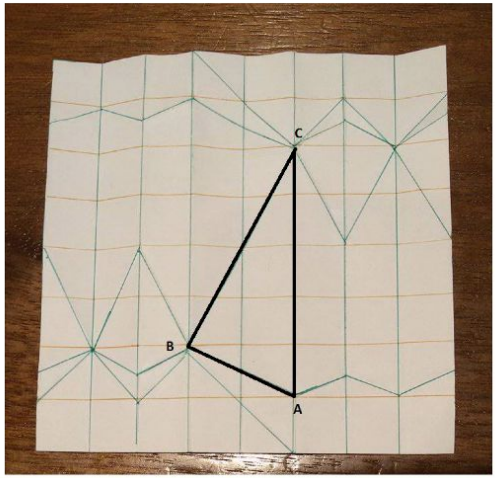
Sample student work 2

Trigonometric Application

After folding the modular piece, I outlined the crease patterns on the paper with different colors. It was helpful to also fold and indicate the 8x8 grid so that I could notice patterns in the folds. What I noticed was that vertex A of the triangle was exactly 1 unit away from the edge of the paper and that vertex C was exactly 2 units away from the edge of the paper. From this, I was able to determine that length AC is exactly 5 units long. I also noticed that the perpendicular dropped from vertex B was exactly 2 units long and the point at which the perpendicular intersects with side AC was 1 unit away from vertex A. Using this information, I used the Pythagorean theorem to find the lengths of sides AB and BC:

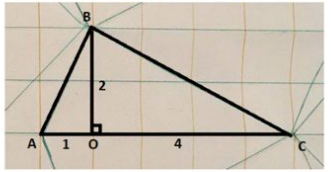


"I used the Pythagorean theorem to find the lengths of side AB and BC."

"After, I used the law of cosines to find the angles of the triangle."

$AB = \sqrt{BO^2 + AO^2} = \sqrt{2^2 + 1^2} = \sqrt{5} = 2.236\ldots$

$BC = \sqrt{BO^2 + CO^2} = \sqrt{2^2 + 4^2} = \sqrt{20} = 2\sqrt{5} = 4.472\ldots$



After, I used the law of cosines to find the angles of the triangle:

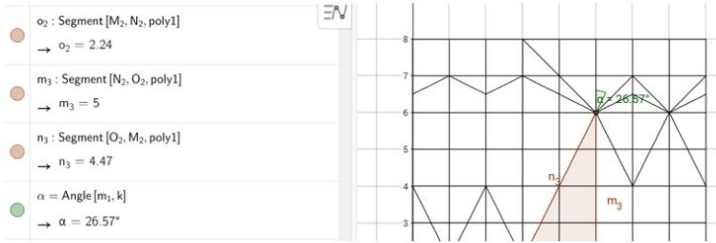
$BC^2 = AB^2 + AC^2 - 2(AB \cdot AC \cdot \cos A)$

$\sqrt{20}^2 = \sqrt{5}^2 + 5^2 - 2(\sqrt{5} \cdot 5 \cdot \cos A)$

$0.4472\ldots = \cos A$

$A = 63.4349\ldots$

I confirmed these calculations by drawing all the crease patterns on Geogebra:



"I confirmed the calculations by drawing all the crease patterns on GeoGebra."

Pre-calculus student, Stuyvesant High School, 2017

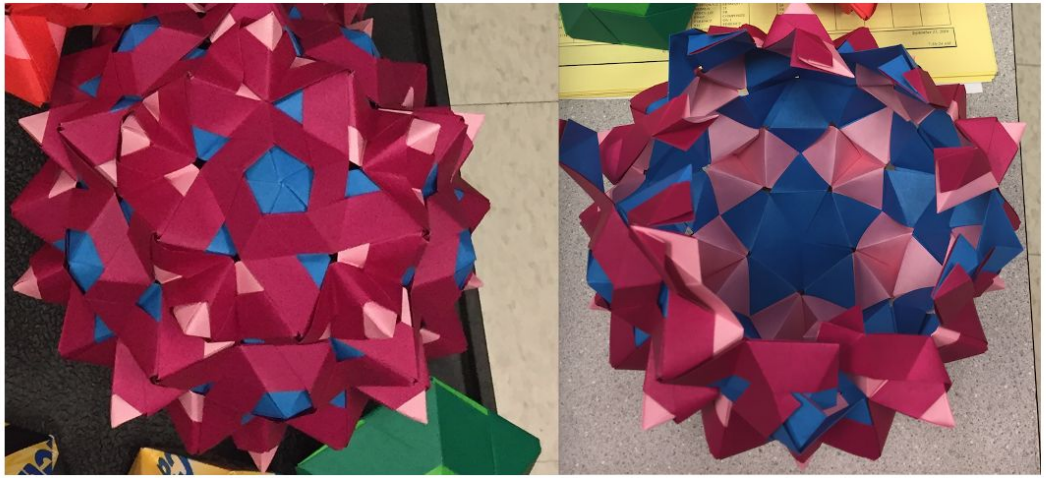
Question:

If the square paper used to fold this model is 1x1, what is the edge length of the finished cube?

