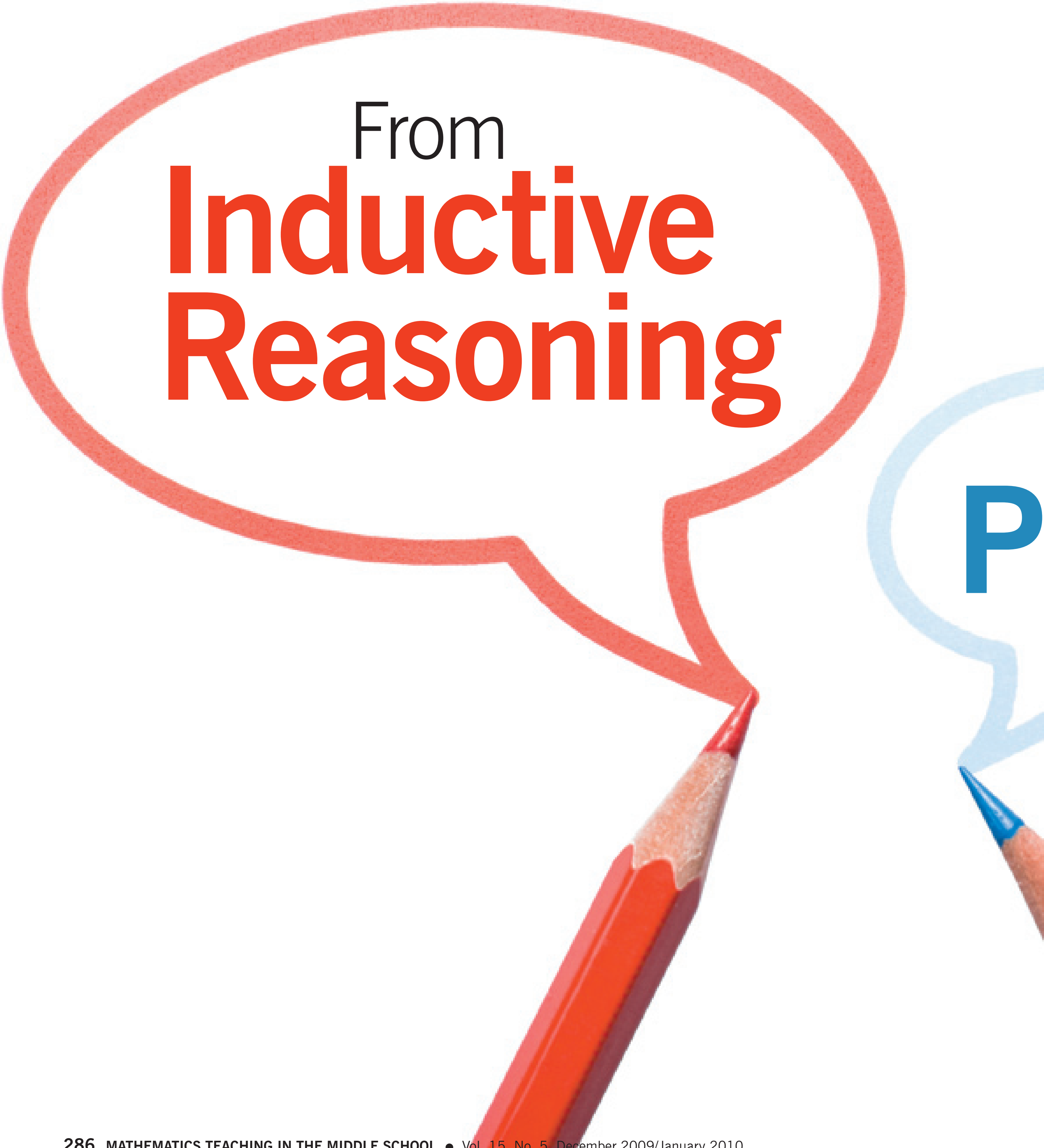
From Inductive Reasoning



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When students learn strategies for identifying key ideas in inductive arguments, these ideas can be extended to provide more formal proofs.

