Problems to Ponder: Worthy of your Effort

NCTM On-Line Session for 100th Anniversary

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Handout 2. Some initial strategies, and extensions for our Problems to Ponder

1. **Covid Café**

In a neighborhood café there are 10 seats in a row at the counter. The owners have planned for what to do when they are allowed to re-open after the Covid 19 virus has subsided. When customers enter the café for their morning coffee, they will have to sit so that they are not next to one another at the counter. Two people enter the café when it opens up.

How any different ways can those two customers sit at the counter with 10 seats so that they are not next to each other?

There are many initial strategies students might try. They might move two customers across the seats with one seat between them, then with two seats between, etc. They might try to count ordered seatings first (which is a tad tedious). They might approach the problem by putting a customer in seat 1, then moving the other customer like this:

X	___	X	___	___	___	___	___	___	___	
X	___	___	X	___	___	___	___	___	___	
X	___	___	___	X	___	___	___	___	___	
X	___	___	___	___	X	___	___	___	___	
X	___	___	___	___	___	X	___	___	___	(8 ways)
X	___	___	___	___	___	___	X	___	___	
X	___	___	___	___	___	___	___	X	___	
X	___	___	___	___	___	___	___	___	X	

___	X	___	X	___	___	___	___	___	___	
___	X	___	___	X	___	___	___	___	___	
___	X	___	___	___	X	___	___	___	___	
___	X	___	___	___	___	X	___	___	___	
___	X	___	___	___	___	___	X	___	___	(7 ways)
___	X	___	___	___	___	___	___	X	___	
___	X	___	___	___	___	___	___	___	X	

___	___	X	___	X	___	___	___	___	___	
___	___	X	___	___	X	___	___	___	___	
___	___	X	___	___	___	X	___	___	___	
___	___	X	___	___	___	___	X	___	___	
___	___	X	___	___	___	___	___	X	___	(6 ways)
___	___	X	___	___	___	___	___	___	X	

And so forth. Continuing this process there would be altogether….

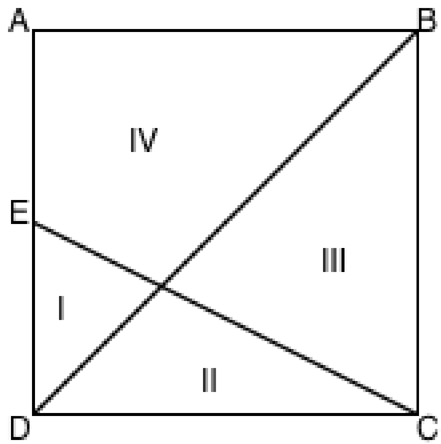
Extensions of the Covid Cafe problem.

1. What if the café had 12 seats instead of 10? Or had 15 seats? Or N seats?
2. How *does* it change the problem if we are concerned with the order that the customers are sitting in?
3. Suppose there were *three* grouchy customers who came in to the café, how many ways could the three be seated so none of them had to sit next to one another?
4. What if the café counter were a round counter, what then?

2. Compare Regions in a Square

In Figure 1. below, quadrilateral ABCD is a square, and E is the midpoint of the side AD. How do the areas of regions I, II, III, and IV compare? That is, what are the respective ratios of the areas of the four regions, I:II:III:IV ?

Figure 1.

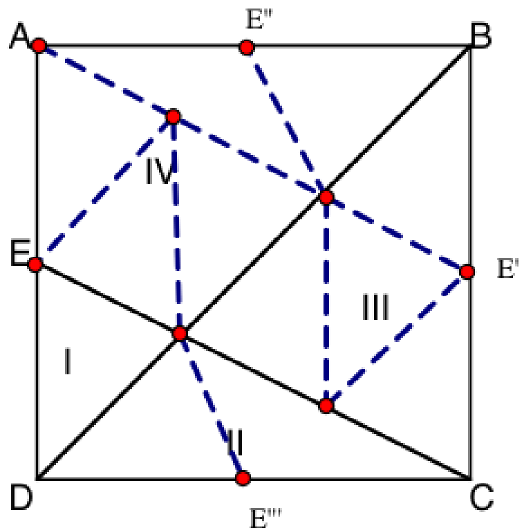


Reasoning path 1: Some students start this problem by noticing that triangular regions I and III are similar figures (AAA), and since their sides are in a ratio of 1:2, the areas are in a ratio of 1:4 for those two pieces.