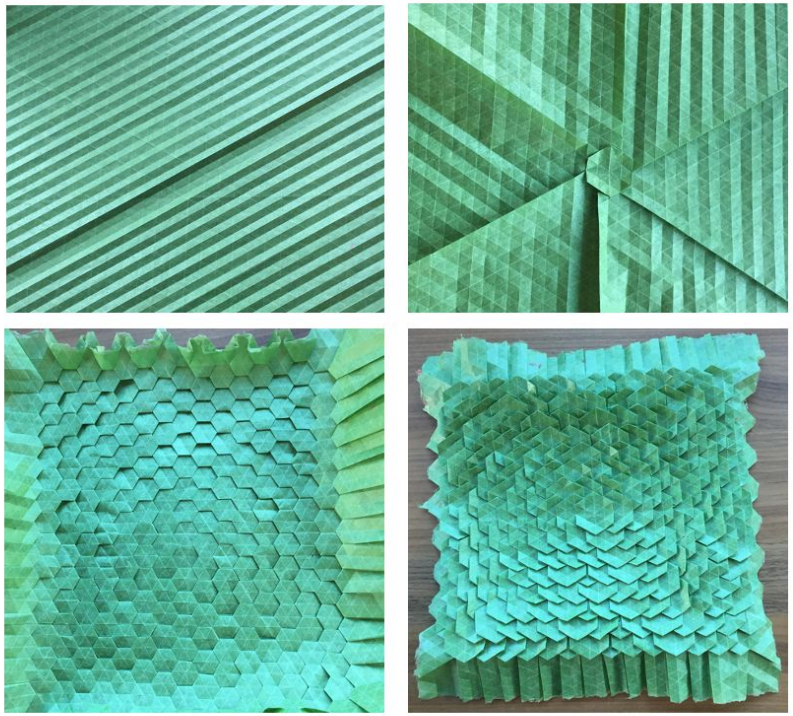
Photos by Joy Hsiao, *Linked Cubes*, 2017

Hidden Beauty –

the folding process, logic, and changing patterns



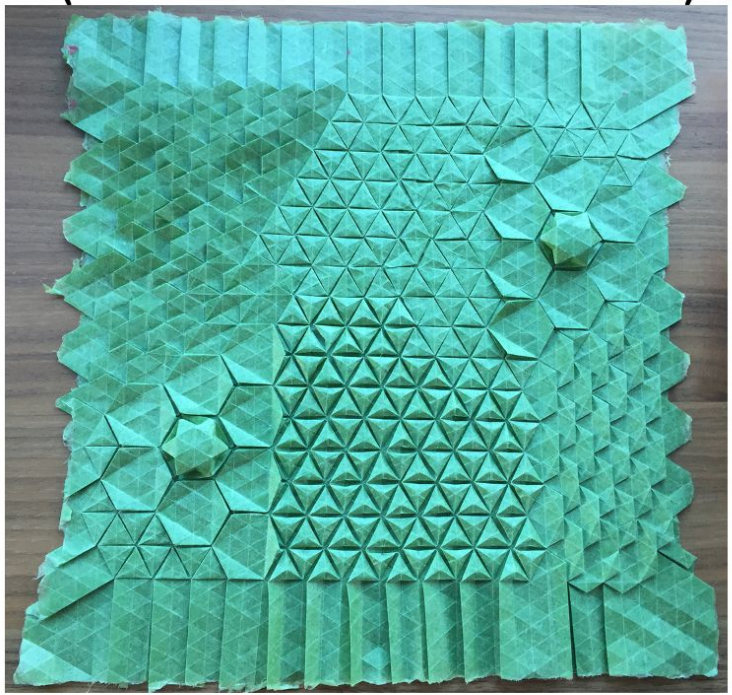
Photos and origami by Joy Hsiao, *Polyhedron*, 2017



Photos and origami by Joy Hsiao, *Four Fish*, 2017

Four Fish (2017)

(A 30 cm x 30 cm tessellation)



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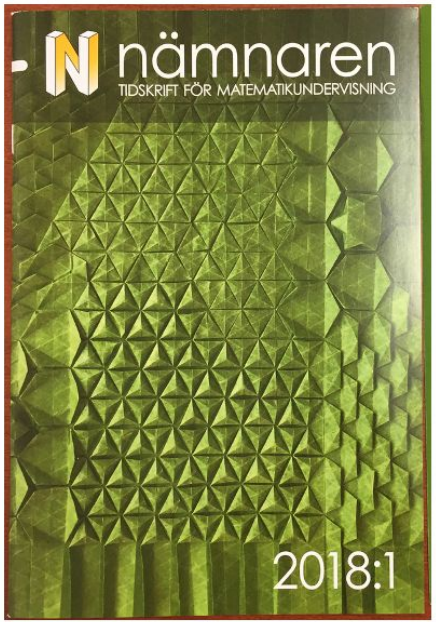
Bridges Conference