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Mathematical proof is an expression of deductive reasoning (drawing conclusions from previous assertions). However, it is often inductive reasoning (conclusions drawn on the basis of examples) that helps learners form their deductive arguments, or proof. In addition, not all inductive arguments generate more formal arguments. This article draws a distinction between inductive reasoning, which successfully lays the foundation for more formal arguments, and other inductive reasoning, which does not. The strategy of identifying inductive arguments that prompt formal arguments is an important skill for both middle school teachers and teachers of prospective middle school teachers.

Principles and Standards for School Mathematics (NCTM 2000) calls for students to develop and evaluate mathematical arguments from prekindergarten through grade 12. In grades 6–8, the reasoning skills developed in earlier grades should be sharpened, and inductive and deductive reasoning should be used to formulate mathematical arguments (NCTM 2000).

This article was triggered by the Morris finding that many junior and senior prospective K–8 teachers in her study "did not appear to understand the relationships among mathematical proof, explaining why, and inductive arguments" (Morris 2007, p. 510). Moreover, many of the prospective teachers who viewed transcripts of student arguments accepted example-based arguments as a valid proof. Accepting empirical evidence as proof was most common among prospective teachers who viewed transcripts that lacked a deductive argument. Morris asserts that preservice teachers could benefit from instruction on how to expand a key idea into a general proof.

A KEY IDEA: INDUCTIVE REASONING

Figure 1 is an example of how a middle school student might convince others of the validity of a formula for counting figurative numbers. To bring the point home, *Principals and Standards* (NCTM 2000, p. 264) states, "Although mathematical argument at this level lacks the formalism and rigor often associated with mathematical proof, it shares many of its important features."

What is it about the argument in **figure 1** that lacks formalism and rigor? The answer is that the argument is inductive; it is a generalization that has been made on the basis of evidence, in this case, empirical evidence. However, the argument is stronger than merely testing Gauss's formula for a few cases, which is purely empirical evidence. That is because the argument is what Morris (2007) calls a "single-case key idea inductive argument." In other words,

Fig. 1 A middle school student might use this argument to convince a classmate of this method for finding the sum of the first *n* counting numbers.

1 + 2 + 3 + 4 + 5 + 6 + 7 7 + 6 + 5 + 4 + 3 + 2 + 1 8 + 8 + 8 + 8 + 8 + 8 + 8	Students can see that the sums of the pairs of addends can be represented as 7×8, or 56.
	Because each number is listed twice, they divide by 56 by 2, resulting in $\frac{7\times8}{2}=\frac{56}{2}=28.$

Fig. 2 These two inductive arguments attempt to show that a number is divisible by 3 if the sum of its digits is divisible by 3. Although the argument shown in (a) provides more examples, the single example in (b) is closer to a proof since it demonstrates a *key idea.*

Number	Sum of Its Digits	Is the Number Divisible by 3?	Is the Sum of the Digits Divisible by 3?
12	3	Yes	Yes
13	4	No	No
360	9	Yes	Yes
361	10	No	No
137,541	21	Yes	Yes
157,541	23	No	No

(a)

$$\begin{aligned} 471 &= 4(100) + 7(10) + 1(1) \\ &= 4(99 + 1) + 7(9 + 1) + 1(1) \\ &= 4(99) + 7(9) + (4 + 7 + 1) \\ &= (4 \times 33 + 7 \times 3) \times 3 + (4 + 7 + 1) \end{aligned}$$

The quantity in the last (right) parentheses is the sum of the digits in 471.

(b)

the argument contains a *key idea*—an understanding or insight—that lays out the reasoning for a more explicit and rigorous argument.

The notion of *key ideas* for proofs was developed by Raman (2003). Studying the concept of proof held by calculus students and their instructors, she found that professors and graduate teaching assistants understood proofs through key ideas, whereas students tended to view proofs simply as representations. She asserts that key ideas might be able to bridge the gap between informal arguments and more formal proofs. This article views key ideas as the link that allows inductive arguments to encourage deductive ones.

INDUCTIVE ARGUMENTS AND KEY IDEAS

Some inductive arguments express key ideas, and some do not. Consider the two arguments in **figure 2**. Both assert that a number is divisible by 3 if and only if the sum of its digits is divisible by 3. Both arguments are empirical. Yet they are very different in terms of the ideas expressed.