Reasoning path 2: Another nice starting observation is that triangular regions I and II have the same altitude from the point of intersection of segments EC and DB, to the opposite sides (ED and DC respectively). However, since base ED is half of base DC, region I is half the area of region II. Triangles I and II have equal altitudes, with their bases in ratio 1:2, so their areas will also be in a ratio of 1:2. These two observations can give you a really good start on this problem.

This is one of those problems that can also be hammered out by using algebra and co-ordinate geometry. I have seen many HS students put this figure into the co-ordinate plane and then write some equations. Sometimes, however, algebraic approaches can make things messier, and this is one of those times.

Another interesting approach I've seen taken by elementary teachers (they are very wise!) is to break up the entire square into equal area triangles, as in the Figure below. The points E, E' E'' and E''' are the respective midpoints of all the sides of the square. We can tessellate the square, first using ten copies of region I (all these are congruent triangles), and then cutting region II and the remaining piece into two triangles with the same area as region I. This makes the problem a "duh"—or as the gestaltist used to say, "Aha!"



3. **Consecutive Sums—Staircases**

Which whole numbers can be written as a sum of consecutive whole numbers?

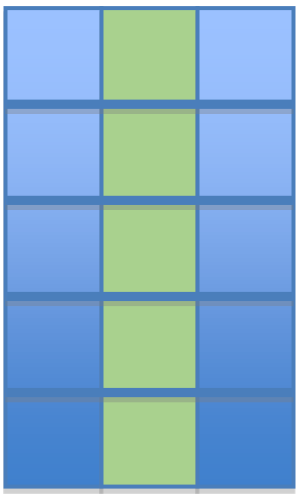
For example, 6 = 1 + 2 + 3 so it can be written as a consecutive sum. Also, 15 = 4 + 5 + 6 is the sum of consecutive whole numbers, and 15 = 7 + 8, so 15 can be written as a consecutive sum in more than one way. So, a natural extension of this problem is: How many different ways can a given whole number be written as a consecutive sum?

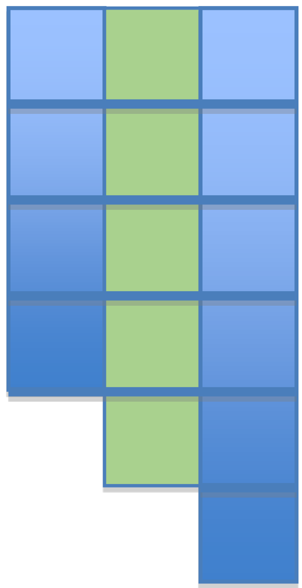
One of the first observations that many people make on this problem is that every odd number can be written as the sum of *two* consecutive whole numbers, for example, 5 = 2 + 3, 15 = 7 + 8, 87 = 43 + 44, and in general, since every odd whole number can be written in the form $2n + 1$ for some other whole number n, $(2n + 1) = n + (n + 1)$ (note: this is both a generalization, and a justification!).