University of Waterloo (2017)

Photo by Joy Hsiao, *Bridges Conference*, 2017

Bridges Conference

University of Waterloo (2017)



Nämnaren. Tidskrift för Matematikundervisning, January 2018

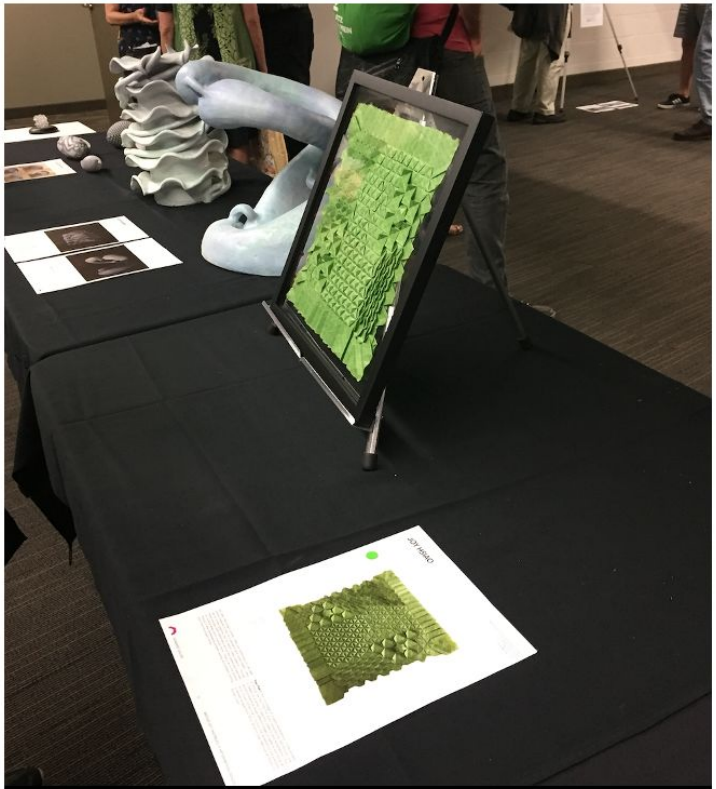


Photo by Joy Hsiao, *Bridges Conference*, 2017

Queens Metropolitan High School

(Arts and Mathematics 2018)

Mini books (one sheet)

Boxes with lids

Photos by Joy Hsiao, *Mini books and boxes with Lids*, 2018

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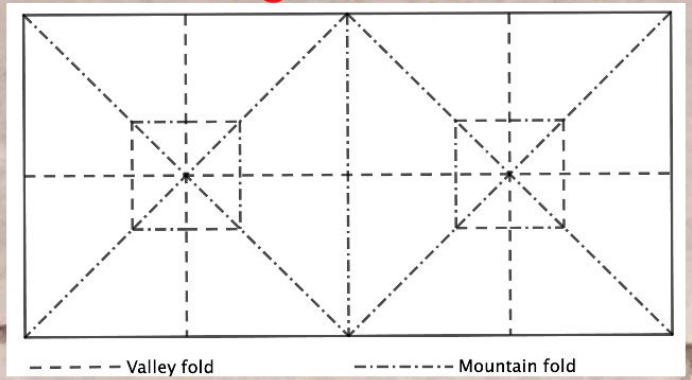
Brooklyn Technical HS (Valentine's Day 2020)

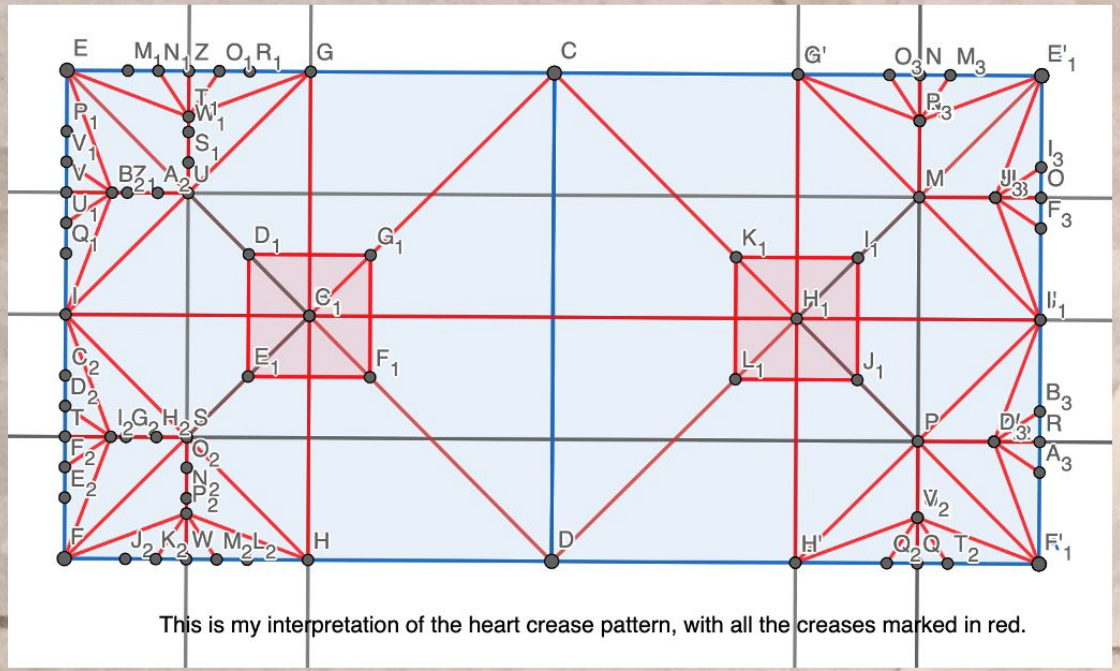
Students' observations:

- Perpendicular and parallel lines
- Angle bisectors
- Diagonals of a square are congruent and perpendicular
- 8 congruent isosceles right triangles
- Rotational symmetry of 90°
- Line symmetries
- Dilation of a square with a constant of dilation equal to 3 (estimated)
- Similar triangles
- 45-45-90 triangles and so on…



Photos and origami by Joy Hsiao, *Hearts and crease pattern*, 2020

Sample student work 3 (origami heart crease pattern)

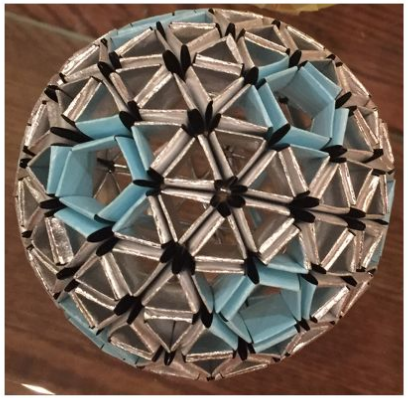


This is my interpretation of the heart crease pattern, with all the creases marked in red.

First attempt
by Lang Ni
Brooklyn Tech 2019

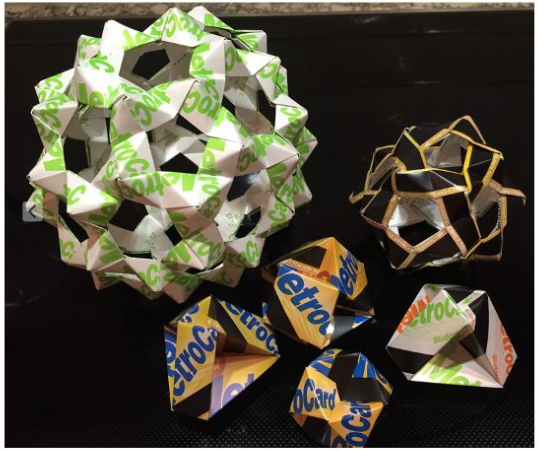
Origami and Math

Polyhedra – regular, semi-regular, truncated, stellated



Photos and origami by Joy Hsiao, *Modular Origami Solids*, 2012-2019

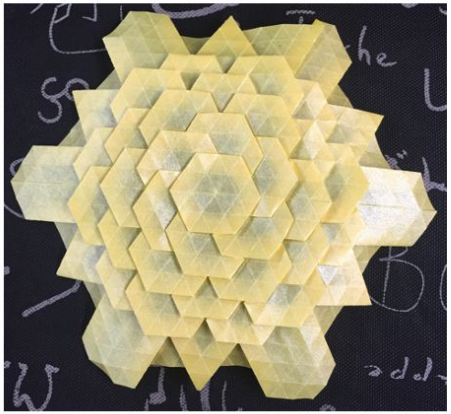
Polyhedra – regular, semi-regular, truncated, stellated

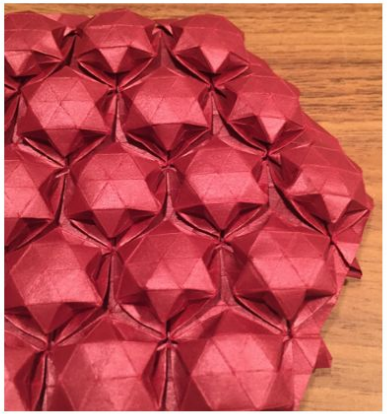
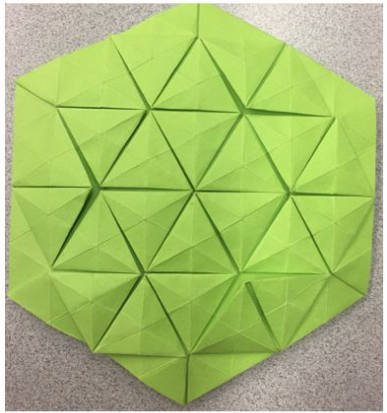