The argument in **figure 2a** verifies the property for several cases, including some large numbers, but it offers no insight into why the divisibility rule works. The argument in **figure 2b** shows only one case, but it gives the reasoning for a more general argument. Explicitly, it expresses the *key idea* behind why the property works—that each of a number's digits times their corresponding place values (in the tens place or higher) can be written as a multiple of 3 plus the digit itself. Although this argument lays the foundation for a more formal argument, it is still a long way from a deductive proof.

EXPANDING KEY IDEAS INTO PROOF

Just because a student expresses an inductive argument with a key idea does not necessarily mean that he or she knows how to expand the key idea into a deductive proof. Because the teacher may have to facilitate this link, we need to be skilled in recognizing key idea inductive arguments *and* expanding the argument into a proof. Expanding a key idea argument means to write it so that it is general; this means the argument works for all cases, not just the ones tested. This can be done using symbols or prose. The following statement is an example that can be used to generalize the argument in **figure 2b**.

> All of a number's digits in the tens place or higher represent a numerical value that can be written as a multiple of 3 (a power of 10 – 1) plus the digit itself.

For practice looking for key ideas and expanding them into proof, consider the arguments in **figure 3**, written by college freshmen enrolled in the author's mathematics-for-prospective-teachers content course. The arguments are in response to this statement:

Fig. 3 These three arguments are constructed for this statement: Every odd whole number is the sum of two consecutive whole numbers. Only two of the three contain key ideas that can lead to proof.

$$0 + 1 = 1$$
$$1 + 2 = 3$$
$$2 + 3 = 5$$
$$3 + 4 = 7$$
$$4 + 5 = 9$$

As the pattern continues, we see that every odd whole number will be the sum of consecutive whole numbers.

(a)

Take any odd number, say, 17. Because 17 is odd, it is an even number plus 1.

$$17 = 16 + 1 =$$
$$8 + 8 + 1 = 8 + 9$$

(b)

$$1 = 0 + 1$$
$$3 = 1 + 2$$
$$5 = 2 + 3$$
$$7 = 3 + 4$$
$$9 = 4 + 5$$

I tried the first five numbers, and it clearly works!

(c)

Every odd whole number is the sum of two consecutive whole numbers.

All three arguments are inductive, but only two express key ideas that can be generalized into deductive arguments. Further, the key ideas are different in the two arguments. Can you identify the key ideas and how they differ?

The argument in **figure 3a**, at first glance, appears to argue the converse, or that *the sum of two consecutive whole numbers is odd*. However, the prose at the end tells us that the argument contains a key idea: If you choose any odd number, we could continue this pattern until we would finally obtain the odd number you chose. The process develops a systematic way to generate any odd number, and this process can be developed into a more general argument.

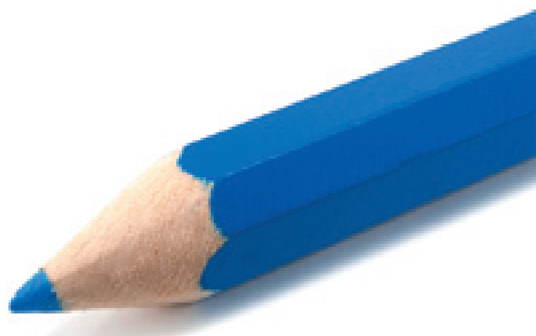
The key idea in **figure 3b** is different. The key idea is that every odd number is an even number + 1. This even number is the double of some whole number, and this double can be written as a sum. Add 1 to one of the summands, and you have the sum of two consecutive numbers. We call this a *constructive argument* because it shows how to construct the consecutive numbers.

Figure 3c, on the other hand, is purely empirical. It is similar to the argument in **figure 3a** but is actually very different. The reverse of the left and right side of the equations is a subtle difference but important. The author of the argument in **figure 3c** may not hold a key idea about the statement, which is supported by the comments at the end. It differs from the argument in **figure 3a** in that it tests several cases (empirical evidence), as opposed to developing a systematic way to produce all cases.