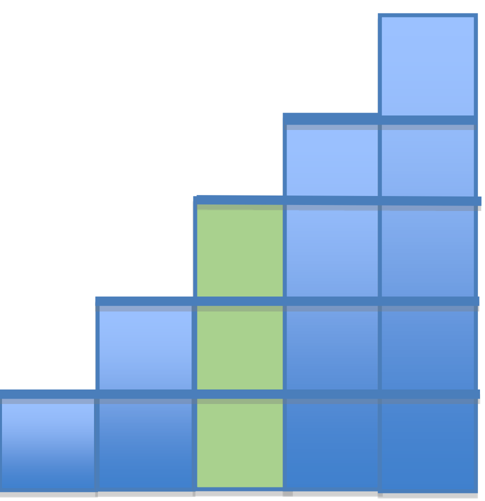
You may also have discovered that some whole numbers cannot be written as any consecutive sum. For example, 2 cannot be written as the consecutive sum of whole numbers, because the only possible choices for addends are 0 and 1. Another number that can't be written as a consecutive sum is 8. 2 + 3 + 4 = 9 (already too big), 1 + 2 + 3 = 6, (too small), all other possibilities are much bigger, or much smaller, so, we can prove that 8 cannot be written as a consecutive sum of whole numbers because we can exhaust all the possibilities that might work.

<u>A visual approach</u>*:* Consecutive Sums are *Staircase numbers*!

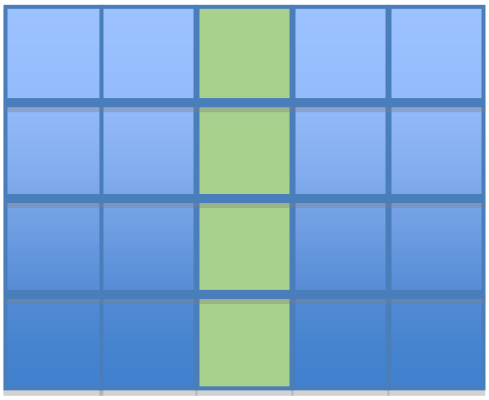
15 = 3 x 5 can be represented in a rectangular grid, as seen below. The area of this figure can then be rearranged into columns 4 + 5 + 6, so 15 is a *staircase number.* Similarly, can be rearranged into another consecutive sum, 1 + 2 + 3 + 4 + 5, as shown below.