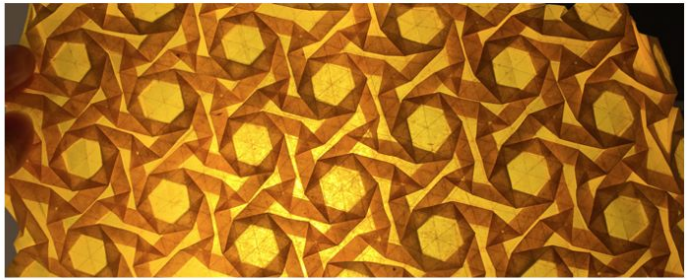
Platonic Solids	# vertices	# edges	# faces
Tetrahedron	4	6	4
Cube	8	12	6
Octahedron	6	12	8
Dodecahedron	20	30	12
Icosahedron	12	30	20

Photos and origami by Joy Hsiao, *Modular Origami Solids*, 2012-2019

Tessellations (made with one piece of paper)



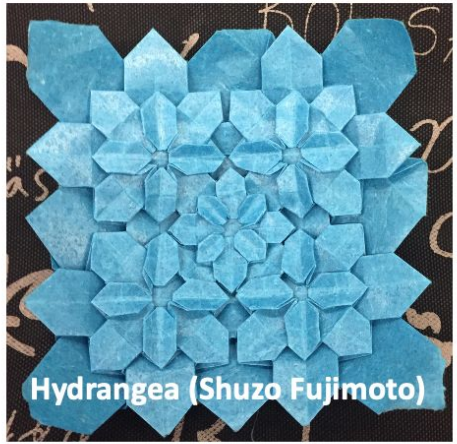



Photos and origami by Joy Hsiao, *Tessellations*, 2016-2019

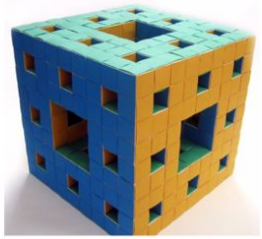
Fractals (self-similar property)

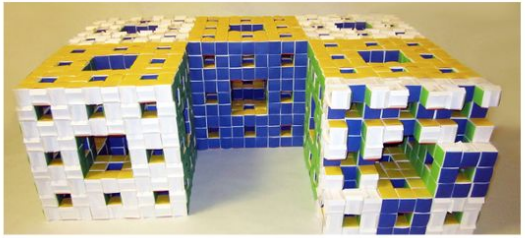

Hydrangea (Shuzo Fujimoto)


Fractal Flower (Roman Diaz)


Hydrangea (Shuzo Fujimoto)

Menger Sponge (Karl Menger)





Photos and origami by Joy Hsiao, *Flowers and Menger sponges*, 2009-2020

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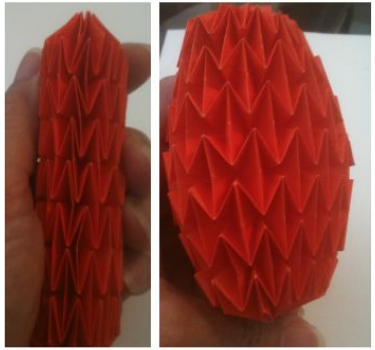
Recycled Materials

Wallets made with subway maps, book cover, and shopping bag

Photos and origami by Joy Hsiao, *Origami made with recycled materials*, 2009-2020