Moving from Inductive Reasoning to Inductive Reasoning and Proof

Arguments involving number theory and geometry problems can easily transition from inductive to deductive. However, some teachers may find it challenging to find good problems for student investigation. Below are some sample statements (and their corresponding key ideas) that teachers can use in their middle school classrooms:

1. If A divides B and A divides C, then A divides $B + C$.
   (Key idea: B and C both have a factor of A.)
2. The sum of two odd numbers is even.
   (Key idea: Odds are even numbers plus 1.)
3. The sum of the angles in a triangle is 180 degrees.
   (Key idea: Three copies of any triangle can be arranged in the form of a trapezoid so that the three angles are supplementary; the parallel postulate is needed.)
4. If a reduced whole-number fraction is equivalent to a terminating decimal, then the fraction's denominator has only 2s and 5s in its prime decomposition.
   (Key idea: Powers of 10 are only divisible by 2s, 5s, or products of 2s and 5s.) (The converse of this result is also true.)



IDENTIFYING KEY IDEAS IN INDUCTIVE ARGUMENTS

How do we consistently recognize the difference between inductive arguments that contain key ideas and those that do not? A deeper look at the arguments in **figure 3** may help. When we consider how they might expand into a more formal argument, the notion of key ideas becomes clearer.

Consider **figure 3c**. At the end of the argument, the reader is left with this notion: If you picked any arbitrary odd number, then you would be able to find the two consecutive numbers. However, we are not given any insight into how we can be certain. Therefore, the conclusion of this inductive argument is uncertain (maybe we were just lucky and happened to pick examples that worked) and, therefore, it does not expand into proof.

The argument in **figure 3a**, however, tells you how to "build" to the number you picked. It is as though you were saying, "With enough time, this process will bring you to your chosen number, no matter how big it is." Although the conclusion remains uncertain, we are at least given a foundation for the logic. (To expand this argument into a formal one would require mathematical induction, which is a strategy for proof that is beyond the scope of middle school. Nevertheless, the logical foundation expressed in **fig. 3a** is accessible.)

The argument in **figure 3b** describes a process for finding the consecutive numbers. The algebraic representations suggest how to expand the key idea in this argument into a deductive argument. Pick any odd number ($2k + 1$). Subtract 1 and divide by 2

$$\frac{(2k+1)-1}{2} = k$$

to produce the smaller of the numbers (k). Add 1 to that number, and you have the other number ($k + 1$). The sum of these two numbers ($k + k + 1$) is your original number. Although the conclusion of the argument presented in **figure 3c** is still uncertain, the use of symbols would make the argument general and therefore an acceptable proof.

KEY IDEAS THAT EXPLAIN WHY AND THOSE THAT DO NOT

Arguments that explain why a conjecture is true are of great pedagogical value because they increase students' understanding of mathematical concepts. Hannah (1990) asserts that even professional mathematicians prefer arguments that explain why and recommends that educators use these types of arguments whenever possible. Not all key-idea inductive arguments explain *why*, and it is important to distinguish those that do from those that do not.

The argument in **figure 3b** explains why every odd number is the sum of two consecutive integers. The *why* is in the construction, which gives us insight into a property of odd numbers that makes the statement true. Any odd number can be written as the double of a whole number + 1.

The argument in **figure 3a** does not explain *why*. It lays out the reasoning for a formal mathematical argument but does not provide insight into why the property is true. Although we may be convinced by the argument, we are not shown why the conjecture is true.

A word of caution: There appears to be some risk in focusing only on evaluating whether an argument explains why mathematical statements are true without discussing whether the argument or explanation generalizes to a deductive mathematical proof. Morris (2007) found that junior and senior preservice K–8 teachers who focused on whether an argument *explained why* tended to accept purely example-based arguments as valid proofs as well. The solution seems to be for educators, whether working with middle school students or preservice teachers, to always expand on key ideas by noting how the argument generalizes to one that considers all cases.