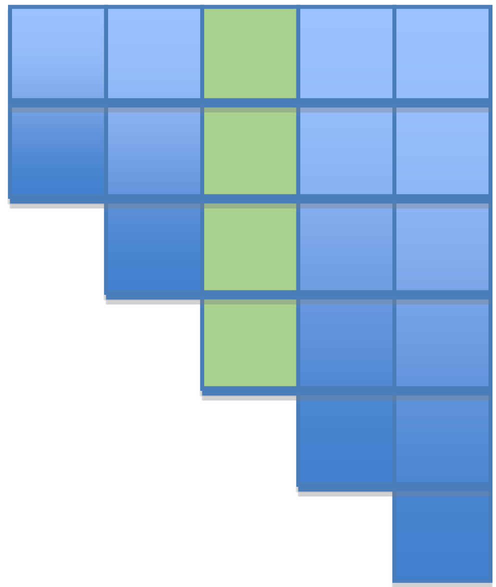






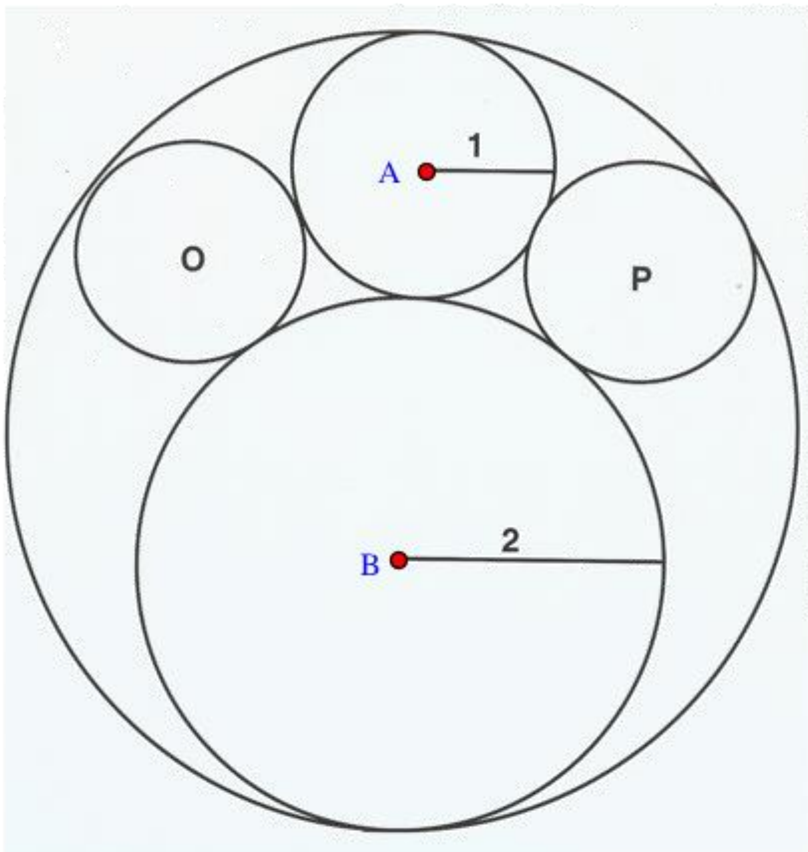
Similarly, 20 = 5 x 4, and from the 5 copies of 4, we can trade cubes across the middle column to write 20 = 2 + 3 + 4 + 5 + 6. So 20 is also a *staircase number.* Now we know any odd number can be written as a consecutive sum in at least one way, and some even numbers can be written as a consecutive sum (a staircase number). What numbers cannot be written as a consecutive sum, and why?



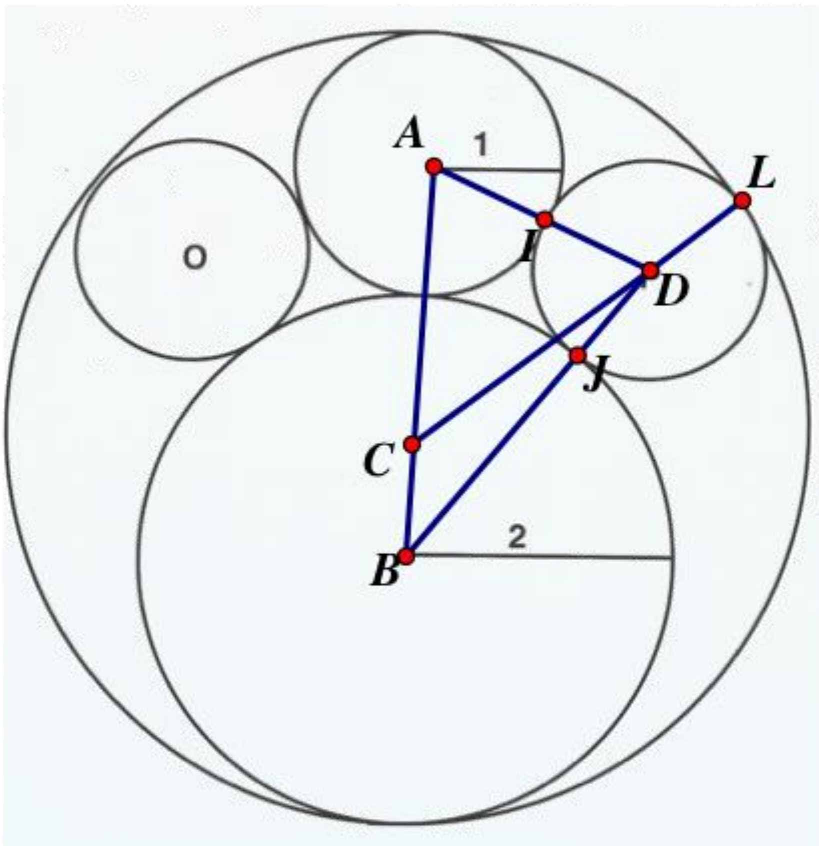


4. **Circling Around**

The circles in the figure below are mutually tangent to one another. The radius of the circle centered at A is one unit, and the radius of the circle centered at B is two units. What are the radii of circles O and P?