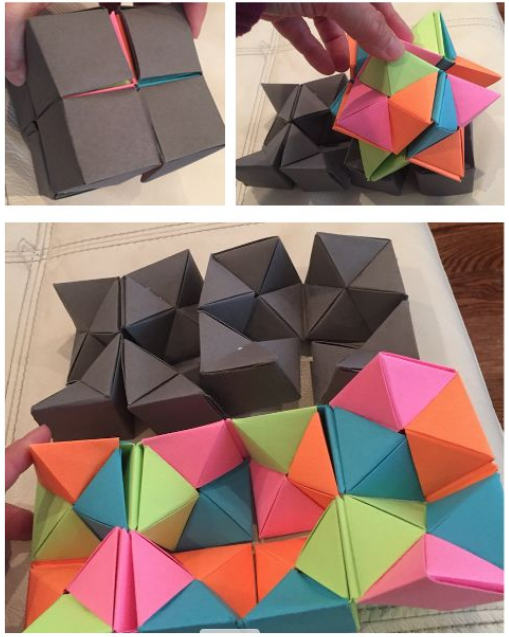
Interactive origami



Magic ball

Clic-clac

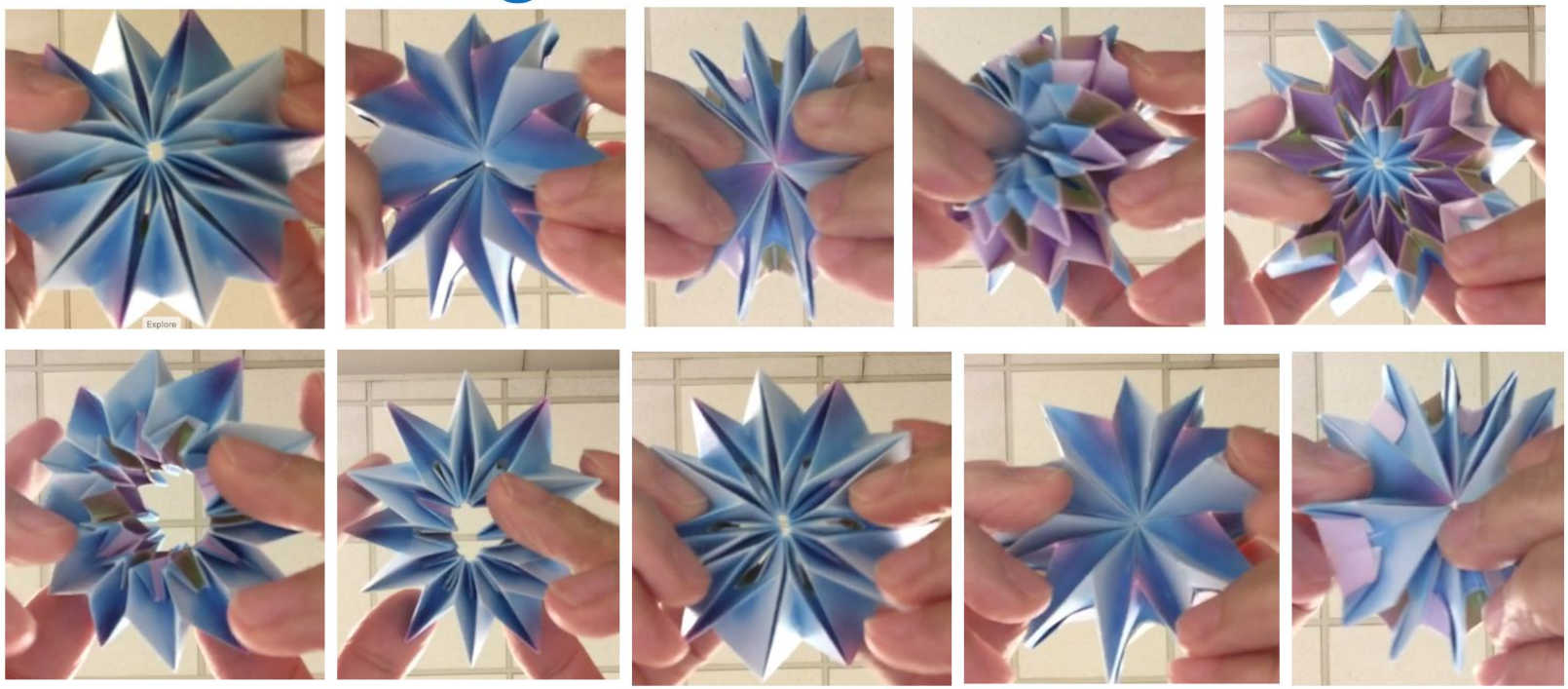
Flexicubes



Double flexicubes

Photos and origami by Joy Hsiao, *Interactive origami*, 2009-2020

Interactive origami Fireworks

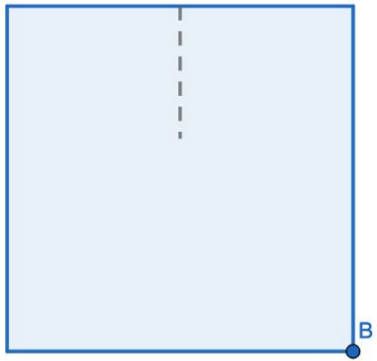


Photos and origami by Joy Hsiao, *Fireworks*, 2020

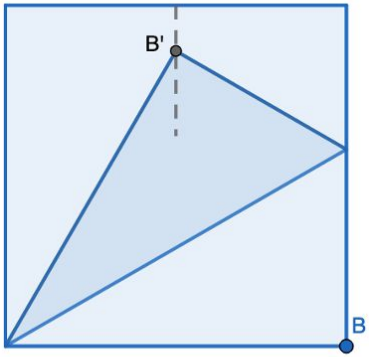
Origami Exercise

- Two-fold angle – conjecture, proof, construction of crease pattern

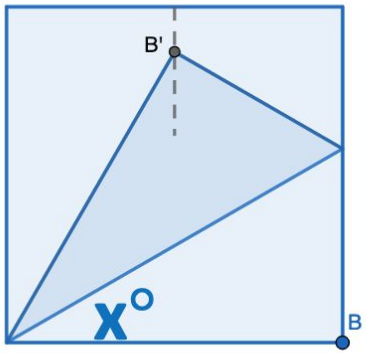
1. Pinch a midline