CAUTIONS ABOUT INDUCTIVE REASONING AND KEY IDEAS

Key idea argument evaluation can be used to avoid making false generalizations through inductive reasoning. The Reasoning and Proof Standard for Grades 6–8 contains two examples to illustrate pitfalls in using inductive reasoning (see **fig. 4**). The first involves a conjecture that a number is divisible by the product of any two of its factors (NCTM 2000, pp. 264–65). This false conjecture might arise from developing a test for divisibility by 6. When discovering a divisibility rule for 6—a number must be even and the sum of its digits must be divisible by 3—students might incorrectly extend this notion of divisibility by products of factors to all numbers and their factors. It should be noted that this false conjecture does not arise from key idea thinking. The inductive reasoning lacks a key idea. The key idea behind the conjecture for 6 involves noting that 2 and 3 are relatively prime, and this notion was not involved in the conjecturing process.

The second cautionary example in *Principles and Standards* arises from a conjecture about a systematic way to determine how many segments of different lengths can be made by connecting pegs on a square geoboard that is n units on each side (NCTM 2000, pp. 266–67). Through inductive reasoning from the pattern generated from considering a 1×1, 2×2, 3×3, and 4×4 geoboard (see **fig. 4b**), students incorrectly conjecture that a 5 × 5 geoboard will have

$$(2 + 3 + 4 + 5) + 6$$

unique segment lengths, failing to note that the length of vertical line segment (5 units) is a repeat of a Pythagorean triple in the 4 × 4 geoboard. This example of inductive reasoning also lacks key-idea thinking. A key idea would include reasoning (beyond actually measuring the segments) that justifies that the new segments formed are not repeated segments. For example, students should note that repeated lengths can appear when

$$\sqrt{a^2+b^2}=\sqrt{c^2+d^2},$$

with $\{a, b\}$ being different from $\{c, d\}$. In the 5 × 5 case, $a = 5$ and $b = 0$ produces a length already considered in the 4 × 4 case, $c = 4$ and $d = 3$. In the 8 × 8 case, $a = 8$ and $b = 1$ produces a repeated length from the 7 × 7 case, $c = 7$ and $d = 4$.

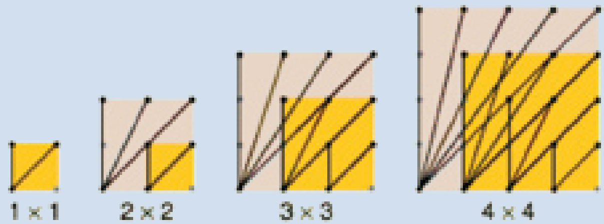
Therefore, this cautionary lesson from *Principles and Standards* is clarified with key-idea thinking. Such pitfalls can be avoided if educators always ask whether the inductive arguments contain a key idea that could be developed into a formal deductive proof as opposed to inductive arguments based purely on empirical evidence.

Fig. 4 Examples from *Principles and Standards* demonstrate the pitfalls of inductive reasoning to provide proof.

Students who encounter the rules of divisibility by 2 and by 3 in number theory know that numbers whose units digits are divisible by 2 are divisible by 2 and numbers whose digits add to a number divisible by 3 are divisible by 3. A teacher might ask students to formulate a rule for divisibility by 6 and develop arguments to support their rule. (NCTM 2000, pp. 264–65)

(a)

A teacher asks students to determine how many segments of different lengths can be made by connecting pegs on a square geoboard that is 5 units on each side (a 5 × 5 square geoboard). Because the number of segments is large and some students will have difficulty being systematic in representing the segments on their geoboards, the teacher encourages the students to examine simpler cases to develop a systematic way to generate the different segments. (NCTM 2000, p. 266)



(b)

SUMMARY

If inductive reasoning is to be used to motivate students when doing proofs in the middle grades, it is important that educators recognize which arguments express key ideas and which do not. When a student expresses a mathematical inductive argument, we should always ask how his or her argument will help the classroom community know with certainty that the rule works for any case, not just those that were chosen. If the argument has a key idea in it, this question can be answered in the affirmative.

Last, this strategy is applicable to all students that sixth- through eighth-grade teachers instruct: regular, gifted and talented, remedial, preadvanced placement, and special education. Because the strategy focuses on helping students expand arguments they produce themselves, it is highly individualized and can be tailored to any student's unique learning needs.

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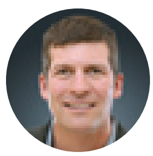

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(*Ed. note:* For more on proof, see the November 2009 *Mathematics Teacher* Focus Issue, titled "Proof: Laying the Foundation.")