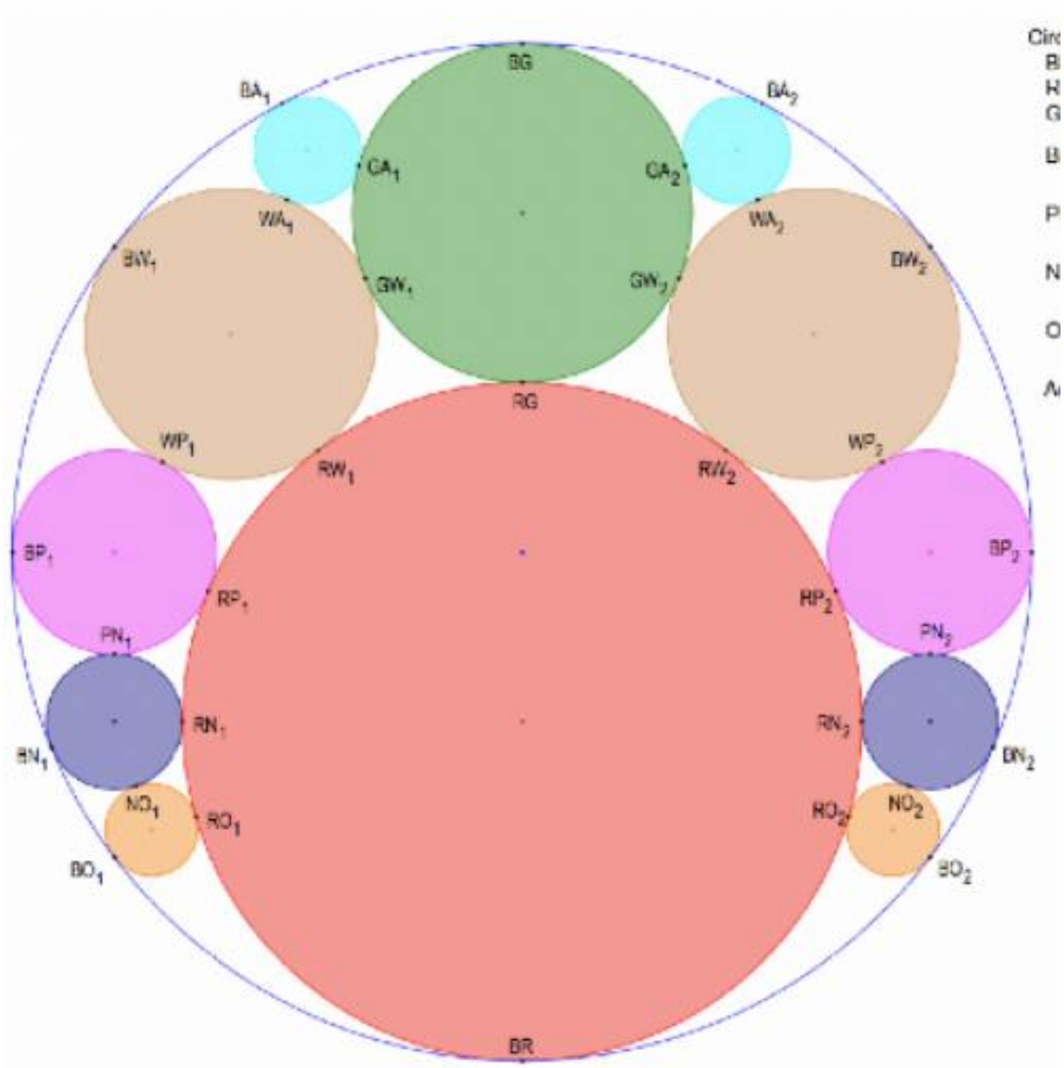


One approach that students often take is to start connecting centers of the circles, as in the figure below. This might be a fruitful approach to finding relationships.