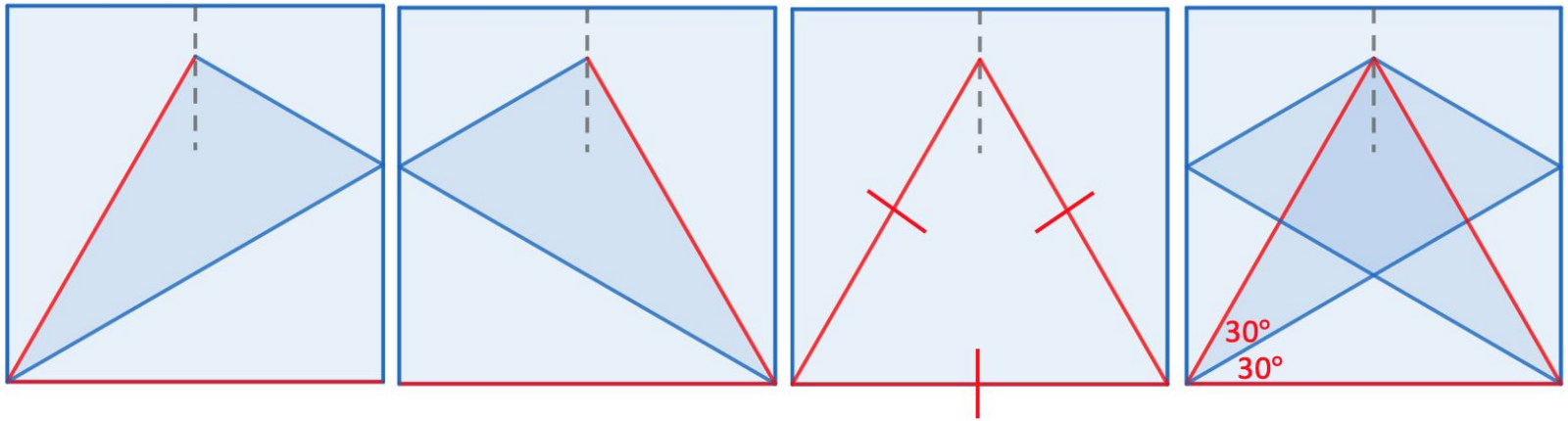
2. Bring the lower right corner to the midline and create a crease that ends at the lower left corner.



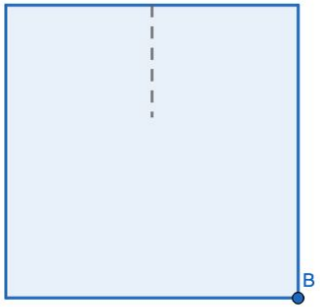
3. Find x.

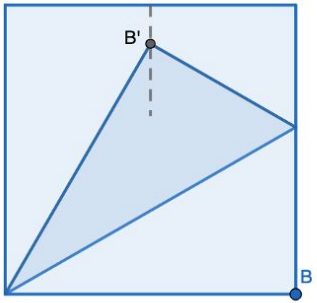


Proof without words



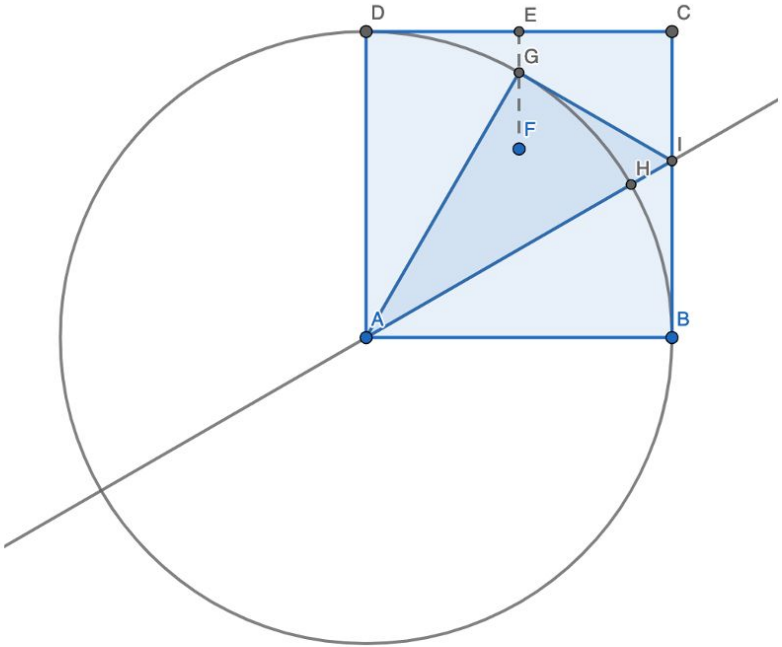
Crease pattern constructions





square
midpoint
circle
perpendicular bisector

intersection
auxiliary line
angle bisector



When I fold origami, I

1. Try to understand the logic
2. Pose questions
3. Think about extensions (variations)
4. Prove observations mathematically