Extensions of Circling Around

Here is a beautiful rendition of an extension of the Circling Around problem by Peggy House, shared with her permission. In Peggy's figure, the outside circle is of radius 3; the red circle of radius 2, and green circle of radius 1, with the brown circle the cotangent circle from our original problem. Then the pink circle will be the next cotangent circle in the pattern, and so forth with her other colors. What are the radii of all these successive circles?



5. **Locks & Keys—protecting the treasure**

In medieval times the inhabitants of a village decided to lock the village valuables in a giant chest to protect them. For insurance the villagers had a number of locks put on the chest, and distributed keys to the villagers so that no two people could open the chest, but any three people always could open it.

How many locks, and how many keys will they need?

Students may first react to this problem by wondering if there is enough information, as we don't know how many people are in the village. The solution to the problem will depend on the number of people in the village. There must be at least 3 people in the village, because 'no two people can open it.'

If there are exactly three people in the village, then we can solve the problem with three distinct locks, with each of the three villagers possessing one of the distinct keys to open a lock. In the table below, there are three pairs of villagers and each pair has to be missing at least one key. Pair AB is missing key 1, AC is missing key 2 and BC is missing key. However, ABC together can open the chest as among them they have keys for all three locks.

3 Villagers/ 3 Locks	1	2	3
A	O	O	X
B	O	X	O
C	X	O	O

Code: O in a cell indicates that Villager is missing a key to that lock, while an X indicates that Villager possesses a key to that lock.

What if there are exactly four villagers? Again, for each pair of villagers, there must be at least one lock for which they are *both* missing a key—one lock that they can't open. Four villagers, A, B, C, & D. The missing key-pairs for a 4 person village form a table like this:

4 Villagers/6 locks	1	2	3	4	5	6
A	O	O	O			
B	O			O	O	
C		O		O		O
D			O		O	O

So, how many locks and how many keys are necessary for a 4 person Village?

Extension of Locks and Keys

A 5-person village? An n person village? (See where generalization and justification arise here?). The general solution to this problem involves a famous mathematical diagram!

6. Water Bucket Conundrums

At a rural cabin water must be drawn from a well using buckets. The cabin has only a 4 - gallon bucket and a 9 - gallon bucket. In one trip to the well, what whole number amounts of water in gallons could you bring back to the cabin in the buckets? There are no other markings on the pails, no estimating allowed, exact whole number amounts only!

I have found that this problem works well for middle school students up through college mathematics majors, it has a lot of 'reach.' Students first reaction is to say that 4, 9, 13, and 0 gallons are possible, and to respond "What, are you serious? How can you get anything else?" But eventually some student will realize that you can pour and make 'trades' with the water, from one bucket to another. For example, two 4 gallon buckets poured into the 9 gallon bucket will give you an 8 gallon return. Three 4 gallon pours into the 9 gallon bucket will leave you with 3 gallons left over in the small bucket to bring back—see what else you can get.