When I solve math problems, I

1. Try to understand the logic
2. Pose questions
3. Think about extensions (variations)
4. Prove observations mathematically

Origami as a research topic

- Easy to pose original questions
- Hands-on, multidimensional
- High school math, applications of prior knowledge in new situations
- Accessible to students on all levels
- Rich in math connections

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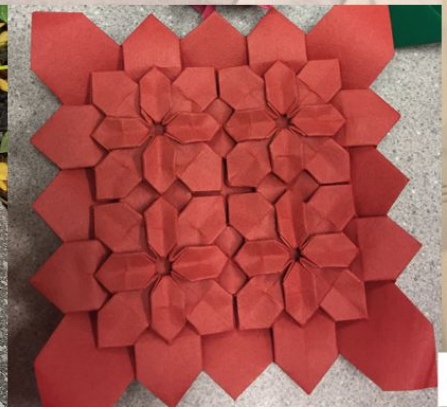
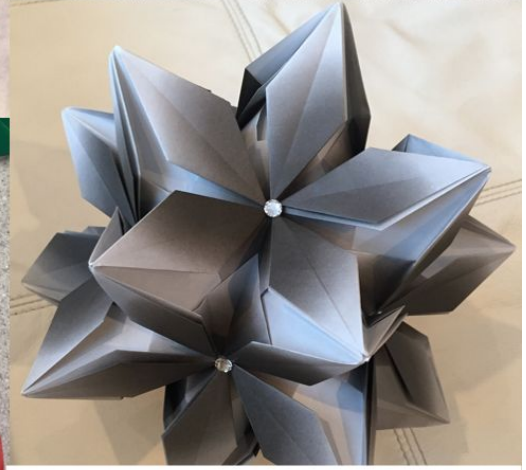
謝謝!

Thank you!

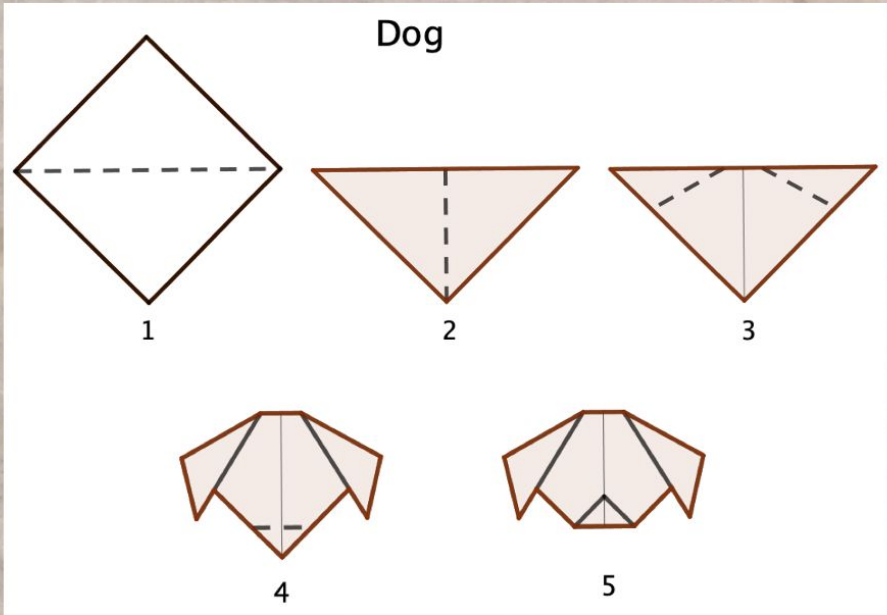
JHsiao@schools.nyc.gov





Photos and origami by Joy Hsiao

Origami Dog



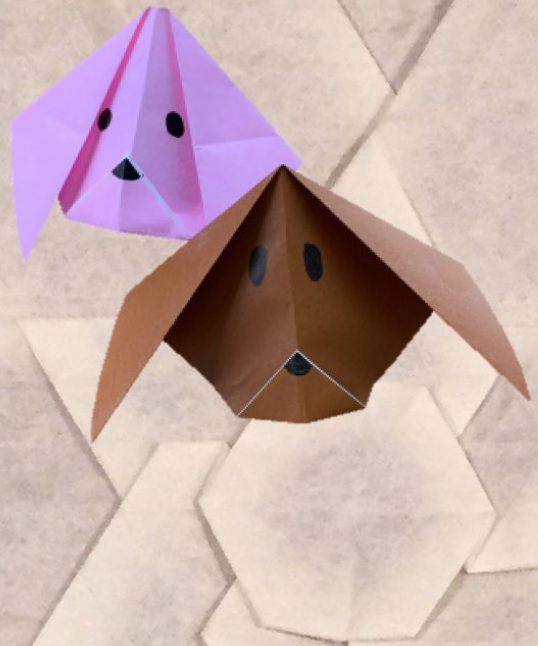


Photo and diagram by Joy Hsiao, *Dog*, 2020