Extensions to Water Buckets

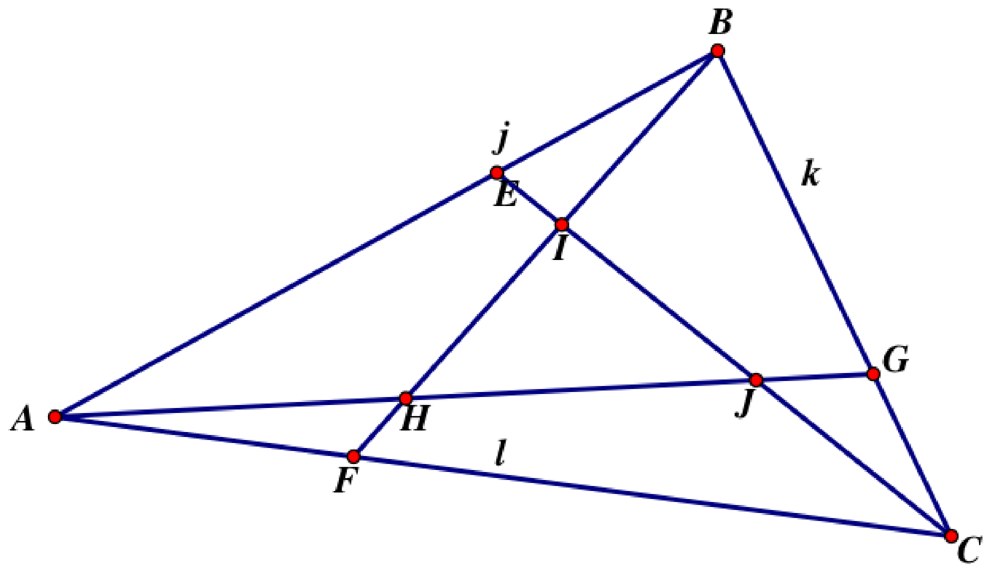
What if you had a 4-gallon bucket and a 10-gallon bucket?
What if you had an n-gallon bucket and an m-gallon bucket?

Note: there is considerable justification involved in the n and m gallon buckets generalization, it's leads to a big idea in number theory!

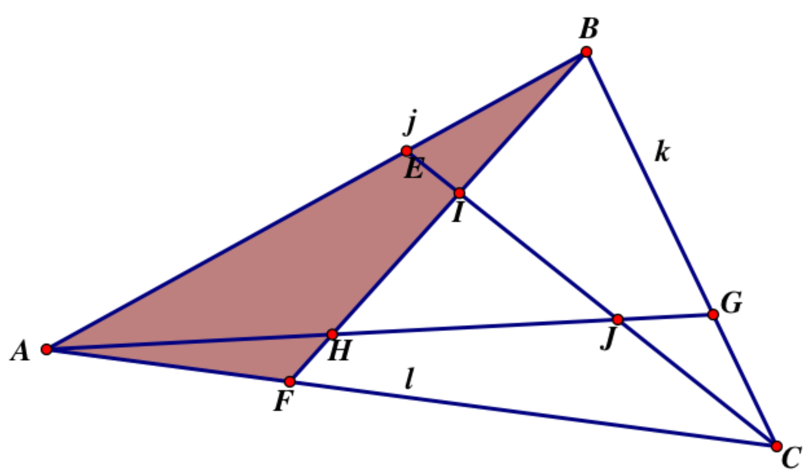
7. **Cevian Triangles**

In the triangle below, segments AG, BF, and CE intersect the opposite sides at the 1/3rd mark on BC, AC, and AB, respectively. The pairwise intersections of these segments then creates another triangle, IJH, in the interior of triangle ABC.

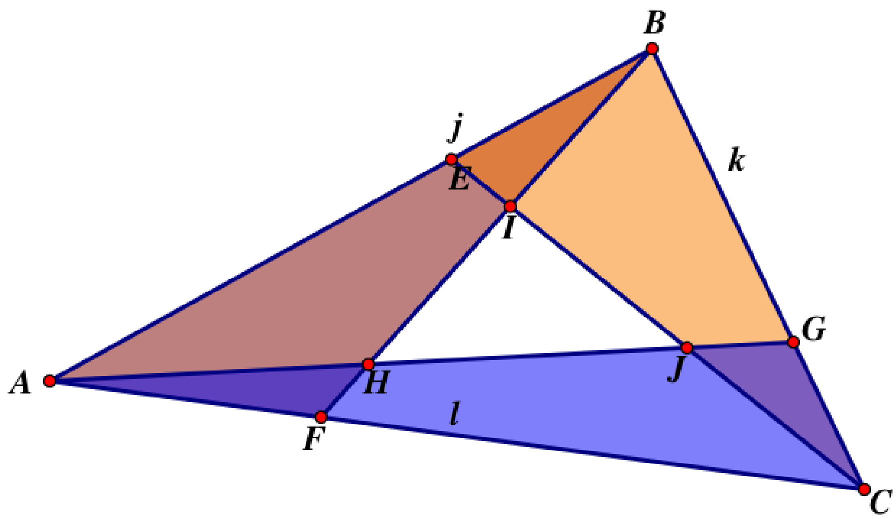
What is the relationship between the areas of triangles ABC and IJH?



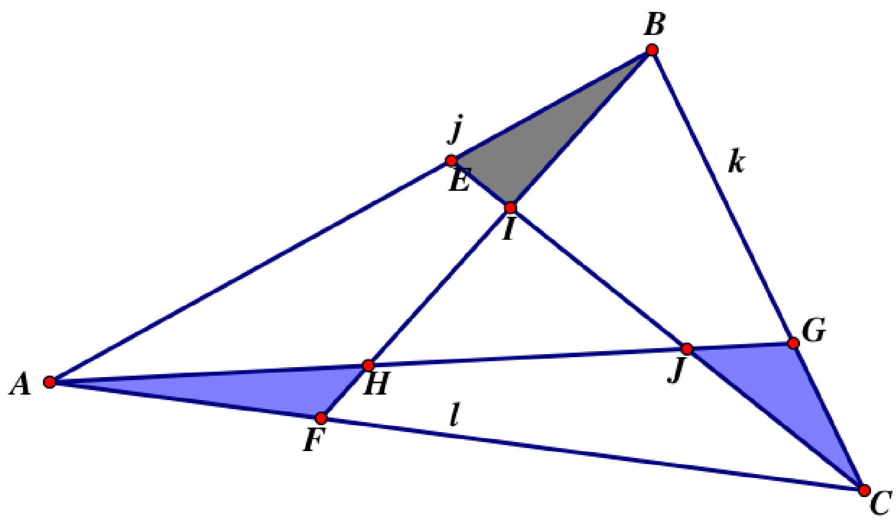
Sometimes with problems like this one it is helpful to look for relationships among various triangles and polygonal regions. Just listing what we *do* know can sometimes help us to get started. For example, segment AF is 1/3 the length of segment AC, so we know that triangle ABF has 1/3 the area of triangle ABC, because both these triangles also have the same altitude from point B.



The same is true for triangles AGC, and CEB, each of them is also 1/3 the area of the original triangle, ABC. If we shade all three of these 1/3area triangles in, we can see more relationships.



Each of the triangles ABF, CBE, and ACG, is 1/3 the area of triangle ABC, so
Area ABF + Area CBE + Area ACG = Area ABC. On the other hand, in the figure above we also see that these three triangles overlap so as to leave the area of triangle IHJ uncovered, and the area of triangle IHJ is exactly what we are trying to find! Since the areas of ABF, ACG, and CBE **do** add up to the area of triangle ABC, this means that the 'missing area' in the picture above can be accounted for in the *overlap* parts, the small triangles AHF, BIE, and CJG, because those are being 'counted twice.'



Keep going….

Extensions of Cevian Triangles.
What is the relationship if the points E, G, and F are at the ¼ th marks on AB, BC, and AC, respectively? How about the 1/nth marks?

More Information on these Problems (and many others)

For further details on solutions to these (and many other) Problems to Ponder, check out this link to the NCTM resource archives. The problems were originally posted in my monthly Presidential Messages in NCTM's publication *Summing UP.* Solution details are spread out over three consecutive Messages. Each month a new problem is posted, initial solution strategies are shared the following month, and more detailed solutions are finally shared in the third month after the original problem was posed.

https://www.nctm.org/News-and-Calendar/Messages-from-the-President/Archive/Problems-to-Ponder/